The Equivalence of Game and Denotational Semantics for the Probabilistic μ-Calculus

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1 Introduction

The modal μ-calculus $L_\mu$ [7] is a very expressive logic obtained by extending classical propositional modal logic with least and greatest fixed point operators. The logic $L_\mu$ has been extensively studied as it provides a very powerful tool for expressing properties of labeled transition systems [14]. Encodings of many important temporal logics such as LTL, CTL and CTL* into $L_\mu$ [1], provided evidence for the very high expressive power of the calculus. A precise expressivity result was given in [6], where the authors showed that every formula of monadic second order logic over transition systems which does not distinguish between bisimilar models is equivalent to a formula of $L_\mu$. The logic $L_\mu$ has a simple denotational interpretation [14] and an elegant proof theory [15]. However it is often very difficult to intuitively grasp the denotational meaning of a $L_\mu$ formula as the nesting of fixed point operators can induce very complicated properties. To alleviate this problem, another complementary semantics for the logic $L_\mu$, based on two player (parity) games, has been studied in [3, 14]. The two semantics have been proven to coincide and this allows us to pick the most convenient viewpoint when reasoning about the logic $L_\mu$. One of the main properties of the games used to give semantics to $L_\mu$ formulae is the so called positional determinacy which asserts that both Players can play optimally in a given game-configuration without knowing the history of the previously played moves.

In the last decade, a lot of research has focused on the study of reactive systems that exhibit some kind of probabilistic behavior, and logics for expressing their properties. Segala systems [13] are a natural generalization of labeled transition systems to the probabilistic scenario. Given a countable set of labels $L$, a Segala System is a pair $\langle P, \{ \xrightarrow{a} \} \rangle_{a \in L}$ where $P$ is a countable set of states and, for each $a \in L$, $\xrightarrow{a} \subseteq P \times \mathcal{D}(P)$ is the $a$-accessibility relation, where $\mathcal{D}(P)$ is the set of probability distributions over $P$. The transition relation models the dynamics of the processes: $(p, d) \in \xrightarrow{a}$ means that the process $p$ can perform the atomic action $a \in L$ and then behave like the process $q$ with probability $d(q)$. The probabilistic modal μ-calculus $pL_\mu$, introduced in [12, 5, 2], is a generalization of $L_\mu$ designed for expressing properties of Segala systems. This logic was originally named quantitative μ-calculus, but since other μ-calculus-like logics, designed for expressing properties of non-probabilistic systems, have been given the same name (e.g. [4]), we adopt the probabilistic adjective.

The denotational semantics for the logic $pL_\mu$ of [12, 2] interprets every formula $F$ as a map $[F] : P \rightarrow [0, 1]$, which assigns to each process $p$ a degree of truth. Actually, in [5] three different possible denotational semantics for $pL_\mu$ (including the one of [12, 2]) have been proposed as there is no, a priori, good reason to prefer one in favour of the others.

In [9, 10], the authors introduce a game semantics for the logic $pL_\mu$. This semantics, given in term of two player stochastic (parity) games, is the natural generalization of the two player (non stochastic) game semantics for the logic $L_\mu$: the key difference being that in the configuration $\langle p, [a] F \rangle$ (respectively $\langle p, F \rangle$) Player 1 (respectively Player 2) chooses a $a$-successor of $p$, i.e. a distribution $d$ such that $(p, d) \in \xrightarrow{a}$, and the next configuration $\langle q, F \rangle$ is then reached with probability $d(q)$. This semantics allows one to interpret formulae as expressing, for each process $p$, the (limit) probability of the property...
specified by the formula to hold in the state $p$. This game semantics suggests a very clear operational interpretation for the logical connectives of the logic which, 	extit{a posteriori}, justifies the denotational interpretation of $[12, 2]$.

In $[10, 9]$, the authors proved the equivalence of the denotational and game semantics for $pL_\mu$ only for finite models. The proof crucially depends on positional determinacy which does not hold, in general, for the infinite $pL_\mu$ stochastic parity games generated by infinite models. The general result, i.e. the equivalence of the game and denotational semantics for arbitrary infinite models, has been left open.

In this workshop paper we show that the equivalence indeed holds for arbitrary infinite models, thus strengthening the connection between denotational and game semantics. Our contribution consists in adapting the technique introduced in $[4]$, where the authors used it to prove a similar result for a $\mu$-calculus-like logic designed to express quantitative properties of (non-probabilistic) labeled transition systems. While this is not a difficult adaption, the result seems worth noticing since the question has been open in literature since $[9]$. Moreover the differences between the games considered in $[4]$ and $pL_\mu$ stochastic games, e.g. the fact that Markov chains are the outcomes of the games rather than just infinite paths, make this result not immediate from $[4]$.

The rest of the paper is organized as follows: in section 2 we define the syntax of the logic $pL_\mu$ and the class of models given by Segala systems; in section 3 we define the denotational semantics of $pL_\mu$ as in $[12, 2]$; in section 4 we define the class of parity games that are going to be used to give game semantics to the logic; in section 5 we define the game semantics of $pL_\mu$ in terms of two player stochastic parity games; in section 6 we state the main theorem which asserts the equivalence of the denotational and game semantics. An extended version of this paper with detailed proofs is available at $[11]$.

2 The Probabilistic Modal $\mu$-Calculus

Given a set $\text{Var}$ of propositional variables ranged over by the letters $X, Y, Z$ and a set of labels $L$ ranged over by the letters $a, b, c$, the formulae of the logic are defined by the following grammar:

$$F, G ::= X \mid \langle a \rangle F \mid [a] F \mid F \lor G \mid F \land G \mid \mu X.F \mid \nu X.F$$

We assume the usual notions of free and bound variables. A formula is closed if it has no free variables.

**Definition 2.1 (Subformulae).** We define the function $Sub(F)$ by case analysis on $F$ as follows:

$$\begin{align*}
Sub(X) & \overset{\text{def}}{=} \{ X \} \\
Sub(\langle a \rangle F) & \overset{\text{def}}{=} \{ \langle a \rangle F \} \cup Sub(F) \\
Sub([a] F) & \overset{\text{def}}{=} \{ [a] F \} \cup Sub(F) \\
Sub(F_1 \land F_2) & \overset{\text{def}}{=} \{ F_1 \land F_2 \} \cup Sub(F_1) \cup Sub(F_2) \\
Sub(F_1 \lor F_2) & \overset{\text{def}}{=} \{ F_1 \lor F_2 \} \cup Sub(F_1) \cup Sub(F_2) \\
Sub(\mu X.F) & \overset{\text{def}}{=} \{ \mu X.F \} \cup Sub(F) \\
Sub(\nu X.F) & \overset{\text{def}}{=} \{ \nu X.F \} \cup Sub(F)
\end{align*}$$

The cases for the connectives $\lor, \langle a \rangle$ and $\mu X$ are defined as for their duals. We say that $G$ is a subformula of $F$ if $G \in Sub(F)$.

**Definition 2.2 (Normal Formula).** A formula $F$ is normal if

- Whenever $*_1X_1$ and $*_2X_2$, with $*_1, *_2 \in \{ \mu, \nu \}$, are two different occurrences of binders in $F$ then $X_1 \neq X_2$.
- No occurrence of a free variable $X$ is also used in a binder $*X$ in $F$.

Every formula can be put in normal form by standard $\alpha$-renaming of the bound variables. We only consider formulae $F$ in normal form. A bound variable $X$ in $F$ is called a $\mu$-variable (respectively a $\nu$-variable) if it is bound in $F$ by a $\mu$ (respectively $\nu$) operator.
Definition 2.3 (Variables subsumption). Given a normal formula $F$ such that $*_1X_1.F_1,*_2X_2.F_2 \in Sub(F)$, we say that the variable $X_1$ subsumes $X_2$ in $F$ if $*_2X_2.F_2 \in Sub(F_1)$.

The formulae of the logic $pL_\mu$ are interpreted over the class of models given by Segala systems.

Definition 2.4. A Segala system is a pair $\langle P, \{ _a \} \rangle$ where $P$ is a countable set of states and for each $a \in L$, $\_a \subseteq P \times D(P)$ is the transition relation where $D(P)$ is the set of all probability distribution over $P$. Given a probability distribution $d \in D(P)$ we denote by $\text{supp}(d)$ the support of $d$ defined as the set $\{ p \in P \mid d(p) > 0 \}$.

3 Denotational Semantics

Given a Segala system $\langle P, \{ _a \} \rangle$ we denote by $(P \rightarrow [0,1])$ and by $(D(P) \rightarrow [0,1])$ the complete lattice of functions from $P$ and from $D(P)$ respectively, to the real interval $[0,1]$ with the component-wise order. Given a function $f \in (P \rightarrow [0,1])$, we denote by $\overline{f} \in (D(P) \rightarrow [0,1])$ the lifted function defined as follows: $\overline{f} \overset{\text{def}}{=} \lambda d. (\sum_{p \in \text{supp}(d)} d(p) \cdot f(p))$.

A function $\rho : \text{Var} \rightarrow (P \rightarrow [0,1])$ is called an interpretation of the variables. Given a function $f \in (P \rightarrow [0,1])$ we denote by $\rho[f/X]$ the interpretation that assigns $f$ to the variable $X$, and $\rho(Y)$ to all other variables $Y$.

Fix a Segala System $\langle P, \{ _a \} \rangle$ and an interpretation $\rho$, the denotational semantics $\llbracket F \rrbracket_\rho : P \rightarrow [0,1]$ of the $pL_\mu$ formula $F$, under the interpretation $\rho$, is defined by structural induction on $F$ as follows:

$$
\llbracket X \rrbracket_\rho = \rho(X)
$$

$$
\llbracket G \lor H \rrbracket_\rho = \llbracket G \rrbracket_\rho \cup \llbracket H \rrbracket_\rho \quad \llbracket G \land H \rrbracket_\rho = \llbracket G \rrbracket_\rho \cap \llbracket H \rrbracket_\rho
$$

$$
\llbracket \langle a \rangle G \rrbracket_\rho = \lambda p. \left( \bigcup \{ \llbracket G \rrbracket_\rho(d) \mid p \overset{a}{\rightarrow} d \} \right) \quad \llbracket [a] H \rrbracket_\rho = \lambda p. \left( \bigcap \{ \llbracket H \rrbracket_\rho(d) \mid p \overset{a}{\rightarrow} d \} \right)
$$

$$
\llbracket [\mu X.G] \rrbracket_\rho = \text{gfp of the functional } \lambda f. \left( \llbracket [\mu X.G] \rrbracket_\rho[f/X] \right) \quad \llbracket [\nu X.H] \rrbracket_\rho = \text{gfp of the functional } \lambda f. \left( \llbracket H \rrbracket_\rho[f/X] \right)
$$

Since the interpretation assigned to every $\mu L_\mu$ operator is monotone, the existence of the least and greatest fixed points is guaranteed by the Knaster-Tarski theorem. Moreover the least and the greatest fixed point can be computed inductively: $\llbracket [\mu X.G] \rrbracket_\rho = \bigcup_{\alpha} \llbracket [\mu X.G] \rrbracket_\rho^\alpha$ and $\llbracket [\nu X.H] \rrbracket_\rho = \bigcap_{\alpha} \llbracket [\nu X.G] \rrbracket_\rho^\alpha$ where

$$
\llbracket [\mu X.G] \rrbracket_\rho^\alpha = \begin{cases} 
\llbracket G \rrbracket_\rho \mid P, 0/X \mid & \text{for } \alpha = 0 \\
\llbracket G \rrbracket_\rho \mid [\mu X.G]_\rho^{\alpha-1}/X \mid & \text{for } \alpha \text{ successor ordinal.} \\
\llbracket G \rrbracket_\rho \mid \sqcap_{a \prec \alpha} [\mu X.G]_\rho^{a}/X \mid & \text{for } \alpha \text{ limit ordinal.}
\end{cases}
$$

4 Two Player Stochastic Parity Games

A turn-based Stochastic Game Arena (or just a $2^1$ Game Arena) is a tuple $A = \langle (S,E), \{ S_1, S_2, S_P \}, \pi \rangle$ where $(S,E)$ is a directed graph with countable set of states $S$ and successor function $E : S \rightarrow 2^S$; the sets $S_1, S_2, S_P$ are a partition of $S$ and $\pi : S_P \rightarrow D(S)$ is called the probabilistic transition function. For every state $s \in S$, $E(s)$ is the (possibly infinite) set of successors of $s$. We require that for all $s \in S_P$, $E(s) = \text{supp}(\pi(s)) \neq \emptyset$. We denote by $S_t$ the set of terminal states, i.e. those $s \in S$ such that $E(s) = \emptyset$.

The states in $S_1$ are Player 1 states; the states in $S_2$ are Player 2 states; the states in $S_P$ are probabilistic, or Player $P$, states. At a state $s \in S_1$ (respectively $s \in S_2$), if $s \notin S_t$ Player 1 (respectively Player 2) chooses
a successor from the set \( E(s) \); if \( s \in S_i \) the game ends. At a state \( s \in S_p \), a successor state is chosen probabilistically according to the distribution \( \pi(s) \).

A finite path \( \vec{s} \) in \( A \) is a finite sequence \( s_0, \ldots, s_n \) of states in \( S \) such that for every \( 0 < i \leq n, s_i \in E(s_{i-1}) \). An infinite path \( \vec{s} \) in \( A \) is an infinite sequence of states \( \{ s_i \}_{i \in \mathbb{N}} \) such that for every \( i > 0, s_i \in E(s_{i-1}) \). We denote by \( \mathcal{P}^o \) and \( \mathcal{P}^<o \) the sets of infinite paths and finite paths in \( A \) respectively. Given a finite path \( \vec{s} \in \mathcal{P}^<o \) we denote by \( \text{last}(\vec{s}) \) the last state \( s \in S \) of \( \vec{s} \). We denote by \( \mathcal{P}^<_1, \mathcal{P}^<_2 \) and \( \mathcal{P}^<_p \) the sets of finite paths having last state in \( S_1, S_2 \) and \( S_p \) respectively. We also denote by \( \mathcal{P}^t \) the set of finite paths ending in a terminal state, i.e. the set of paths \( \vec{s} \) such that \( E(\text{last}(\vec{s})) = \emptyset \); the paths in \( \mathcal{P}^t \) are called \text{terminated} paths. We denote by \( \mathcal{P} \) the set \( \mathcal{P}^o \cup \mathcal{P}^t \) and we refer to this set as the set of the possible Games in \( A \). Given a finite path \( \vec{s} \in \mathcal{P}^<o \), we denote by \( O_\vec{s} \) the set of all plays having \( \vec{s} \) as prefix. We consider the standard topology on \( \mathcal{P} \), where the basis for the open sets is given by the cones in \( G \), i.e. the sets \( O_\vec{s} \) for \( \vec{s} \in \mathcal{P}^<o \).

As usual in Game Theory, Players’ moves are determined by strategies. A (full memory) deterministic strategy \( \sigma_i \) for Player 1 in the Game Arena \( A \) is defined as usual as a function \( \sigma_1 : \mathcal{P}^<_1 \rightarrow S \cup \{ \bullet \} \) such that \( \sigma_1(\vec{s}) \in E(\text{last}(\vec{s})) \) if \( E(\text{last}(\vec{s})) \neq \emptyset \) and \( \sigma_1(\vec{s}) = \bullet \) otherwise. Similarly a strategy \( \sigma_2 \) for Player 2 is defined as a function \( \sigma_2 : \mathcal{P}^<_2 \rightarrow S \cup \{ \bullet \} \).

A pair \( \langle \sigma_1, \sigma_2 \rangle \) of strategies, one for each player, is called a strategy profile and determines the behaviors of both players. Fix a strategy profile \( \langle \sigma_1, \sigma_2 \rangle \) and an initial state \( s \in S \) we denote by \( M^a_{\sigma_1, \sigma_2} \) the Markov Chain obtained by pruning \( A \), starting from \( s \), accordingly with \( \sigma_1 \) and \( \sigma_2 \). We often refer to this Markov Chain as the Markov Play generated by the strategy profile \( \langle \sigma_1, \sigma_2 \rangle \) from \( s \in S \). Each Markov Play \( M^a_{\sigma_1, \sigma_2} \) has an associated probability measure on the set \( \mathcal{P} \) of plays in \( A \), which we denote by \( \mathcal{M}^a_{\sigma_1, \sigma_2} \). The probability measure \( \mathcal{M}^a_{\sigma_1, \sigma_2} \) is defined as usual as the unique probability measure which assigns to every basic open set \( O_\vec{s} \) the multiplication of all the probabilities associated with the probabilistic transitions in \( \vec{s} \).

A priority assignment \( P \) (of rank \( n \in \mathbb{N} \)) for the arena \( A \) is a function assigning a natural number in \( \{0, \ldots, n\} \) to every state in \( S \), i.e. \( P : S \rightarrow \{0, \ldots, n\} \). Given a priority assignment \( P \) of any rank, and an infinite path \( \vec{s} = \{ s_i \}_{i \in \mathbb{N}} \), we denote by \( P(\vec{s}) \) the greatest natural number appearing infinitely often in the infinite sequence \( \{ P(s_i) \}_{i \in \mathbb{N}} \). A reward assignment \( B \) for the arena \( A \) is a function assigning a value in the real interval \([0,1]\) to each terminal state \( s \in S_t \), i.e. \( B : S_t \rightarrow [0,1] \). A pair \( \langle P, B \rangle \) of a priority assignment \( P \) and a reward assignment \( B \) for the arena \( A \), determines a unique measurable function \( \Phi_{\langle P, B \rangle} : \mathcal{P} \rightarrow [0,1] \) defined as follows:

\[
\Phi_{\langle P, B \rangle}(\vec{s}) = \begin{cases} 
B(\text{last}(\vec{s})) & \text{if } \vec{s} \in \mathcal{P}^t \\
0 & \text{if } \vec{s} \in \mathcal{P}^o \text{ and } P(\vec{s}) \text{ is even} \\
1 & \text{if } \vec{s} \in \mathcal{P}^o \text{ and } P(\vec{s}) \text{ is odd}
\end{cases}
\]

The value \( \Phi_{\langle P, B \rangle}(\vec{s}) \) should be understood as the payoff assigned to Player 1 when \( \vec{s} \) is the outcome of the game. The payoff function \( \Phi_{\langle P, B \rangle} \) is called the Parity payoff determined by \( \langle P, B \rangle \). We say that \( \Phi_{\langle P, B \rangle} \) has rank \( n \) if \( P \) has rank \( n \). We often omit the subscript \( \langle P, B \rangle \) in \( \Phi_{\langle P, B \rangle} \) if the context is clear enough. Note that, since the priority assigned by \( P \) to the terminal states \( s \in S_t \) does not affect the induced payoff function, we can assume, without any loss of generality, that \( P(s) = 0 \) for any terminal state \( s \in S_t \). From now on we assume that this condition holds for every priority assignment \( P \).

A Two Player Stochastic Parity Game is a tuple \( G = \langle A, P, B \rangle \) of a 2\( \frac{1}{2} \) Game Arena \( A \), a priority assignment \( P \) and a \( B \) assignment for the arena \( A \). Given a Parity game \( G = \langle A, P, B \rangle \), a state \( s \in S \) and a strategy profile \( \langle \sigma_1, \sigma_2 \rangle \), we denote by with \( \mathcal{M}^a_{\sigma_1, \sigma_2}(\Phi_{\langle P, B \rangle}) \) the value defined as:
\[
\int_{\rho} \Phi_{\{\mathbb{P}, \mathbb{B}\}} d\mathcal{M}_{\sigma_1, \sigma_2}^s
\]

which corresponds to the expected payoff for Player 1 when the game starts in the state \(s\) and Player 1 and Player 2 follow the strategies \(\sigma_1\) and \(\sigma_2\) respectively. We denote by \(Val_1(\mathcal{G}) : S \rightarrow [0,1]\) and \(Val_2(\mathcal{G}) : S \rightarrow [0,1]\) the functions defined as follows:

\[
Val_1(\mathcal{G})(s) = \sqcup_{\sigma_1} \sqcap_{\sigma_2} \mathcal{M}_{\sigma_1, \sigma_2}^s(\Phi) \quad Val_2(\mathcal{G})(s) = \sqcap_{\sigma_2} \sqcup_{\sigma_1} \mathcal{M}_{\sigma_1, \sigma_2}^s(\Phi).
\]

\(Val_1(\mathcal{G})(s)\) represents the limit (expected) payoff that Player 1 can get, when the game begins in \(s\), by choosing his strategy \(\sigma_1\) first and then letting Player 2 pick an appropriate counter strategy \(\sigma_2\). Similarly \(Val_2(\mathcal{G})(s)\) represents the limit (expected) payoff that Player 1 can get, when the game begins in \(s\), by first letting Player 2 choose a strategy \(\sigma_2\) and then picking an appropriate counter strategy \(\sigma_1\). Clearly \(Val_1(\mathcal{G})(s) \leq Val_2(\mathcal{G})(s)\) for every \(s \in S\).

**Theorem 4.1 (Determinacy [8]).** For every Two Player Stochastic Parity Game \(\mathcal{G} = (A, \mathbb{P}, \mathbb{B})\) the following equality holds:

\[
\forall s \in S, Val_1(\mathcal{G})(s) = Val_2(\mathcal{G})(s).
\]

Intuitively the determinacy Theorem states that the players do not get any advantage by letting the opponent choose his strategy first. We just write \(\forall(\mathcal{G})\) for the value function of the Game defined as \(Val_1(\mathcal{G}) = Val_2(\mathcal{G})\). As a corollary of the Determinacy theorem we have the following Lemma:

**Lemma 4.2 (\(\varepsilon\)-optimal strategies).** Given a Two Player Stochastic Parity Game \(\mathcal{G} = (A, \mathbb{P}, \mathbb{B})\), for every \(\varepsilon > 0\) the following assertions hold:

- there exists a strategy \(\sigma_1^\varepsilon\) for Player 1 such that for every \(s \in S\), \(\sqcup_{\sigma_1} \sqcap_{\sigma_2} \mathcal{M}_{\sigma_1, \sigma_2}^s(\Phi) \geq \forall(\mathcal{G})(s) - \varepsilon\).
- there exists a strategy \(\sigma_2^\varepsilon\) for Player 2 such that for every \(s \in S\), \(\sqcap_{\sigma_2} \sqcup_{\sigma_1} \mathcal{M}_{\sigma_1, \sigma_2}^s(\Phi) \leq \forall(\mathcal{G})(s) + \varepsilon\).

5 Game Semantics

Fix a Segala System \(\langle P, \{a \in L\}\rangle\), a formula \(F\) and an interpretation \(\rho : \text{Var} \rightarrow (P \rightarrow [0,1])\) of the variables, we denote by \(\mathcal{G}^\rho\) the parity game \((A, \mathbb{P}, \mathbb{B})\) formally defined as described below.

The state space of the arena \(A = \langle (S, \mathbb{E}), \{S_1, S_2, S_p\}, \pi \rangle\), is the set \(S = (P \cup \mathcal{D}(P)) \times \text{Sub}(F)\) of pairs of states \(p \in P\) or distributions \(d \in \mathcal{D}(P)\) and subformulas \(G \in \text{Sub}(F)\); the transition relation \(E\) is defined as \(E(\langle d, G \rangle) = \{\langle p, G \rangle \mid p \in \text{supp}(d)\}\) for every \(d \in \mathcal{D}(P)\); \(E(\langle p, G \rangle)\) is defined by case analysis on the outermost connective of \(G\) as follows:

1. if \(G = X\), with \(X\) free in \(F\), then \(E(\langle p, G \rangle) = \emptyset\).
2. if \(G = X\), with \(X\) bound in \(F\) by the subformula \(*X.H\), with \(* \in \{\mu, \nu\}\), then \(E(\langle p, G \rangle) = \{\langle p, *X.H \rangle\}\).
3. if \(G = *X.H\), with \(* \in \{\mu, \nu\}\), then \(E(\langle p, G \rangle) = \{\langle p, H \rangle\}\).
4. if \(G = \langle a \rangle H\) or \(G = [a] H\) then \(E(\langle p, G \rangle) = \{\langle d, H \rangle \mid p \overset{a}{\rightarrow} d\}\).
5. if \(G = H \lor H'\) or \(G = H \land H'\) then \(E(\langle p, G \rangle) = \{\langle p, H \rangle, \langle p, H' \rangle\}\).

The partition \(\{S_1, S_2, S_p\}\) is defined as follows: every state \(\langle p, G \rangle\) with \(G\)’s main connective in \(\{\langle a \rangle, \lor, \mu X\}\) or with \(G = X\) where \(X\) is a \(\mu\)-variable, is in \(S_1\); dually every state \(\langle p, G \rangle\) with \(G\)’s main connective in \(\{[a], \land, \nu X\}\) or with \(G = X\) where \(X\) is a \(\nu\)-variable, is in \(S_2\). Finally every state \(\langle d, G \rangle\) is in \(S_p\). The terminal states \(\langle p, X \rangle\), with \(X\) free in \(F\), are in \(S_1\) by convention. The probability transition function \(\pi : S_p \rightarrow S\) is defined as \(\pi(\langle d, G \rangle)(\langle p, G \rangle) = d(p)\). The priority assignment \(\mathbb{P}\) is defined as
usual in \( \mu \)-calculus model checking games [14]: the priority assigned to the states \( \langle p, X \rangle \), with \( X \) a \( \mu \)-variable is even; dually the priority assigned to the states \( \langle p, X \rangle \), with \( X \) a \( \nu \)-variable is odd. Moreover \( \mathbb{P}(\langle p, X \rangle) > \mathbb{P}(\langle p', X' \rangle) \) if \( X \) subsumes \( X' \) in \( F \). All other states get priority 0. The reward assignment \( \mathbb{B} \) is defined as \( \mathbb{B}(\langle p, X \rangle) = \rho(X)(p) \) for every terminal state \( \langle p, X \rangle \) with \( X \) free in \( F \). All other terminal states in \( \mathcal{G}_\rho^F \) are either of the form \( \langle p, (a)H \rangle \) or \( \langle p, [a]H \rangle \). The reward assignment \( \mathbb{B} \) is defined on these terminal states as follows: \( \mathbb{B}(\langle p, (a)H \rangle) = 0 \) and \( \mathbb{B}(\langle p, [a]H \rangle) = 1 \).

Fix a Segala system \( \langle P, \{ a \rightarrow \}_a \in L \rangle \), the game semantics of the formula \( F \) under the interpretation \( \rho \), is the function \( \llbracket F \rrbracket_\rho : P \to [0, 1] \) defined as \( \llbracket F \rrbracket_\rho \triangleq \lambda p. \forall \nu(\mathcal{G}_\rho^F)(\langle p, F \rangle) \).

6 Equivalence of Denotational and Game Semantics for \( pL_\mu \)

We are now ready to give the main theorem which states that given any Segala system \( \langle P, \{ a \rightarrow \}_a \in L \rangle \), the denotational and game semantics of any \( pL_\mu \) formula coincide.

**Theorem 6.1.** Given a Segala system \( \langle P, \{ a \rightarrow \}_a \in L \rangle \), for every \( pL_\mu \) formula \( F \) and interpretation \( \rho \) for the variables, the following equality holds: \( \llbracket F \rrbracket_\rho = \llbracket F \rrbracket_\rho \).

References