Complete Axiomatizations of Fragments of MSO on Finite Trees

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(joint work with Balder ten Cate)
Why Trees?

- **XML**-documents are modeled by trees
  - renewed interest in finite (labelled ordered unranked) trees

- close connection between logic and tree automata
  - low complexity of the model checking problem on trees

- some intractable problems become **tractable** when restricting attention to trees
  - 3-COLORABILITY: from NP-complete to constant time
  - FO/MSO-SATISFIABILITY: from undecidable to decidable
Theorem (Thatcher & Wright 1968; Doner 1970)

A language of finite trees is recognizable by a finite tree automaton $\iff$ it is MSO-definable.

Corollary

The SATISFIABILITY problem for MSO on finite trees is decidable.

Proof: On input $\varphi$, construct the finite tree automaton $A_\varphi$ and check if $L(A_\varphi) = \emptyset$.

Other well-behaved classes of structures

Similar classical results hold for finite words, $\omega$-words, infinite trees...
Complete axiomatizations of MSO on trees

Background

- Axiomatizations of monadic second-order theories:
  - the theory of all countable ordinals (Büchi & Siefkes 1973).
  - the weak theory of two successors (Siefkes 1978).
  - the theory of all ordinals $< \omega^2$ (Zaiontz 1983).
  - techniques based on tree automata

Here: model-theoretic techniques

⇒ Composition of models, using Ehrenfeucht-Fraïssé games.
The basic tool: Ehrenfeucht-Fraïssé games

Duplicator has a **winning strategy** in the \( n \)-round game if he can win in \( n \) rounds no matter how Spoiler plays.

**Theorem**

Duplicator has a **winning strategy** in the \( n \)-round Ehrenfeucht-Fraïssé game on \( A \) and \( B \)

\[ \iff \]

\( A \) and \( B \) cannot be distinguished by FO-sentences of quantifier rank up to \( n \).

**Extension to MSO**

Players can also pick sets.
Complete axiomatization of MSO on finite trees

Completeness on finite trees

\[ \varphi \in MSO \] (or some fragment) valid on finite trees

\[ \iff \]

\[ \varphi \] provable using our axioms

Uniform proof that works for several logics

- **Step 1**: show completeness over a class of Henkin structures that *could be larger* than the class of finite trees.
  \[ \Rightarrow \] relatively easy

- **Step 2**: use composition methods to show that every structure in this class is in fact a finite tree.
  \[ \Rightarrow \] not easy

- **Uniform method**: works also for FO(TC\(^1\)), FO(LFP\(^1\)) etc + other classes of finite structures with bounded tree-width.
The induction axiom

Every node satisfies a given MSO-property, whenever:

*if all descendants and siblings to the right of a node satisfy the property, then this node satisfies it too.*

On finite words:

\[ \forall x (\forall y ((x < y \rightarrow \varphi(y)) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x) ) \]

The issue of infinite models

- Some models of the FO-theory of finite trees are infinite.
- Here, only finite models, but we still need to show that they coincide with the models of our axioms.
Eliminating infinite models

The trick
We show that each of our Henkin models is MSO $n$-equivalent to a finite tree, for all $n$.

The tool
MSO Ehrenfeucht-Fraïssé games give us tools to deal with the notion of $n$-equivalence.

The main Theorem
Composition of MSO $n$-equivalent Henkin structures preserves MSO $n$-equivalence.

⇒ Relying on our induction axiom, we can show MSO $n$-equivalence with a finite tree by induction on sub-forests in the tree.
Our “Feferman-Vaught” theorem

- We consider a general operation of fusion on finite sets of models.

\[ \Rightarrow \text{we reduce the MSO } n\text{-theory of the fusion structure to the MSO } n\text{-theory of the components structures.} \]

If you want to know more

### Follow ups

**The \(\mu\)-Calculus on finite trees (2010)**

Gaëlle Fontaine and Balder ten Cate  
*An Easy Completeness Proof for the Modal \(\mu\)-Calculus on Finite Trees*

**MSO on \(\omega\)-words (2012)**

Colin Riba  
*A Model Theoretic Proof of Completeness of an Axiomatization of Monadic Second-Order Logic on Streams*

**The future**

What about a nice model theoretic proof of completeness for the \(\mu\)-Calculus?
Shelah: *The monadic theory of order* (1975)

Proof of the Büchi Theorem and of many more results (e.g., on dense orderings) using games only, no automata.

Wolfgang Thomas: *Ehrenfeucht Games, the Composition Method, and the Monadic Theory of Ordinal Words* (1997)

*Preference was (and still is) given to the automata theoretic method: by its connection with a computational model it looks more intuitive, it incorporates “programs” in the form of state-transition systems, and it does not involve frightening logical technicalities as one finds them in [Shelah, 1975]. Thus there is a tendency that the merits of the model theoretic approach are overlooked.*