Łukasiewicz $\mu$-calculus

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The paper explores properties of Łukasiewicz $\mu$-calculus, a version of the quantitative/probabilistic modal $\mu$-calculus containing both weak and strong conjunctions and disjunctions from Łukasiewicz (fuzzy) logic. We show that this logic encodes the well-known probabilistic temporal logic PCTL. And we give a model-checking algorithm for computing the rational denotational value of a formula at any state in a finite rational probabilistic nondeterministic transition system.

1 Introduction

Among logics for expressing properties of nondeterministic (including concurrent) processes, represented as transition systems, Kozen’s modal $\mu$-calculus [15] plays a fundamental rôle. It subsumes other temporal logics of processes, such as LTL, CTL and CTL*. It does not distinguish bisimilar processes, but separates (finite) non-bisimilar ones. More generally, by a remarkable result of Janin and Walukiewicz [14], it is exactly as expressive as the bisimulation-invariant fragment of monadic second-order logic. Furthermore, there is an intimate connection with parity games, which offers an intuitive reading of fixed-points, and underpins the existing technology for model-checking $\mu$-calculus properties.

For many purposes, it is useful to add probability to the computational model, leading to probabilistic nondeterministic transition systems, cf. [23]. Among the different approaches that have been followed to developing analogues of the modal $\mu$-calculus in this setting, the most significant is that introduced independently by Huth and Kwiatkowska [12] and by Morgan and McIver [22], under which a quantitative interpretation is given, with formulas denoting values in $[0, 1]$. This quantitative setting permits several variations. In particular, three different quantitative extensions of conjunction from booleans to $[0, 1]$ (with 0 as false and 1 as true) arise naturally [12]: minimum, min$(x, y)$; multiplication, $xy$; and the strong conjunction (a.k.a. Łukasiewicz t-norm) from Łukasiewicz fuzzy logic, max$(x + y - 1, 0)$. In each case, there is a dual operator giving a corresponding extension of disjunction: maximum, max$(x, y)$; comultiplication, $x + y - xy$; and Łukasiewicz strong disjunction, min$(x + y, 1)$. The choice of min and max for conjunction and disjunction is particularly natural, since the corresponding $\mu$-calculus, called qL$\mu$ in [13], has an interpretation in terms of 2-player stochastic parity games, which extends the usual parity-game interpretation of the ordinary modal $\mu$-calculus. This allows the real number denoted by a formula to be understood as the value of the associated game [18, 20].

The present paper contributes to a programme of ongoing research, one of whose overall aims is to investigate the extent to which quantitative $\mu$-calculi play as fundamental a rôle in the probabilistic setting as that of Kozen’s $\mu$-calculus in the nondeterministic setting. The logic qL$\mu$, with min/max as conjunction/disjunction, is insufficiently expressive. For example, it cannot encode the standard probabilistic temporal logic PCTL of [2]. Nevertheless, richer calculi can be obtained by augmenting qL$\mu$ with the other alternatives for conjunction/disjunction, to be used in combination with max and min. Such extensions were investigated by the first author in [21, 19], where the game-theoretic interpretation was generalized to accommodate the new operations.
In this paper, we focus on a calculus containing two different interpretations of conjunction and disjunction: min and max (written as $\sqcap$ and $\sqcup$) and the Łukasiewicz operations (written as $\odot$ and $\oplus$). In addition, as is natural in the quantitative setting, we include a basic operation for multiplying the value of a formula by a rational constant in $[0,1]$. Since these operations are all familiar from Łukasiewicz fuzzy logic (see, e.g., [11]), we call the resulting logic Łukasiewicz $\mu$-calculus ($\mu\ell\mu$).

As our first contribution, we show that the standard probabilistic temporal logic PCTL [2] can be encoded in $\mu\ell\mu$. A similar translation was originally given in the first author’s PhD thesis [19], where PCTL was translated into a quantitative $\mu$-calculus containing all three pairs of quantitative conjunction/disjunction operations in combination. Here, we streamline the treatment by implementing the observation that the (co)multiplication operations are not required once the Łukasiewicz operations are in place. In fact, all that is needed is the encodability of certain threshold modalities, see Remark 3.6 below.

An advantage of the Łukasiewicz $\mu$-calculus considered in the present paper is that it enjoys the property that the value of a formula in a finite rational model is rational, a property which does not hold when the (co)multiplication operations are included in the logic. As our second contribution, we exploit this property by giving a (quantitative) model-checking algorithm that computes the value of a $\mu\ell\mu$ formula at a state in a finite rational probabilistic nondeterministic transition system. The algorithm adapts the approximation-based approach to nested fixed-point calculation to our quantitative calculus.

One could combine our two contributions and obtain a new model-checking algorithm for PCTL. But this is not advisable since the complexity bounds we obtain for model-checking $\mu\ell\mu$ are abysmal. The positive messages of this paper are rather that PCTL fits into the conceptually appealing framework of quantitative $\mu$-calculi, and that this framework is itself algorithmically approachable.

2 Technical background

Definition 2.1. Given a set $S$ we denote with $\mathcal{D}(S)$ the set of (discrete) probability distributions on $S$ defined as $\mathcal{D}(S) = \{d : S \rightarrow [0,1] | \sum_{s \in S} d(s) = 1 \}$. We say that $d \in \mathcal{D}(S)$ is rational if $d(s)$ is a rational number, for all $s \in S$.

Definition 2.2. A probabilistic nondeterministic transition system (PNTS) is a pair $(S,\rightarrow)$ where $S$ is a set of states and $\rightarrow \subseteq S \times \mathcal{D}(S)$ is the accessibility relation. We write $s \not\rightarrow d$ if $\{d | s \rightarrow d\} = \emptyset$. A PNTS $(S,\rightarrow)$ is finite rational if $S$ is finite and $\bigcup_{s \in S} \{d | s \rightarrow d\}$ is a finite set of rational probability distributions.

We now introduce the novel logic $\ell\mu$ which extends the probabilistic (or quantitative) modal $\mu$-calculus (qL$\mu$) of [12,22,18,5].

Definition 2.3. The logic $\ell\mu$ is generated by the following grammar:

$$\phi ::= X | P | \overline{P} | q \phi | \phi \sqcap \phi | \phi \sqcup \phi | \phi \ominus \phi | \phi \oplus \phi | \Diamond \phi | \Box \phi | \mu X.\phi | \nu X.\phi,$$

where $q$ ranges over rationals in $[0,1]$, $X$ over a countable set $\mathcal{V}ar$ of variables and $P$ over a set $\mathcal{P}rop$ of propositional letters which come paired with associated complements $\overline{P}$. As a convention we denote $\bot$ the formula $\nu X.X$ and with $\overline{q}$ the formula $q \bot$.

Thus, $\ell\mu$ extends the syntax of the probabilistic modal $\mu$-calculus by the new pair of connectives ($\ominus$, $\oplus$), which we refer to as Łukasiewicz conjunction and disjunction, respectively, and a form of scalar multiplication ($q \phi$) by rationals numbers in $[0,1]$. For mild convenience in the encoding of PCTL below, we consider a version with unlabelled modalities and propositional letters. However, the approach of this paper easily adapts to a labeled version of $\ell\mu$.

Formulas are interpreted over PNTS’s as we now describe.
Definition 2.4. Given a PNTS $(S, \rightarrow)$, an interpretation for the variables and propositional letters is a function $\rho : (\Var \cup \Prop) \rightarrow (S \rightarrow [0, 1])$ such that $\rho(\overline{P})(x) = 1 - \rho(P)(x)$. Given a function $f : S \rightarrow [0, 1]$ and $X \in \Var$ we define the interpretation $\rho[f/X]$ as $\rho[f/X](x) = f$ and $\rho[f/X](X) = \rho(X)$, for $X \neq Y$.

Definition 2.5. The semantics of a Ł formula $\phi$ interpreted over $(S, \rightarrow)$ with interpretation $\rho$ is a function $[\phi]_\rho : S \rightarrow [0, 1]$ defined inductively on the structure of $\phi$ as follows:

$$
[X]_\rho = \rho(X) \\
[P]_\rho = \rho(P) \\
[\phi \land \psi]_\rho(x) = \min\{[\phi]_\rho(x), [\psi]_\rho(x)\} \\
[\phi \lor \psi]_\rho(x) = \max\{1, [\phi]_\rho(x) + [\psi]_\rho(x)\} \\
[\phi \Rightarrow \psi]_\rho(x) = \sup\{0, [\phi]_\rho(x) + [\psi]_\rho(x)\} \\
[\phi \Leftrightarrow \psi]_\rho(x) = \sup\{0, [\phi]_\rho(x) + [\psi]_\rho(x) - 1\} \\
[\Box \phi]_\rho(x) = \bigcap_{y \sim x} (\sum_{y \in X} d(y)[\phi]_\rho(y)) \\
[\mu X. \phi] = \text{gfp}(f \mapsto [\phi]_\rho[f/X])
$$

It is straightforward to verify that the interpretation of every operator is monotone, thus the existence of least and greatest points in the last two clauses is guaranteed by the the Knaster-Tarski theorem.

As customary in fixed-point logics, we presented the logic Ł in positive normal form. A negation operator $\mu X. \phi$ can be defined on closed formulas by replacing every connective with its dual and $(\neg \phi)$ with $(1 - \phi)$. It is simple to verify that $[\mu X. \phi]_\rho(x) = 1 - [\phi]_\rho(x)$.

Next, we introduce the syntax and the semantics of the logic PCTL. We refer to [1] for an extensive presentation of this logic.

The notions of paths, schedulers and Markov runs in a PNTS are at the basis of the logic PCTL.

Definition 2.6. For a given PNTS $\mathcal{L} = (S, \rightarrow)$ the binary relation $\sim_\mathcal{L} \subseteq S \times S$ is defined as follows: $\sim_\mathcal{L} = \{ (s, t) \mid \exists d. (s \rightarrow d \land d(t) > 0) \}$. Note that $s \not\sim$ if and only if only if $s \not\sim^*$. We refer to $\langle S, \sim \rangle$ as the graph underlying $\mathcal{L}$.

Definition 2.7. A path in a PNTS $\mathcal{L} = (S, \rightarrow)$ is an ordinary path in the graph $(S, \sim)$, i.e., a finite or infinite sequence $\{s_i\}_{i \in I}$ of states such that $s_i \sim s_{i+1}$, for all $i + 1 \in I$. We say that a path is maximal if either it is infinite or it is finite and its last entry is a state $s_n$ without successors, i.e., such that $s_n \not\sim^*$. We denote with $P(\mathcal{L})$ the set of all maximal paths in $\mathcal{L}$. The set $P(\mathcal{L})$ is endowed with the topology generated by the basic open sets $U_\vec{s} = \{ \vec{\tau} \mid \vec{s} \sqsubseteq \vec{\tau} \}$ where $\vec{s}$ is a finite sequence of states and $\sqsubseteq$ denotes the prefix relation on sequences. The space $P(\mathcal{L})$ is always 0-dimensional, i.e., the basic sets $U_\vec{s}$ are both open and closed and thus form a Boolean algebra. We denote with $P(s)$ the open set $U_{\{s\}}$ of all maximal paths having $s$ as first state.

Definition 2.8. A scheduler in a PNTS $(S, \rightarrow)$ is a partial function $\sigma$ from non-empty finite sequences $s_0, \ldots, s_n$ of states to probability distributions $d \in \mathcal{D}(S)$ such that $\sigma(s_0, \ldots, s_n)$ is not defined if and only if $s_n \not\rightarrow$ and, if $\sigma$ is defined at $s_0, \ldots, s_n$ with $\sigma(s_0, \ldots, s_n) = d$, then $s_n \rightarrow d$ holds. A pair $(s, \sigma)$ is called a Markov run in $\mathcal{L}$ and denoted by $M^s_\sigma$. It is clear that each Markov run $M^s_\sigma$ can be identified with a (generally) infinite Markov chain (having a tree structure) whose vertices are finite sequences of states and having $\{s\}$ as root.

Markov runs are useful as they naturally induce probability measures on the space $P(\mathcal{L})$.

Definition 2.9. Let $\mathcal{L} = (S, \rightarrow)$ be a PNTS and $M^s_\sigma$ a Markov run. We define the measure $m^s_\sigma$ on $P(\mathcal{L})$ as the unique (by Carathéodory extension theorem) measure specified by the following assignment of basic open sets:

$$
m^s_\sigma(U_{s_0, \ldots, s_n}) = \prod_{i=0}^{n-1} d_i(s_{i+1})
$$
where $d_i = \sigma(s_0 \ldots s_i)$ and $\prod \emptyset = 1$. It is simple to verify that $m^s_\sigma$ is a probability measure, i.e., $m^s_\sigma(P(\mathcal{L})) = 1$. We refer to $m^s_\sigma$ as the probability measure on $P(\mathcal{L})$ induced by the Markov run $M^s_\sigma$.

We are now ready to specify the syntax and semantics of PCTL.

**Definition 2.10.** Let the letter $P$ range over a countable set of propositional symbols $\mathsf{Prop}$. The class of PCTL state-formulas $\phi$ is generated by the following two-sorted grammar:

$$\phi ::= \text{true} \mid P \mid \neg \phi \mid \phi \lor \psi \mid \exists \psi \mid \forall \psi \mid \mathbb{P}^3_{\ast q} \psi \mid \mathbb{P}^\triangledown_{\ast q} \psi$$

with $q \in \mathbb{Q} \cap [0,1]$ and $\triangledown \in \{>, \geq\}$, where path-formulas $\psi$ are generated by the simple grammar:

$$\psi ::= \circ \phi \mid \phi \mathbin{\triangledown} \phi_2.$$ Adopting standard terminology, we refer to the connectives $\circ$ and $\mathbin{\triangledown}$ as the next and until operators, respectively.

**Definition 2.11.** Given a PNTS $(S, \to)$, a PCTL-interpretation for the propositional letters is a function $\rho : \mathsf{Prop} \to 2^S$, where $2^S$ denotes the powerset of $S$.

**Definition 2.12.** Given a PNTS $(S, \to)$ and a PCTL-interpretation $\rho$ for the propositional letters, the semantics $\langle \phi \rangle_{\rho}$ of a PCTL state-formula $\phi$ is a subset of $S$ (i.e., $\langle \phi \rangle_{\rho} : S \to \{0,1\}$) defined by induction on the structure of $\phi$ as follows:

- $\langle \text{true} \rangle_{\rho} = S$, $\langle P \rangle_{\rho} = \rho(P)$, $\langle \phi_1 \lor \phi_2 \rangle_{\rho} = \langle \phi_1 \rangle_{\rho} \cup \langle \phi_2 \rangle_{\rho}$, $\langle \neg \phi \rangle_{\rho} = S \setminus \langle \phi \rangle_{\rho}$,
- $\langle \exists \psi \rangle_{\rho}(s) = 1$ if and only there exists $\bar{s} \in P(s)$ such that that $\bar{s} \in \langle \psi \rangle_{\rho}$,
- $\langle \forall \psi \rangle_{\rho}(s) = 1$ if and only forall $\bar{s} \in P(s)$ it holds that $\bar{s} \in \langle \psi \rangle_{\rho}(\bar{s})$,
- $\langle \mathbb{P}^3_{\ast q} \psi \rangle_{\rho}(s) = 1$ if and only $(\bigcup_{\sigma} m^s_{\sigma}(\langle \psi \rangle_{\rho})) \times q$,
- $\langle \mathbb{P}^\triangledown_{\ast q} \psi \rangle_{\rho}(s) = 1$ if and only $(\bigcap_{\sigma} m^s_{\sigma}(\langle \psi \rangle_{\rho})) \times q$,

where $\sigma$ ranges over schedulers and the semantics $\langle \psi \rangle_{\rho}$ of path formulas, defined as a subset of $P(\mathcal{L})$ (i.e., as a map $\langle \psi \rangle_{\rho} : P(\mathcal{L}) \to \{0,1\}$) is defined as:

- $\langle \circ \phi \rangle_{\rho}(\bar{s}) = 1$ if and only if $|\bar{s}| \geq 2$ (i.e., $\bar{s} = s_0, s_1 \ldots$) and $s_1 \notin \langle \phi \rangle_{\rho}$,
- $\langle \phi_1 \mathbin{\triangledown} \phi_2 \rangle_{\rho}(\bar{s}) = 1$ if and only if $\exists n \cdot (s_n \in \langle \phi_2 \rangle_{\rho}) \land \forall m < n \cdot (s_m \in \langle \phi_1 \rangle_{\rho})$.

It is simple to verify that, for all path-formulas $\psi$, the set $\langle \psi \rangle_{\rho}$ is Borel measurable [1]. Therefore the definition is well specified. Note how the logic PCTL can express probabilistic properties, by means of the connectives $\mathbb{P}^\triangledown_{\ast q}$ and $\mathbb{P}^3_{\ast q}$, as well as (qualitative) properties of the graph underlying the PNTS by means of the quantifiers $\forall$ and $\exists$.

3 Encoding of PCTL

We prove in this section how PCTL can be seen as a simple fragment of $\mathcal{L}\mu$ by means of an explicit encoding. We first introduce a few useful macro formulas in the logic $\mathcal{L}\mu$ which, crucially, are not expressible in the probabilistic $\mu$-calculus (qL$\mu$).

**Definition 3.1.** Let $\phi$ be a (possibly open) $\mathcal{L}\mu$ formula. We define:

- $P_{>0} \phi = \mu X(X \lor \phi)$
- $P_{=1} \phi = \nu X(X \land \phi)$
- $P_{>q} \phi = P_{>0}(\phi \land 1 - q)$
- $P_{\geq q} \phi = P_{=1}(\phi \lor 1 - q)$

for $q \in \mathbb{Q} \cap (0,1)$. We write $P_{\ast q} \phi$, for $q \in \mathbb{Q} \cap [0,1]$, to denote one of the four cases.

The following proposition describes the denotational semantics of these macro formulas.

**Proposition 3.2.** Let $(S, \to)$ be a PNTS, $\phi$ a $\mathcal{L}\mu$ formula and $\rho$ an interpretation of the variables. Then it holds that:
\[ [\mathbb{P}_{=q} \phi]_\rho(s) = \begin{cases} 1 & \text{if } [\phi]_\rho(s) \times q \\ 0 & \text{otherwise} \end{cases} \]

**Proof.** For the case \( \mathbb{P}_{>0} \phi \), observe that the map \( x \mapsto q \circ x \), for a fixed \( q \in [0, 1] \), has 1 as unique fixed point when \( q > 0 \), and 0 as the least fixed point when \( q = 0 \). The result then follows trivially. Similarly for \( \mathbb{P}_{=1} \phi \). The other cases are trivial. \( \square \)

The following lemma is also useful.

**Lemma 3.3.** Let \( (S, \rightarrow) \) be a PNTS, \( \phi \) a \( \mathcal{L} \mu \) formula and \( \rho \) an interpretation of the variables. Then:

1. \( [\mathbb{P}_{>0} (\Diamond X)]_\rho(s) = 1 \) iff \( \exists t. (s \rightarrow t \land \rho(X)(t) > 0) \)
2. \( [\mathbb{P}_{=1} (\Box X)]_\rho(s) = 1 \) iff \( \forall t. (s \rightarrow t \land \rho(X)(t) = 1) \)

**Proof.** Note that \( [\Diamond X]_\rho(s) > 0 \) iff there exists \( s \rightarrow d \) such that \( \sum_{t \in S} d(t) \rho(X)(t) > 0 \) holds. This is the case iff \( d(t) > 0 \) (i.e., \( s \rightarrow t \)) and \( \rho(X)(t) > 0 \), for some \( t \in S \). The result then follows by Proposition 3.2. The case for \( \mathbb{P}_{=1} (\Box X) \) is similar. \( \square \)

**Remark 3.4.** When considering \( \{0, 1\} \)-valued interpretations for \( X \), the macro formula \( \mathbb{P}_{>0} \Diamond \) expresses the meaning of the diamond modality in classical modal logic with respect to the graph \( (S, \rightarrow) \) underlying the PNTS. Similarly, \( \mathbb{P}_{=1} \Box \) corresponds to the classical box modality.

We are now ready to define the encoding of \( \text{PCTL} \) into \( \mathcal{L} \mu \).

**Definition 3.5.** We define the encoding \( \textbf{E} \) from \( \text{PCTL} \) formulas to closed \( \mathcal{L} \mu \) formulas (where \( \Box \phi \) stands for the \( \mathcal{L} \mu \) formula \( \Box \phi \land \Box 1 \)), by induction on the structure of the \( \text{PCTL} \) formulas \( \phi \) as follows:

1. \( \textbf{E}(P) = P \),
2. \( \textbf{E}(\text{true}) = \bot \),
3. \( \textbf{E}(\phi_1 \lor \phi_2) = \textbf{E}(\phi_1) \cup \textbf{E}(\phi_2) \),
4. \( \textbf{E}(\neg \phi) = \text{dual}(\textbf{E}(\phi)) \),
5. \( \textbf{E}(\exists \phi) = \mathbb{P}_{>0} (\Diamond \textbf{E}(\phi)) \),
6. \( \textbf{E}(\forall \phi) = \mathbb{P}_{=1} (\Box \text{E}(\phi)) \),
7. \( \textbf{E}(\exists (\phi_1 \forall \phi_2)) = \mu X. \left( \textbf{E}(\phi_2) \cup (\textbf{E}(\phi_1) \cap \mathbb{P}_{>0} \Diamond X) \right) \),
8. \( \textbf{E}(\forall (\phi_1 \forall \phi_2)) = \mu X. \left( \textbf{E}(\phi_2) \cup (\textbf{E}(\phi_1) \cap \mathbb{P}_{=1} \Box X) \right) \),
9. \( \textbf{E}(\mathbb{P}_{>q} \Diamond \phi) = \mathbb{P}_{=q} (\Diamond \textbf{E}(\phi)) \),
10. \( \textbf{E}(\mathbb{P}_{=q} \Diamond \phi) = \mathbb{P}_{=q} (\Box \text{E}(\phi)) \),
11. \( \textbf{E}(\mathbb{P}_{>q} (\phi_1 \forall \phi_2)) = \mathbb{P}_{=q} \left( \mu X. \left( \textbf{E}(\phi_2) \cup (\textbf{E}(\phi_1) \cap \Diamond X) \right) \right) \),
12. \( \textbf{E}(\mathbb{P}_{=q} (\phi_1 \forall \phi_2)) = \mathbb{P}_{=q} \left( \mu X. \left( \textbf{E}(\phi_2) \cup (\textbf{E}(\phi_1) \cap \Box X) \right) \right) \),

Note that Case 4 is well defined since \( \textbf{E}(\phi) \) is closed by construction.
Remark 3.6. The only occurrences of Łukasiewicz operators \{\oplus, \odot\} and scalar multiplication \((q\phi)\) in encoded PCTL formulas appear in the formation of the macro formulas \(\mathbb{P}_{\lambda q}()\) which we refer to as threshold modalities. Thus, PCTL can be also seen as a fragment of qL\(\mu\) extended with threshold modalities as primitive operations. With the aid of these modalities the encoding is, manifestly, a straightforward adaption of the standard encoding of CTL into the modal \(\mu\)-calculus (see, e.g., [24]).

We are now ready to prove the correctness theorem which holds for arbitrary models.

Theorem 3.7. For every PNTS \((S, \rightarrow)\), PCTL-interpretation \(\rho: \text{Prop} \rightarrow (S \rightarrow \{0, 1\})\) of the propositional letters and PCTL formula \(\phi\), the equality \(\langle \phi \rangle_\rho(s) = [E(\phi)]_\rho(s)\) holds, for all \(s \in S\).

Proof (outline). The proof goes by induction on the complexity of \(\phi\). Cases 1-4 of Definition 3.5 are trivial. Case 5 follows directly from Lemma 3.3. Observing that \([\Box \phi]_\rho(s) = 0\) if \(s \not \rightarrow\) and \([\Box \phi]_\rho(s) = [\Box \phi]_\rho(s)\) otherwise, also Case 6 is a consequence of Lemma 3.3. Consider cases 7 and 8. The encoding is of the form \(\mu X.(F \sqcup (G \sqcap H(X)))\), where \(F\) and \(G\) (by induction hypothesis) and \(H(X)\) (by Proposition 3.2) are all \{0, 1\}-valued. Therefore the functor \(f \mapsto [F \sqcup (G \sqcap H(X))]_{\rho/F(X)}\) maps \{0, 1\}-valued functions to \{0, 1\}-valued functions and has only \{0, 1\}-valued fixed-points. It then follows by Remark 3.4 that the correctness of the encoding for these two cases can be proved with the standard technique used to prove the correctness of the encoding of CTL into Kozen’s \(\mu\)-calculus (see, e.g., [24]). Consider Case 9. It is immediate to verify that \(\bigcup_\sigma \{m^\sigma_\rho(U)\}\) holds, for all \(s \in S\). Let \(\phi = \mathbb{P}_{\lambda q}^\sigma(\Psi)\) and \(\psi = \phi_1 \nabla \phi_2\). We denote by \(\Psi\) the set of paths \(\langle \psi \rangle_\rho\). Denote by \(F(X)\) the formula \(E(\phi_1) \sqcup (E(\phi_1) \sqcap \Diamond X)\). It is clearly sufficient to prove that the equality \(\bigcup_\sigma \{m^\sigma_\rho(\Psi)\} = [\mu X.F(X)]_\rho(s)\) holds. Note that \(\mu X.F(X)\) can be expressed as an equivalent qL\(\mu\) formulas by substituting the closed subformulas \(E(\phi_1)\) and \(E(\phi_2)\) with two fresh atomic predicates \(P_i\) with interpretations \(\rho(P_i) = [E(\phi_i)]_\rho\). The equality can then be proved by simple arguments based on the game-semantics of qL\(\mu\) (see, e.g., [18] and [20]), similar to the ones used to prove that the Kozen’s \(\mu\)-calculus formula \(\mu X.(P_2 \lor (P_1 \land \Diamond X))\) has the same denotation of the CTL formula \(\exists(P_1 \nabla P_2)\) (see, e.g., [24]). □

4 Łukasiewicz \(\mu\)-terms

The aim of the second half of this paper is to show how to compute the (rational) denotational value of a \(\Lambda\mu\) formula at any state in a finite rational probabilistic transition system. In this section, we build the main machinery for doing this, based on a system of fixed-point terms for defining monotone functions from \([0, 1]^n\) to \([0, 1]\). The syntax of (Łukasiewicz) \(\mu\)-terms is specified by the grammar:

\[
t := x \mid qt \mid t \sqcup t \mid t \sqcap t \mid t \oplus t \mid t \odot t \mid \mu x.t \mid \nu x.t
\]

Again, \(q\) ranges over rationals in \([0, 1]\). As expected, the \(\mu\) and \(\nu\) operators bind their variables. We write \(t(x_1, \ldots, x_n)\) to mean that all free variables of \(t\) are contained in \(\{x_1, \ldots, x_n\}\).

The value \(t(\vec{r})\) (we eschew semantic brackets) of a \(\mu\)-term \(t(x_1, \ldots, x_n)\) applied to a vector \((r_1, \ldots, r_n) \in [0, 1]^n\) is defined inductively in the obvious way, cf. Definition 2.5. (Indeed, \(\mu\)-terms form a fragment of \(\Lambda\mu\) of formulas whose value is independent of the transition system in which they are interpreted.)

In Section 6 the model-checking task will be reduced to the problem of computing the value of \(\mu\)-terms. The fundamental property that allows such values to be computed is that, for any \(\mu\)-term \(t(x_1, \ldots, x_n)\) and vector of rationals \((q_1, \ldots, q_n)\), the value of \(t(\vec{q})\) is rational and can be computed from
and \( \varrho \). One way of establishing this result is by a simple reduction to the first-order theory of rational linear arithmetic, which provides an indirect means of computing the value of \( t(\varrho) \). The current section presents a brief outline of this approach. After this, in Section 5, we provide an alternative direct algorithm for computing \( t(\varrho) \).

A linear expression in variables \( x_1, \ldots, x_n \) is an expression

\[
q_1x_1 + \cdots + q_nx_n + q
\]

where \( q_1, \ldots, q_n, q \) are real numbers. In the sequel, we only consider rational linear expressions, in which \( q_1, \ldots, q_n, q \) are all rational, and we henceforth assume this property without mention. We write \( e(x_1, \ldots, x_n) \) if \( e \) is a linear expression in \( x_1, \ldots, x_n \), in which case, given real numbers \( r_1, \ldots, r_n \), we write \( e(\bar{r}) \) for the value of the expression when the variables \( \bar{x} \) take values \( \bar{r} \). We also make use of the closure of linear expressions under substitution: given \( e(x_1, \ldots, x_n) \) and \( e_1(y_1, \ldots, y_m), \ldots, e_n(y_1, \ldots, y_m) \), we write \( e(e_1, \ldots, e_n) \) for the evident substituted expression in variables \( y_1, \ldots, y_m \) (which is defined formally by multiplying out and adding coefficients).

The first-order theory of rational linear arithmetic has linear expressions as terms, and strict and non-strict inequalities between linear expressions,

\[
e_1 < e_1 \quad e_1 \leq e_2 \; ,
\]

as atomic formulas. Equality can be expressed as the conjunction of two non-strict inequalities and the negation of an atomic formula can itself be expressed as an atomic formula. The truth of a first-order formula is given via its interpretation in the reals, or equivalently in the rationals since the inclusion of the latter in the former is an elementary embedding. The theory enjoys quantifier elimination [8].

**Proposition 4.1.** For every \( \Lukasiewicz \) \( \mu \)-term \( t(x_1, \ldots, x_n) \), its graph \( \{ (\bar{x}, y) \in [0,1]^{n+1} \mid t(\bar{x}) = y \} \) is definable by a formula \( F_t(x_1, \ldots, x_n, y) \) in the first-order theory of rational linear arithmetic, where \( F_t \) is computable from \( t \).

**Proof.** The proof is a straightforward induction on the structure of \( t \). We consider two cases, in order to illustrate the simple manipulations used in the construction of \( F_t \).

If \( t \) is \( t_1 \oplus t_2 \) then \( F_t \) is the formula

\[
\exists z_1, z_2. F_{t_1}(\bar{x}, z_1) \land F_{t_2}(\bar{x}, z_2) \land ((z_1 + z_2 \leq 1 \land z = z_1 + z_2) \lor (1 \leq z_1 + z_2 \land z = 1))
\]

If \( t \) is \( \mu x_{n+1}. t' \) then \( F_t \) is the formula

\[
F_t(x_1, \ldots, x_n, y, y) \land \forall z. F_{t'}(x_1, \ldots, x_n, z, z) \rightarrow y \leq z .
\]

\( \square \)

Proposition 4.1 provides the following method of computing the value \( t(\varrho) \) of \( \mu \)-term \( t(x_1, \ldots, x_n) \) at a rational vector \( (q_1, \ldots, q_n) \in [0,1]^n \). First construct \( F_t(x_1, \ldots, x_n, y) \). Next, perform quantifier elimination to obtain an equivalent quantifier-free formula \( G_t(x_1, \ldots, x_n, y) \), and consider its instantiation \( G_t(q_1, \ldots, q_n, y) \) at \( \varrho \). (Alternatively, obtain an equivalent formula \( G^\varrho_t(y) \) by performing quantifier elimination on \( F_t(q_1, \ldots, q_n, y) \).) By performing obvious simplifications of atomic formulas in one variable, \( G_t(q_1, \ldots, q_n, y) \) reduces to a boolean combination of inequalities each having one of the following forms

\[
y \leq q \quad y < q \quad y \geq q \quad y > q \; .
\]
By the correctness of $G_i$ there must be a unique rational satisfying the boolean combination of constraints, and this can be extracted in a straightforward way from $G_i(q_1, \ldots, q_n, y)$.

We give a crude (but sufficient for our purposes) complexity analysis of the above procedure. In general, for a $\mu$-term $t$ of length $n$ containing $\nu$ fixed points, the length of $F_i$ is bounded by $2^{\nu uc}$, for some constant $c$. The quantifier-elimination procedure in \cite{leivant95}, when given a formula of length $l$ as input produces a formula of length at most $2^{dl}$ as output, for some constant $d$, and takes time at most $2^{2dl}$. Thus the length of the formula $G_i(x_1, \ldots, x_n, y)$ is bounded by $2^{\nu uc}$, and the computation time for $t(\vec{q})$ is $O(2^{{\nu uc}d})$, using a unit cost model for rational arithmetic.

5 A direct algorithm for evaluating $\mu$-terms

Our direct approach to computing the values of $\mu$-terms is based on a simple explicit representation of the functions defined by such terms. A conditioned linear expression is a pair, written $C \vdash e$, where $e$ is a linear expression, and $C$ is a finite set of strict and non-strict inequalities between linear expressions; i.e., each element of $C$ has one of the forms in \cite{boros99}. We write $C(\vec{r})$ for the conjunction of the inequations obtained by instantiating $\vec{r}$ for $\vec{x}$ in $C$. Clearly, if $\vec{q}$ is a vector of rationals then it is decidable if $C(\vec{q})$ is true or false. The intended meaning of a conditioned linear expression $C \vdash e$ is that it denotes the value $e(\vec{r})$ when applied to a vector of reals $\vec{r}$ for which $C(\vec{r})$ is true, otherwise it is undefined. A basic property we exploit in the sequel is that every conditioning set $C(x_1, \ldots, x_n)$ defines a convex subset $\{(r_1, \ldots, r_n) \mid C(\vec{r})\}$ of $\mathbb{R}^n$.

Let $\mathcal{P}$ be a system (i.e., finite set) of conditioned linear expressions in variables $x_1, \ldots, x_n$. We say that $\mathcal{P}$ represents a function $f : [0, 1]^n \rightarrow [0, 1]$ if the following conditions hold:

1. For all $d_1, \ldots, d_n \in [0, 1]$, there exists a conditioned linear expression $(C \vdash e) \in \mathcal{P}$ such that $C(\vec{d})$ is true, and
2. for all $d_1, \ldots, d_n \in [0, 1]$, and every conditioned linear expression $(C \vdash e) \in \mathcal{P}$, if $C(\vec{d})$ is true then $e(\vec{d}) = f(\vec{d})$.

Note that, for two conditioned linear expressions $(C_1 \vdash e_1), (C_2 \vdash e_2) \in \mathcal{P}$, we do not require different conditioning sets $C_1$ and $C_2$ to be disjoint. However, $e_1$ and $e_2$ must agree on any overlap.

Obviously, the function represented by a system of conditioned linear expressions is unique, when it exists. But not every system represents a function. One could impose syntactic conditions on a system to ensure that it represents a function, but we shall not pursue this.

While conditioned linear expressions provide a syntax more directly tailored to expressing functions than general logical formulas, their expressivity in this regard coincides with rational linear arithmetic.

**Proposition 5.1.** A function $f : [0, 1]^n \rightarrow [0, 1]$ is representable by a system of conditioned linear expressions if and only if its graph $\{(\vec{x}, y) \in [0, 1]^{n+1} \mid f(\vec{x}) = y\}$ is definable by a formula $F(x_1, \ldots, x_n, y)$ in the first-order theory of rational linear arithmetic. Moreover, a defining formula and a representing system of conditioned linear equations can each be computed from the other.

We believe this result to be folklore. The proof is a straightforward application of quantifier elimination.

Combining Propositions\cite{boros99} and \cite{leivant95} we obtain:

**Corollary 5.2.** For every Łukasiewicz $\mu$-term $t(x_1, \ldots, x_n)$, the function

$$\vec{r} \mapsto t(\vec{r}) : [0, 1]^n \rightarrow [0, 1]$$

is representable by a system of conditioned linear expressions in variables $x_1, \ldots, x_n$. Furthermore a representing system can be computed from $t$. 
The computation of a representing system for \( t \) via quantifier elimination, provided by the proofs of Propositions \[4.1\] and \[5.1\] is indirect. The goal of this section is to present an alternative algorithm for calculating the value \( t(\bar{r}) \) of a \( \mu \)-term at rationals \( r_1, \ldots, r_n \in [0, 1] \), which is directly based on manipulating conditioned linear expressions. Rather than computing an entire system of conditioned linear expressions representing \( t \), the algorithm works locally to provide a single conditioned expression that applies to the input vector \( \bar{r} \).

The algorithm takes, as input, a \( \mu \)-term \( t(x_1, \ldots, x_n) \) and a vector of rationals \( (r_1, \ldots, r_n) \in [0, 1]^n \), and returns a conditioned linear expression \( C \vdash e \), in variables \( x_1, \ldots, x_n \), with the following two properties.

(P1) \( C(\bar{r}) \) is true.

(P2) For all \( s_1, \ldots, s_n \in \mathbb{R} \), if \( C(\bar{s}) \) is true then \( s_1, \ldots, s_n \in [0, 1] \) and \( e(\bar{s}) = t(\bar{s}) \).

It follows that \( e(\bar{r}) = t(\bar{r}) \), so \( e \) can indeed be used to compute the value \( t(\bar{r}) \).

5.1 The algorithm

The algorithm takes, as input, a \( \mu \)-term \( t(x_1, \ldots, x_n) \) and a vector of rationals \( (r_1, \ldots, r_n) \in [0, 1]^n \), and returns a conditioned linear expression \( C \vdash e \), in variables \( x_1, \ldots, x_n \), with the properties (P1) and (P2) above. For the purposes of the correctness proof in Section 5.3, it is convenient to consider the running of the algorithm in the more general case that \( r_1, \ldots, r_n \) are arbitrary real numbers in \([0, 1] \). This more general algorithm can be understood as an algorithm in the Real RAM (a.k.a. BSS) model of computation \[3\]. When the input vector is rational, all real numbers encountered during execution of the algorithm are themselves rational, and so the general Real RAM algorithm specialises to a bona fide (Turing Machine) algorithm in this case. Moreover, even in the case of irrational inputs, all linear expressions constructed in the course of the algorithm are rational.

The algorithm works recursively on the structure of the term \( t \). We present illustrative cases for terms \( t_1 \oplus t_2 \) and \( \mu x_{n+1}.t' \). The latter is the critical case. The algorithm for \( \forall x_{n+1}.t' \) is an obvious dualization.

If \( t \) is \( t_1 \oplus t_2 \) then recursively compute \( C_1 \vdash e_1 \) and \( C_2 \vdash e_2 \). If \( e_1(\bar{r}) + e_2(\bar{r}) \leq 1 \) then return

\[
C_1, C_2, e_1 + e_2 \leq 1 \vdash e_1 + e_2 .
\]

Otherwise, return

\[
C_1, C_2, e_1 + e_2 \geq 1 \vdash 1 .
\]

In the case that \( t \) is \( \mu x_{n+1}.t' \), enter the following loop starting with \( D = \emptyset \) and \( d = 0 \).

\textbf{Loop:} At the entry of the loop we have a finite set \( D \) of inequalities between linear expressions in \( x_1, \ldots, x_n \), and we have a linear expression \( d(x_1, \ldots, x_n) \). The loop invariant that applies is:

(1) \( D(\bar{r}) \) is true; and

(2) for all \( \bar{s} \in [0, 1]^n \), if \( D(\bar{s}) \) then \( d(\bar{s}) \leq (\mu x_{n+1}.t')(\bar{s}) \).

We think of \( D \) as constraints propagated from earlier iterations of the loop, and of \( d \) as the current approximation to the least fixed point subject to the constraints.

Recursively compute \( t'(x_1, \ldots, x_{n+1}) \) at \( (\bar{r}, d(\bar{r})) \) as \( C \vdash e \), where \( e \) has the form:

\[
q_1 x_1 + \cdots + q_n x_n + q_{n+1} x_{n+1} + q .
\]

(2)
In the case that \( q_{n+1} \neq 1 \), define the linear expression:

\[
f := \frac{1}{1 - q_{n+1}} (q_1 x_1 + \cdots + q_n x_n + q)
\]

Test if \( C(\vec{r}, f(\vec{r})) \) is true. If it is, exit the loop and return:

\[
D \cup C(x_1, \ldots, x_n, d(x_1, \ldots, x_n)) \cup C(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \vdash f
\]

as the result of the algorithm for \( \mu x. t' \) at \( \vec{r} \). Otherwise, if \( C(\vec{r}, f(\vec{r})) \) is false, define \( N(x_1, \ldots, x_n) \) to be the negation of the inequality \( e_1(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) < e_2(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \) (using < to stand for either \( < \) or \( \leq \)), where \( e_1(x_1, \ldots, x_n+1) < e_2(x_1, \ldots, x_n+1) \) is a chosen inequality in \( C \) for which \( e_1(\vec{r}, f(\vec{r})) \), \( e_2(\vec{r}, f(\vec{r})) \) is false, and go to find next approximation below.

In the case that \( q_{n+1} = 1 \), test the equality \( q_1 r_1 + \cdots + q_n r_n + q = 0 \). If true, exit the loop with result:

\[
D \cup C(x_1, \ldots, x_n, d(x_1, \ldots, x_n)) \cup \{q_1 x_1 + \cdots + q_n x_n + q = 0\} \vdash d
\]

If instead \( q_1 r_1 + \cdots + q_n r_n + q \neq 0 \), choose \( N(x_1, \ldots, x_n) \) to be whichever of the inequalities

\[
q_1 x_1 + \cdots + q_n x_n + q < 0 \quad \text{or} \quad 0 < q_1 x_1 + \cdots + q_n x_n + q
\]

is true for \( \vec{r} \), and proceed with find next approximation below.

**Find next approximation:** Arrange the inequalities in \( C \) so they have the following structure.

\[
C' \cup \{x_{n+1} > a_i \}_{1 \leq i \leq p} \cup \{x_{n+1} \geq a_i \}_{p < i \leq q} \cup \{x_{n+1} < b_i \}_{1 \leq i \leq m'} \cup \{x_{n+1} < b_i \}_{m' < i \leq m}
\]

such that the only variables in the inequalities \( C' \), and linear expressions \( a_i, b_i \) are \( x_1, \ldots, x_n \). Choose \( j \) with \( 1 \leq j \leq m \) such that \( b_j(\vec{r}) \leq b_j(\vec{r}) \) for all \( i \) with \( 1 \leq i \leq m \). Then go back to loop, taking

\[
D \cup C(x_1, \ldots, x_n, d(x_1, \ldots, x_n)) \cup \{N(x_1, \ldots, x_n)\} \cup \{b_j \leq b_i \mid 1 \leq i \leq m\} \quad e(\vec{x}, b_j(\vec{x}))
\]

to replace \( D \) and \( d \) respectively.

### 5.2 A simple example

Consider the \( \mathbb{L}_\mu \) term \( t = \mu x. (\mathbb{P}_{\geq \frac{1}{2}} x \cup \frac{1}{2}) \), where \( \mathbb{P}_{\geq \frac{1}{2}} x \) is the macro formula as in Definition 3.1 that is \( \mathbb{P}_{\geq \frac{1}{2}} x = \mathbb{P}_{=1} (x \oplus \frac{1}{2}) = vy.(y \oplus (x \oplus \frac{1}{2})) \). Thus,

\[
t = \mu x. (vy.(y \oplus (x \oplus \frac{1}{2})) \cup \frac{1}{2})
\]

Here, \( t'(x) = vy.(y \oplus (x \oplus \frac{1}{2})) \cup \frac{1}{2} \) is a discontinuous function, and the value of \( t \) is 1.

We omit giving a detailed simulation of the algorithm on the subexpression \( t'(x) \) at \( x = r \). The result it produces, however, is \( \{0 \leq x < \frac{1}{2}\} \vdash \frac{1}{2} \) if \( r < \frac{1}{2} \), and \( \{\frac{1}{2} \leq x \leq 1\} \vdash 1 \) if \( r \geq \frac{1}{2} \).

We run the algorithm on input \( \mu x.t'(x) \). Set \( D = \emptyset \) and \( d = 0 \). Calculating \( t'(x) \) at \( x = 0 \) we obtain \( C \vdash e \) as \( \{0 \leq x < \frac{1}{2}\} \vdash \frac{1}{2} \). We now need to calculate \( f := \frac{1}{1-0}(\frac{1}{2}) = \frac{1}{2} \). The constraint \( C(\frac{1}{2}) \) does not hold. Thus we need to improve the approximation \( d = 0 \). Since \( e = \frac{1}{2} \) is constant, the next approximation is \( \frac{1}{2} \). The new set of constraints is still the empty set. Thus we iterate the algorithm with \( D = \emptyset \) and \( d = \frac{1}{2} \). Calculating \( t'(x) \) at \( x = \frac{1}{2} \) produces \( C \vdash e \) as \( \{\frac{1}{2} \leq x \leq 1\} \vdash 1 \). Compute \( f := \frac{1}{1-\frac{1}{2}}(1) = 1 \). Since \( C(1) \) holds, the algorithm terminates with \( 0 \vdash 1 \), as desired.
5.3 Correctness of the algorithm

Theorem 5.3. Let \( t(x_1, \ldots, x_n) \) be any Łukasiewicz \( \mu \)-term. Then, for every input vector \((r_1, \ldots, r_n) \in [0, 1]^n\), the above (Real RAM) algorithm terminates with a conditioned linear expression \( C_\varphi \vdash e_\varphi \) satisfying properties (P1) and (P2). Moreover, the set of all possible resulting conditioned linear expressions

\[
\{ C_\varphi \vdash e_\varphi \mid \varphi \in [0, 1]^n \}
\]

is finite, and thus provides a representing system for the function \( t : [0, 1]^n \rightarrow [0, 1] \).

Proof. By induction on the structure of \( t \). We verify the critical case when \( t = \mu x_{n+1}.t' \).

We show first that the loop invariants (I1), (I2) guarantee that any result returned via (4) or (5) satisfies (P1) and (P2). By induction hypothesis, the recursive computation of \( t'(x_1, \ldots, x_{n+1}) \) at \((\vec{r}, d(\vec{r}))\) as \( C \vdash e \), where \( e \) has the form \( q_1 x_1 + \cdots + q_n x_n + q_{n+1} x_{n+1} + q \) as in (2), satisfies: \( C(\vec{r}, d(\vec{r})) \); and, for all \( s_1, \ldots, s_{n+1} \in \mathbb{R} \), if \( C(s_1, \ldots, s_{n+1}) \) then \( \vec{s} \in [0, 1]^n \) and \( t'(s_1, \ldots, s_{n+1}) = e(s_1, \ldots, s_{n+1}) \).

In the case that \( q_{n+1} \neq 1 \), the linear expression \( f \), defined in (3), maps any \( s_1, \ldots, s_n \in \mathbb{R} \) to the unique solution \( f(\vec{s}) \) to the equation \( x_{n+1} = e(s_1, \ldots, s_n, x_{n+1}) \) in \( \mathbb{R} \). Suppose that \( D(\vec{s}) \) holds. Then, by loop invariant (I2), \( d(\vec{s}) \leq (\mu x_{n+1}.t')(\vec{s}) \). Suppose also that \( C(\vec{s}, f(\vec{s})) \). Then \( t'(\vec{s}, f(\vec{s})) = e(\vec{s}, f(\vec{s})) = f(\vec{s}), \) i.e., \( f(\vec{s}) \) is a fixed point of \( x_{n+1} \mapsto t'(\vec{s}, x_{n+1}) \); whence, \( (\mu x_{n+1}.t')(\vec{s}) \leq f(\vec{s}) \). Suppose, finally, that \( C(\vec{s}, d(\vec{s})) \) also holds. Then, because both \( C(\vec{s}, f(\vec{s})) \) and \( C(\vec{s}, f(\vec{s})) \), and \( d(\vec{s}) \leq (\mu x_{n+1}.t')(\vec{s}) \leq f(\vec{s}) \), we have, by the convexity of constraints, that \( t'(\vec{s}, s_{n+1}) = e(\vec{s}, s_{n+1}) \) for all \( s_{n+1} \in [d(\vec{s}), f(\vec{s})] \). So \( f(\vec{s}) \) is the unique fixed-point of \( x_{n+1} \mapsto t'(\vec{s}, x_{n+1}) \) on \([d(\vec{s}), f(\vec{s})]\). Since, \( d(\vec{s}) \leq (\mu x_{n+1}.t')(\vec{s}) \), we have \( f(\vec{s}) = (\mu x_{n+1}.t')(\vec{s}) \). This argument justifies that the conditioned linear expression of (4) satisfies (P2). It satisfies (P1) just if \( C(\vec{r}, f(\vec{r})) \), which is exactly the condition under which (4) is returned as the result.

In the case that \( q_{n+1} = 1 \) then, for any \( s_1, \ldots, s_n \in \mathbb{R} \), the equation \( x_{n+1} = e(s_1, \ldots, s_n, x_{n+1}) \) has a solution if and only if \( q_1 s_1 + \cdots + q_n s_n + q = 0 \), in which case any \( x_{n+1} \in \mathbb{R} \) is a solution. Suppose that \( q_1 s_1 + \cdots + q_n s_n + q = 0 \) and \( C(\vec{s}, d(\vec{s})) \) both hold. Then \( t'(s_1, \ldots, s_n, d(\vec{s})) = e(\vec{s}, d(\vec{s})) = d(\vec{s}) \), so \( d(\vec{s}) \) is a fixed point of \( x_{n+1} \mapsto t'(\vec{s}, x_{n+1}) \). If also \( D(\vec{s}) \) holds then, by loop invariant (I2), \( d(\vec{s}) = (\mu x_{n+1}.t')(\vec{s}) \). We have justified that the conditioned linear expression of (5) satisfies (P2). It satisfies (P1) just if \( q_1 r_1 + \cdots + q_n r_n + q = 0 \), which is exactly the condition under which (5) is returned as the result.

Next we show that the loop invariants are preserved through the computation. Properties (I1) and (I2) are trivially satisfied by the initial values \( D = 0 \) and \( d = 0 \). We must show that they are preserved when \( D \) and \( d \) are modified via (7), which happens when execution passes to find next approximation. In this subroutine, the inequalities in \( C \) are first arranged as in (6) where, as \( C(\vec{r}, d(\vec{r})) \), we must have \( m \geq 1 \), as otherwise \( C(\vec{r}, s) \) would hold for all real \( s \geq d(\vec{r}) \), contradicting that \( C(\vec{r}, s) \) implies \( s \in [0, 1] \). (Similarly, \( l \geq 1 \) ) Thus there indeed exists \( j \) with \( 1 \leq j \leq m \) such that \( b_j(\vec{r}) \leq b_i(\vec{r}) \) for all \( i \) with \( 1 \leq i \leq m \). It is immediate that the constraints in the modified \( D \) of (7) are true for \( \vec{r} \). Thus (I1) is preserved. To show (I2), suppose \( s_1, \ldots, s_n \) satisfy the constraints, i.e.,

\[
D(\vec{s}) \quad C(\vec{s}, d(\vec{s})) \quad N(\vec{s}) \quad \{ b_j(\vec{s}) \leq b_i(\vec{s}) \mid 1 \leq i \leq m \}.
\]

Defining \( t' = (\mu x_{n+1}.t')(\vec{s}) \), by (I2) for \( D, d \) we have \( d(\vec{s}) \leq t' \). We must show that \( e(\vec{s}, b_j(\vec{s})) \leq t' \). By the definition of \( N(x_1, \ldots, x_n) \), in either the \( q_{n+1} \neq 1 \) or \( q_{n+1} = 1 \) case, \( N(\vec{s}) \) implies that \( C(\vec{s}, t') \) does not hold. Because \( C(\vec{s}, d(\vec{s})) \) and by the choice of \( j \), it holds that \( C(\vec{s}, s) \), for all \( s \in [0, 1] \) such that \( s = d(\vec{s}) \) or
\[ d(\bar{s}) < s < b_j(\bar{s}). \] Since \( C(\bar{s}, r') \) is false and \( d(\bar{s}) < r' \), it follows from the convexity of the conditioning set \( C \) that, for every \( s \) with \( s = d(\bar{s}) \) or \( d(\bar{s}) < s < b_j(\bar{s}) \), we have \( s < r' \). Whence, since \( r' \) is the least prefixed point for \( x_{n+1} \mapsto t'(\bar{s},x_{n+1}), \) also \( s < t'(\bar{s},s) \leq r' \), i.e.,

\[ s < e(\bar{s},s) \leq r'. \] (9)

Thus, \( e(\bar{s},b_j(\bar{s})) = \sup \{ e(\bar{s},s) \mid s = d(\bar{s}) \text{ or } d(\bar{s}) < s < b_j(\bar{s}) \} \leq r' \). Thus, \( e(\bar{s},b_j(\bar{s})) \leq r' \), i.e., it is an approximation to the fixed point. Moreover, it is a good new approximation to choose in the sense that:

\[ d(\bar{s}) < e(\bar{s},b_j(\bar{s})) \text{ and not } C(\bar{s},e(\bar{s},b_j(\bar{s}))). \] (10)

The former holds because \( d(\bar{s}) < e(\bar{s},d(\bar{s})) \), by (9), and \( d(\bar{s}) \leq b_j(\bar{s}) \). The latter because if \( C(\bar{s},e(\bar{s},b_j(\bar{s}))) \) then, in particular, \( e(\bar{s},b_j(\bar{s})) \leq b_j(\bar{s}) \), so \( b_j(\bar{s}) = e(\bar{s},b_j(\bar{s})) = r' \), contradicting that not \( C(\bar{s},r') \).

To show termination, by induction hypothesis, collecting all possible results of running the algorithm on \( t' \) produces a representing system for \( t' : [0, 1]^{n+1} \mapsto [0, 1] : C_1 \vdash e_1 \ldots C_{k'} \vdash e_{k'} \),

\[ k' \text{ is the basis size of } t'. \] (11)

where \( k' \) is the basis size of \( t' \). We now analyse the execution of the algorithm for \( \mu x_{n+1}.t' \) on a given input vector \((r_1, \ldots, r_n)\). On iteration number \( i \), the loop is entered with constraints \( D_i \) and approximation \( d_i \) (where \( D_1 = \emptyset \) and \( d_1 = 0 \)), after which the recursive call to the algorithm for \( t' \) yields one of the conditioned linear expressions, \( C_k \vdash e_k \), from (11) above, such that \( C_k(\vec{r},d_i(\vec{r})) \) holds. Then, depending on conditions involving only \( C_k \vdash e_k \) and \( \vec{r} \), either a result is returned, or \( D_{i+1} \) and \( d_{i+1} \) are constructed for the loop to be repeated. By (10), at iteration \( i + 1 \) of the loop, we have \( d_{i+1}(\vec{r}) > d_i(\vec{r}) \) and also \( C_k(\vec{r},d_{i+1}(\vec{r})) \) is false. Since each conditioning set is convex, it follows that no \( C_j \) can occur twice in the list \( C_{k_1}, C_{k_2}, \ldots \). Hence the algorithm must exit the loop after at most \( k' \) iterations. Therefore, the computation for \( \mu x.t' \) at \( \vec{r} \) terminates.

It remains to show that the algorithm for \( \mu x.t' \) produces only finitely many conditioned linear expressions \( C_\vec{r} \vdash e_\vec{r} \). The crucial observation is that the vector \( \vec{r} \) is used only to determine the control flow of the algorithm, i.e., which branches of conditional statements are followed, the choices made in selecting \( N \) and \( b_j \) in (7), and the order in which the different \( C_j \vdash e_j \), from (11) are visited (given by the sequence \( k_1, k_2, \ldots \) of values taken by \( j \)). Using this, if \( l' \) is the condition size of \( t' \), then a loose upper bound is that the number of possible results \( C_\vec{r} \vdash e_\vec{r} \) for the algorithm for \( \mu x_{n+1}.t' \) is at most \( (k'(l')^2)^{k'} \), and the number of inequalities in \( C_\vec{r} \) is at most \( 2k'l' \).

The above proof gives a truly abysmal complexity bound for the algorithm. Let the basis and condition size for the term \( t'(x_1, \ldots, x_{n+1}) \) be \( k' \) and \( l' \) respectively. Then, as in the proof, the basis and condition size for \( \mu x_{n+1}.t' \) are respectively bounded by:

\[ k \leq (k'(l')^2)^{k'} \text{ and } l \leq 2k'l'. \]

Using these bounds, the basis and condition size have non-elementary growth in the number of fixed points in a term \( t \).

### 5.4 Comparison

According to the crude complexity analyses we have given, the evaluation of Łukasiewicz \( \mu \)-terms via rational linear arithmetic is (in having doubly- and triply-exponential space and time complexity bounds)
preferable to the (non-elementary space and hence time) evaluation via the direct algorithm. Nevertheless, we expect the direct algorithm to work better than this in practice. Indeed, a main motivating factor in the design of the direct algorithm is that the algorithm for $\mu x_{n+1}.t'$ only explores as much of the basis set for $t'$ as it needs to, and does so in an order that is tightly constrained by the monotone improvements made to the approximating $d$ expressions along the way. In contrast, the crude complexity analysis is based on a worst-case scenario in which the algorithm is assumed to visit the entire basis for $t'$, and, moreover, to do so, for different input vectors $\vec{r}$, in every possible order for visiting the different basis sets. Perhaps better bounds can be obtained by a more careful analysis of the algorithm.

6 Model checking

Let $\phi$ be a closed $\mu$ formula and $(S, \rightarrow)$ a finite rational PNTS. We wish to compute the value $\lbrack \phi \rbrack(s)$ at any given state $s \in S$. We do this by effectively producing a closed $\mu$-term $t_\phi$, with the property that $t_\phi(\phi) = \lbrack \phi \rbrack(s)$, whence the rational value of $\lbrack \phi \rbrack(s)$ can be calculated by the algorithm in Section 5.

We assume, without loss of generality, that all fixed-point operators in $\phi$ bind distinct variables. Let $X_1, \ldots, X_m$ be the variables appearing in $\phi$. We write $\sigma_i X_i. \psi_i$ for the unique subformula of $\phi$ in which $X_i$ is bound. The strict (i.e., irreflexive) domination relation $X_i \triangleright X_j$ between variables is defined to mean that $\sigma_j X_j. \psi_j$ occurs as a subformula in $\psi_i$.

Suppose $|S| = n$. For each $s \in S$, we translate $\phi$ to a $\mu$-term $t_\phi$ containing at most $mn$ variables $x_{i,s}$, where $1 \leq i \leq m$ and $s' \in S$. The translation is defined using a more general function $t_\phi^\Gamma$, defined on subformulas of $\phi$, where $\Gamma \subseteq \{1, \ldots, m\} \times S$ is an auxiliary component keeping track of the states at which variables have previously been encountered. Given $\Gamma$ and $(i, s) \in \{1, \ldots, m\} \times S$, we define:

$$\Gamma \triangleright (i, s) = (\Gamma \cup \{(i, s)\}) \setminus \{(j, s') \in \Gamma | X_i \triangleright X_j\} .$$

This operation is used in the definition below to ‘reset’ subordinate fixed-point variables whenever a new variable that dominates them is declared.

$$t_\phi^\Gamma(X_i) = \begin{cases} x_{i,s} & \text{if } (i, s) \in \Gamma \\ \sigma_{i,1,s}. t_\phi^{\triangleright (i, s)}(\psi_i) & \text{otherwise} \end{cases}$$

$$t_\phi^\Gamma(P) = \rho(P)(s)$$

$$t_\phi^\Gamma(\neg P) = 1 - \rho(P)(s)$$

$$t_\phi^\Gamma(q \phi) = qt_\phi^\Gamma(\phi)$$

$$t_\phi^\Gamma(\phi_1 \bullet \phi_2) = t_\phi^\Gamma(\phi_1) \bullet t_\phi^\Gamma(\phi_2) \quad \bullet \in \{\sqcup, \sqcap, \oplus, \odot\}$$

$$t_\phi^\Gamma(\hat{\phi}) = \bigcup_{s \rightarrow d} \bigoplus_{s' \in S} d(s') t_\phi^\Gamma(\phi)$$

$$t_\phi^\Gamma(\bigotimes i \phi) = \bigcap_{s \rightarrow d} \bigoplus_{s' \in S} d(s') t_\phi^\Gamma(\phi)$$

$$t_\phi^\Gamma(\sigma_i X_i. \psi_i) = \sigma_{i,1,s}. t_\phi^{\triangleright \{(i, s)\}}(\psi_i)$$

This is well defined because changing from $\Gamma$ to $\Gamma \triangleright (i, s)$ or to $\Gamma \cup \{(i, s)\}$ strictly increases the function

$$i \mapsto |\{(i, s) | (i, s) \in \Gamma\}| : \{1, \ldots, m\} \rightarrow \{0, \ldots, n\}$$

under the lexicographic order on functions relative to $\triangleright$. 
Proposition 6.1. For any closed Łµ formula φ, finite PNTS (S, →) and s ∈ S, it holds that $[\phi](s) = t^s_0(\phi)$.

We omit the laborious proof. It is reminiscent of the reduction of modal μ-calculus model checking to a system of nested boolean fixed-point equations in Section 4 of [17].

7 Related and future work

The first encodings of probabilistic temporal logics in a probabilistic version of the modal μ-calculus were given in [4], where a version PCTL∗, tailored to processes exhibiting probabilistic but not non-deterministic choice, was translated into a non-quantitative probabilistic variant of the μ-calculus, which included explicit (probabilistic) path quantifiers but disallowed fixed-point alternation.

In their original paper on quantitative μ-calculi [12], Huth and Kwiatkowska attempted a model checking algorithm for alternation-free formulas in the version of Łµ with ⊕ and ⊙ but without ⊓,⊔ and scalar multiplication. Subsequently, several authors have addressed the problem of computing (sometimes approximating) fixed points for monotone functions combining linear (sometimes polynomial) expressions with min and max operations; see [10] for a summary. However, such work has focused on (efficiently) finding outermost (simultaneous) fixed-points for systems of equations whose underlying monotone functions are continuous. The nested fixed points considered in the present paper give rise to the complication of non-continuous functions, as the example of Section 5.2 demonstrates.

As future work, it is planned to run an experimental comparison of the direct algorithm against the reduction to linear arithmetic. As suggested in Section 5.4, we expect the direct algorithm to work better in practice than the non-elementary upper bound on its complexity, given by our crude analysis, suggests. Furthermore, as a natural generalization of the approximation approach to computing fixed points, the direct algorithm should be amenable to optimizations such as the simultaneous solution of adjacent fixed points of the same kind, and the reuse of previous approximations when applicable due to monotonicity considerations. Unlike the black-box reduction to linear arithmetic, based on quantifier elimination, the linear-constraint-based approach of the direct algorithm should also offer a flexible machinery helpful in the design of optimized procedures for calculating values of particular subclasses of Łµ-terms. An important example is given by the fragment of Łµ capable of encoding PCTL (see Remark 3.6).

Our results on Łµ are a contribution towards the development of a robust theory of fixed-point probabilistic logics. The simplicity of the proposed encoding of PCTL (see Remark 3.6 above) suggests that the direction we are following is promising. In a follow-up paper, by the first author, it will be shown that the process equivalence characterised by Łukasiewicz μ-calculus is the standard notion of probabilistic bisimilarity [23]. Thus the quantitative approach to probabilistic μ-calculi may be considered equally suitable as a mechanism for characterising process equivalence as the non-quantitative μ-calculi advocated for this purpose in [4] and [7].

Further research will have to explore the relations between quantitative μ-calculi such as Łµ and other established frameworks for verification and design of probabilistic systems. Important examples include the abstract probabilistic automata of [6], the compositional assume-guarantee techniques of [16,9] and the recent p-automata of [13]. In particular, with respect to the latter formalism, we note that the acceptance condition of p-automata is specified in terms of stochastic games whose configurations may have preseeded threshold values whose action closely resembles that of the threshold modalities considered in this work (Definition 3.1). Exploring the relations between p-automata games and Łµ-games [19] could shed light on some underlying fundamental ideas.
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References


We add detail to the outlined proof of Theorem 3.7, by supplying the omitted argument for the equality

\[ \bigwedge_{\sigma} \{m_{\sigma}^{r}(\Psi)\} = \left[\mu X . F(X)\right]_{\rho}(s), \]

which appears as case 11. Although game semantics provides the most intuitive justification, we instead give a direct denotational proof, in order to avoid introducing game-theoretic machinery.

**Expanded proof of Theorem 3.7**

**Case 11 (\(\leq\)).** We first show that

\[ \bigwedge_{\sigma} \{m_{\sigma}^{r}(\Psi)\} \leq \left[\mu X . F(X)\right]_{\rho}(s) \quad (12) \]

Define \(\Psi_{k} = \{s_{0}.s_{1}.s_{2} \ldots \mid s_{0} = s \text{ and } \exists n \leq k. (s_{n} \in \langle \phi_{2} \rangle_{\rho} \wedge \forall m < n. (s_{m} \in \langle \phi_{1} \rangle_{\rho}))\}.\) Clearly \(\Psi = \bigcup_{k} \Psi_{k}.\)

Suppose Inequality (12) does not hold. Then there exists some \(k\) and scheduler \(\sigma\) such that

\[ m_{\sigma}^{r}(\Psi_{k}) > \left[\mu X . F(X)\right]_{\rho}(s) \quad (13) \]

We prove that this is not possible by induction on \(k.\) In the \(k = 0\) case, since we are assuming \(m_{\sigma}^{r}(\Psi_{0}) > 0,\) it holds that \(s \in \langle \phi_{2} \rangle_{\rho}.\) By inductive hypothesis on \(\phi_{2},\) we know that \(\left[\mathbf{E}(\phi_{2})\right](s) = 1\) and this implies that \(\mu X . F(X) = 1,\) which is a contradiction with the assumed strict inequality (13). Consider the case \(k + 1.\)

Note that if \(s \in \langle \phi_{2} \rangle_{\rho}\) then, \(\left[\mu X . F(X)\right]_{\rho}(s) = 1\) as before, contradicting Inequality (13). So assume \(s \not\in \langle \phi_{2} \rangle_{\rho}.\) Since we are assuming \(m_{\sigma}^{r}(\Psi_{k+1}) > 0\) it must be the case that \(s \in \langle \phi_{1} \rangle_{\rho}.\) Similarly, \(m_{\sigma}^{r}(\Psi_{k+1}) > 0\) and \(s \not\in \langle \phi_{2} \rangle_{\rho}\) imply that \(s \not\in \langle \phi_{2} \rangle_{\rho}\) does not hold. This means (see Definition 2.8) that \(\sigma(\{s\})\) is defined. Let \(d = \sigma(\{s\})\) and observe that \(m_{\sigma}^{r}(\Psi_{k+1}) = \sum_{i \in S} d(i) m_{\sigma}^{r}(\Psi_{k}),\) where \(\sigma'(s_{0}, s_{1}, \ldots, s_{n}) = \sigma(s, s_{0}, s_{1}, \ldots, s_{n}).\)

By induction on \(k\) we know that the inequality \(m_{\sigma}^{r}(\Psi_{k}) \leq \left[\mu X . F(X)\right]_{\rho}(t)\) holds for every \(t \in S.\) Thus,
by definition of the semantics of ♦, we obtain $m^\sigma_\alpha(L_\phi) \leq [\lbrack\lbrack \mu X.F(X)\rbrack\rbrack_\rho].$ Recall that we previously assumed $s \not\in [\lbrack\lbrack \phi_2\rbrack\rbrack_\rho$ and $s \in [\lbrack\lbrack \phi_1\rbrack\rbrack_\rho.$ Hence the equality

$$[\lbrack\lbrack \mu X.F(X)\rbrack\rbrack_\rho(s) = [\lbrack\lbrack E(\phi_2) \cap (E(\phi_1) \cap (\lbrack\lbrack \mu X.F(X)\rbrack\rbrack_\rho(s)
$$

holds. The formula on the right is just the unfolding $F(\mu X.F(X))$ of $\mu X.F(X).$ This implies the desired contradiction.

Case $[\lbrack\lbrack \geq\rbrack\rbrack.$ We now prove that also the inequality

$$[\lbrack\lbrack m^\sigma_\alpha(\psi)\rbrack\rbrack_s \geq [\lbrack\lbrack \mu X.F(X)\rbrack\rbrack_\rho(s)
$$

holds. By Knaster-Tarski theorem, $[\lbrack\lbrack [F(X)]_\rho = [\bigsqcup_{\alpha} [F(X)]^\alpha_\rho,$ where $\alpha$ ranges over the ordinals and $[F(X)]^\alpha_\rho$ with $\rho^\alpha = \rho [\bigsqcup_{\beta < \alpha} [F(X)]^\beta_\rho / \rho].$ We prove Inequality $[\lbrack\lbrack 14\rbrack\rbrack$ by showing, by transfinite induction, that for every ordinal $\alpha$ and $\varepsilon > 0,$ the inequality

$$[\lbrack\lbrack m^\sigma_\alpha(\psi)\rbrack\rbrack_s > [\lbrack\lbrack \mu X.F(X)\rbrack\rbrack_\rho(s) - \varepsilon
$$

holds, for all $s \in S.$ The case for $\alpha = 0$ is immediate since $[F]_\rho(s) > 0 \text{ if and only if } [E(\phi_2)]_\rho(s) = 1$ and this implies $[\bigsqcup_{s} \{m^\sigma_\alpha(\psi)\} = 1.$ Consider $\alpha = \beta + 1.$ If $[E(\phi_2)]_\rho(s) = 1$ then Inequality $[\lbrack\lbrack 14\rbrack\rbrack$ holds as above. Thus assume $[E(\phi_2)]_\rho(s) = 0.$ Note that $[F]_\rho(s) > 0 \text{ only if } s \in [E(\phi_1)].$ Thus assume $[E(\phi_1)]^\beta_\rho(s) = 1.$ Under these assumption, $[F(X)]^\alpha_\rho = [\lbrack\lbrack F(X)\rbrack\rbrack^\beta_\rho$ as it is immediate to verify. By definition of the semantics of ♦ we have:

$$[\lbrack\lbrack \mu X.F(X)\rbrack\rbrack_\rho(s) = [\bigsqcup_{s \rightarrow d} \left( \sum_{t \in S} d(t) [F(X)]^\beta_\rho(t) \right)
$$

By induction hypothesis on $\beta$ we know that for every $\varepsilon,$

$$[\lbrack\lbrack \mu X.F(X)\rbrack\rbrack_\rho(s) < [\bigsqcup_{s \rightarrow d} \left( \sum_{t \in S} d(t) \left( [\bigsqcup_{\sigma} \{m^\sigma_\alpha(\psi)\} + \varepsilon \right) \right)
$$

For each $s \rightarrow d$ and $\sigma$ define $\sigma^d$ as $\sigma^d(\{s\}) = d$ and $\sigma^d(s,t_0,\ldots) = \sigma(t_0,\ldots).$ A simple argument shows that

$$[\bigsqcup_{s \rightarrow d} \left( \sum_{t \in S} d(t) \left( [\bigsqcup_{\sigma} \{m^\sigma_\alpha(\psi)\} + \varepsilon \right) \right) = [\bigsqcup_{\sigma^d} \{m^\sigma_\alpha(\psi)\} + \varepsilon
$$

and this conclude the proof for the case $\alpha = \beta + 1.$ Lastly, the case for $\alpha$ a limit ordinal follows straightforwardly from the inductive hypothesis on $\beta < \alpha.$

**Proof of Proposition 5.1.** Suppose we have a system of $k$ conditioned linear expressions representing $F.$ Each conditioned expression $C \vdash e$ is captured by the implication $(\land C) \rightarrow y = e,$ so the whole system translates into a conjunction of $k$ such implications. To this conjunction, one need only add the range constraints $0 \leq z$ and $z \leq 1$ for each variable $z,$ as further conjuncts. In this way, the graph is easily expressed as a quantifier free formula. (Since the implications are equivalent to disjunctions of atomic formulas, the resulting formula is naturally in conjunctive normal form.)

Conversely, suppose $F(x_1,\ldots,x_n,y)$ defines the graph of $f.$ By quantifier elimination, we can assume that $F$ is quantifier free and in disjunctive normal form. Then $F$ is a disjunction of conjunctions, where each conjunction, $K,$ can be easily rewritten in the form

$$\left( \land C \right) \land \left( \land_{1 \leq i \leq h} y > a_i \right) \land \left( \land_{1 \leq i \leq k} y \geq b_i \right) \land \left( \land_{1 \leq i \leq l} y < c_i \right) \land \left( \land_{1 \leq i \leq m} y < d_i \right),$$

(16)
such that the only variables in the finite set of atomic formulas $C$, and linear expressions $a_i, b_i, c_i, d_i$ are $x_1, \ldots, x_n$. Since $F$ is the graph of a function, for all reals $r_1, \ldots, r_n$, there is at most one $s$ such that $K(\vec{r}, s)$ holds, and, if it does, then all of $r_1, \ldots, r_n, s$ are in $[0, 1]$. Given such an $s$, we therefore have:

$$\max\{a_i(\vec{r}) \mid 1 \leq i \leq h\} < \max\{b_i(\vec{r}) \mid 1 \leq i \leq k\} = s = \min\{c_i(\vec{r}) \mid 1 \leq i \leq l\} < \min\{d_i(\vec{r}) \mid 1 \leq i \leq m\}.$$  

A system of conditioned linear expressions for $f$ is thus obtained as follows. For each conjunct $K$ in $F$, written in the form of (16) above, and each $j$ with $1 \leq j \leq k$, include the conditioned linear expression:

$$ C, \{b_j > a_j\}_{1 \leq j \leq h}, \{b_j \geq b_j\}_{1 \leq j \leq k}, \{b_j \leq c_j\}_{1 \leq j \leq l}, \{b_j < d_j\}_{1 \leq j \leq m}, \vdash b_j. $$

We supplement the proof of Theorem 5.3 with more detail on the bounds on basis and condition size.

**Expanded proof of Theorem 5.3** We analyse the control flow in the algorithm for $\mu x_{n+1}, t'$ on a given input vector $(r_1, \ldots, r_n)$. On iteration number $i$, the loop is entered with constraints $D_i$ and approximation $d_i$, after which the recursive call to the algorithm for $t'$ yields one of the conditioned linear expressions, $C_k \vdash e_k$. Suppose that $C_k$ and $D_i$ contain $u$ and $v$ inequalities respectively. If the loop is exited producing (4) as result then the resulting $C_{r}$ has $2u + v$ inequalities. If it is exited producing (5) as result then $C_{r}$ has $u + v + 2$ inequalities (where $u + v + 2 \leq 2u + v$ because $C_k$ has to enforce the range constraint $0 \leq x_{n+1} \leq 1$). Otherwise, the algorithm repeats the loop, entering iteration $i + 1$ with $D_{i+1}$, given by (7), having at most $2u + v$ inequalities ($N$ contributes $1$ inequality, and there are at most $u - 1$ inequalities $b_j \leq b_j$ in (7) since $l \geq 1$).

Therefore, if $l'$ is now maximum number of occurrences in any $C_j$ from (11) (i.e., if it is the condition size for $t'$) the algorithm for $\mu x_{n+1}, t'$ at $\vec{r}$, which runs for at most $k'$ iterations, results in $C_{\vec{r}}$ containing at most $2k'l'$ inequalities.

To bound the number of results $C_{\vec{r}} \vdash e_{\vec{r}}$, we count the possible control flows of the algorithm. At iteration $i$, the algorithm uses $C_k \vdash e_k$ from (11), using which it might terminate with either (4) or (5), or it might repeat the loop, entering iteration $i + 1$ with $D_{i+1}$, given by (7), which can arise from $C_k$ in a number of ways determined by the possible pairs of choices for $N$ and $b_j$ in (7). In the case that the variable vector $(x_1, \ldots, x_n)$ is empty (i.e., the term $\mu x_{n+1}, t'$ is closed) the constraints in $D$ are redundant (they are simply true inequalities between rationals) and so can be discarded. In the case that $n \geq 1$, there are at least $2$ inequalities in $C$ giving range constraints on $x_1$, so there are at most $l'$ choices for $N$ ($l' - 2$ choices in the case that $q_{n+1} \neq 1$, and $2$ in the case $q_{n+1} = 1$). Irrespective of $n$, there are at most $l' - 1$ choices for $b_j$ (taking $n$ into account this can be improved to $l' - 2n - 1$). Therefore, the execution of the algorithm, is determined by the sequence:

$$k_1, u_1, k_2, u_2, \ldots, k_m, v$$

where: $m \leq k'$ is the number of loop iterations performed; each $u_i$, where $1 \leq u_i \leq l'(l' - 1)$, represents the choice of $N$ and $b_j$ used in the construction of $D_{i+1}$ (7), and $v$ is $1$ or $2$ according to whether the resulting $C_{\vec{r}} \vdash e_{\vec{r}}$ is returned via (4) or (5). Since each number $k_i$ is distinct, the number of different such sequences is bounded by:

$$2 \sum_{m=1}^{k'} \frac{k!}{(k'-m)!} (l'(l' - 1))^{m-1} \leq (k'(l')^2)^{k'}$$

(17)

where the right-hand-side gives a somewhat loose upper bound. Therefore, the number of possible results $C_{\vec{r}} \vdash e_{\vec{r}}$ for the algorithm for $\mu x_{n+1}, t'$ is at most $(k'(l')^2)^{k'}$. 

\[ \square \]