Constructive Set Theories and their Category-theoretic Models

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Abstract

We advocate a pragmatic approach to constructive set theory, using axioms based solely on set-theoretic principles that are directly relevant to (constructive) mathematical practice. Following this approach, we present theories ranging in power from weaker predicative theories to stronger impredicative ones. The theories we consider all have sound and complete classes of category-theoretic models, obtained by axiomatizing the structure of an ambient category of classes together with its subcategory of sets. In certain special cases, the categories of sets have independent characterizations in familiar category-theoretic terms, and one thereby obtains a rich source of naturally occurring mathematical models for (both predicative and impredicative) constructive set theories.

1 Introduction

1.1 Constructive set theories

The modern era of constructive mathematics began with the publication of Bishop's seminal book [9], in which he set out to redress "the well-known scandal ... that classical mathematics is deficient in numerical meaning." In pursuit of this aim, Bishop undertook the ambitious programme of reformulating the core of mathematical analysis in such a way that all statements and proofs would have direct numerical content. To achieve this, Bishop adopted Brouwer's constructive interpretation of the logical primitives, a step which, in turn, demanded that proofs be carried out using the restricted principles of intuitionistic logic. In addition, Bishop banished from his mathematical ontology much of the machinery of abstract set theory. Instead, he insisted on a very concrete (and intensional) notion of set,[1] and he made use of only the simplest and most tangible constructions on sets (e.g. product, function space). Even quotient sets

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[1] "A set is defined by describing exactly what must be done in order to construct an element of the set and what must be done in order to show that two elements are equal," [*ibid.* p.6].

were apparently considered harmful.[2] Bishop's motivation for such parsimony was to make his mathematics as straightforward and realistic as possible, and few would argue against his outstanding success in this enterprise.

Nevertheless, experience in the decades since the publication of Bishop's book has taught us that there is, after all, no *a priori* conflict between the concepts of abstract set theory and the numerical meaningfulness demanded by constructive mathematics. During this period, a wide range of set theories, based on comfortably familiar axioms, and variously labelled as *constructive* or *intuitionistic*, have been thoroughly investigated in the literature, see e.g. [23, 13, 1, 28, 4]. While not all such theories are accepted as truly constructive by every constructive mathematician, each can, via the definition of an appropriate realizability interpretation, be shown to meet the basic requirement of numerical meaningfulness. Furthermore, Aczel [1] has shown how even the most strident constructivist can be reconciled to those set theories that only contain axioms based on predicative principles, on account of the possibility of translating such set theories into Martin-Löf's type theory [21], a foundation whose constructive credentials are universally accepted.

The *formal* compatibility of set theory and constructive mathematics licenses the introduction of abstract set-theoretic methods into the practice of *informal* constructive mathematics. A first goal of this paper is to survey, in as straightforward a way as possible, the formal principles for reasoning about sets constructively that are actually relevant to the ways in which sets are used in ordinary informal constructive mathematics. This leads to a different emphasis from the usual presentations of constructive set theories. In standard axiomatizations, the contents of the axiomatized universe are rigidly delimited,[3] and theories are often strongly prescriptive about the nature of sets themselves.[4] In contrast, we leave our mathematical universe entirely open (it may contain whatever entities the constructive mathematician ever feels the need to construct), and we axiomatize the situation in which sets are included in the universe alongside everything else that may be there. Also, we leave our notion of set as unconstrained as possible, while remaining consistent with the ways in which sets are actually used in mathematical practice.

In Sections 3–5, we present axiomatizations of a number of set theories, motivated by the above desire to meet the pragmatic needs of constructive mathematicians who wish to use abstract set-theoretic methods in their mathematics. The theories range in power from weaker predicative theories to stronger theories incorporating full-blown impredicativism. Accordingly, not every constructivist will accept all the theories we discuss, but our aim is to survey the range of options available rather than to restrict attention to any one fixed interpretation of constructivism. From a formal point of view, the theories we consider are all intertranslatable with existing intuitionistic set theories studied in the literature.

[2] "The axiom of choice is [in classical mathematics] used to extract elements from equivalence classes where they should never have been put in the first place," [*ibid.* p.9].

[3] In Myhill's CST [23], every element of the universe is required to be a set, a function or a number; in Aczel's CZF [1], as in classical ZF, all elements are sets.

[4] E.g., in CZF (again as in classical ZF) sets are required to be well-founded.

The novelty in our formulation is due entirely to taking seriously the pragmatic motivation, discussed above, of basing our theories solely on principles that are directly useful for the purpose of incorporating set-theoretic reasoning into the informal practice of constructive mathematics.

1.2 Category-theoretic models

The usual motivation for the activity of constructive mathematics is philosphical. Constructive mathematicians question the very meaningfulness of classical mathematics, and constructivization is pursued in order to restore the numerical meaning that is lacking in classical maths. Depending upon the personal beliefs of the mathematician in question, any one of many different brands of constructivism may be adopted. For example, a sceptical constructive mathematician will probably accept only methods that are predicative in nature.

While such philosophical motivations are undoubtedly important (indeed crucial to those who are in possession of constructivist beliefs), one wonders what further rationale there is for the constructivization of mathematics beyond the purely solipsistic one of wishing to achieve personal reassurance over the justifiability of mathematical methods. To address this, we consider two questions in the style of Kreisel.

1. What are the positive mathematical consequences that arise as a result of developing mathematics constructively as opposed to classically?
2. What additional benefits are obtained if one develops constructive mathematics using predicative methods only?

The first question has an obvious answer. Constructive mathematics is a form of mathematics that is computationally meaningful in its entirety. In particular, a constructive proof of a statement about numbers (whether integer, real or complex) provides direct algorithmic evidence for the statement. Thus constructive mathematics can be usefully performed, even by a classical mathematician, as a means for extracting computational information from proofs.

The second question may also be answered along similar lines. Predicative methods are (significantly) weaker in proof-theoretic strength than impredicative ones. Thus the use of predicative methods guarantees, for example, that any number-theoretic function extracted from a proof will grow slowly in comparison with the wildly inflationary functions definable using impredicative methods. To some, the existence of such bounds on growth may provide an adequate non-philosophical reason for considering predicative constructive mathematics as being of interest. However, as the basic systems of predicative constructive mathematics are at least as rich as intuitionistic first-order arithmetic, the bounds they guarantee are hopelessly infeasible from a computational point of view. In the author's view, the existence of such infeasible bounds does not provide a compelling reason for preferring predicative brands of constructivism to impredicative ones.

There is, nonetheless, a second, entirely different, approach to answering the questions above. Various constructive (or intuitionistic) formal systems arise as *internal languages* of natural notions of category with structure. For example, (extensional) Martin-Löf type theory corresponds to the internal language of locally cartesian closed categories [25], and higher-order intuitionistic type theory is the the internal language of elementary toposes [18]. In the special case of Grothendieck toposes, which first arose in algebraic geometry, one can even interpret full Intuitionistic Zermelo-Fraenkel set theory, see e.g. [12]. Locally cartesian closed categories, elementary toposes and Grothendieck toposes are ubiquitous in mathematics. Thus the intuitionistic theories mentioned above are given a non-ideological *raison d'être* through their relationship with such natural kinds of category.

The second goal of this paper is to demonstrate that there is analogous non-ideological motivation for the various constructive set theories considered in Sections 3–5. To address this, we present, in Section 6, a general category-theoretic framework for modelling the full range of different constructive set theories. The framework is based on the *algebraic set theory* of [17], as further developed in [27, 5, 7]. Within this framework, each of the constructive set theories axiomatized in Sections 3–5 has a corresponding notion of model defined by a reasonable collection of category-theoretic axioms.

It is a standard view that the use of weaker constructive set theories, such as predicative theories, represents a minimalist approach to mathematics, in the sense that it is based on reducing the assumptions on which mathematics is based to a minimal core. The consideration of category-theoretic models presents the same practice in an alternative "maximalist" light. It is a triviality that the weaker a set theory the more inclusive its associated class of category-theoretic models. Thus, from a non-ideological viewpoint, the use of predicative constructive methods can be justified on the grounds of maximalizing the class of available models within which mathematical arguments can be interpreted. In the author's view, this provides a quite acceptable answer to the second question above.

We have argued that constructive set theories can be justified as being of non-ideological mathematical interest because they have associated classes of category-theoretic models. The force of this argument is strengthened considerably in cases in which one can show that the category-theoretic models include familiar mathematical structures that occur in ordinary mathematics. In Section 7, we state results in this direction, taken from [5, 7]. These results characterize, in familiar category-theoretic terms, the possible categories of sets associated with certain different constructive set theories. In particular, for the theory we call BCST+Pow, an impredicative constructive set theory, the associated categories of sets are exactly the elementary toposes (with natural numbers object). Similarly, for the predicative theory BCST + Exp, the associated categories of sets are exactly the locally cartesian closed (lcc) pretoposes (again with natural numbers object). Lcc pretoposes generalize elementary toposes, and there are naturally occurring mathematical examples of lcc pretoposes that are not toposes. It follows that there exist naturally occurring mathematical models of the predicative set theory BCST + Exp that do not model the impredicative powerset axiom. The possibility of using constructive set-theoretic methods for reasoning within such naturally occurring models provides the desired non-ideological justification for the use of predicative constructive methods. It would be most interesting to see a convincing mathematical application of the soundness of these methods within such models.

Acknowledgements This paper presents a non-technical survey of recent results developed, in part, by the author in collaboration with Steve Awodey, Carsten Butz and Thomas Streicher [27, 5], and further developed by Steve Awodey, Henrik Forssell and Michael Warren [6, 7]. Explicit attributions are given in the text where appropriate. I would like to thank Peter Aczel, Carsten Butz, Nicola Gambino, Thomas Streicher and, especially, Steve Awodey for discussions on the subject of this paper.

2 Sets in constructive mathematics

As traditionally practised, the process of constructive mathematics involves: the construction of mathematical objects; and, of course, the proof of properties of these mathematical objects. The mathematical universe should thus be viewed as an open-ended collection of entities, to which one, from time to time, adds new objects, as and when such objects are created.[5]

In addition to working with mathematical objects individually, it is frequently useful to consider collections of such objects, that is subcollections of the universe, each determined by a given defining property. We shall refer to such collections as *classes*. Of course, the membership of a class is not fixed, once and for all by its defining property, but rather has to be understood within the context of an open-ended universe (note, for example, that the entire universe is itself a class). The use of quantification over such open-ended classes is common mathematical practice. For example, one may perfectly well state and prove a property of all groups, without rigidly fixing the membership of the class of all groups.

Sets are a central ingredient in the definition of many kinds of abstract mathematical structure (for example, group). Indeed, the inclusion of sets in the mathematical universe is indispensible for the pursuit of abstract constructive mathematics. Our aim is to isolate pragmatic principles governing the way in which sets can be usefully included in the universe.

Sets can be motivated in many ways; most commonly via the *iterative conception of set* [26], which, if adopted, leads to (a constructive version of) von Neumann's cumulative hierarchy as the universe of sets. This familiar notion of set does not, however, correspond to how sets are used in mathematical practice. According to the iterative conception, sets are given by well-founded membership trees in which each node is itself a set, whereas, in practice, sets

[5]It should be remarked that this view of an open-ended universe may also be seen as consistent with the practice of classical mathematics.

are merely distinguished collections of mathematical objects. It need not be the case that every element of a set is itself a set. Furthermore, the presumption that membership trees are necessarily well-founded is never used in ordinary mathematics.[6]

We suggest that the use of sets, as actually occurs in mathematical practice, is governed by just three basic laws.

1. A set is a distinguished collection of mathematical objects, itself rendered as a mathematical object.

2. A set is uniquely determined by its collection of members.

3. The class of sets is closed under a basic range of useful operations on collections.

The first law above is naturally considered alongside a mathematical universe that may already contain preconstructed mathematical objects that are not themselves sets (for example, natural numbers), and to which other newly constructed non-sets[7] may possibly be added in future. Of course, the law is also consistent with having a mathematical universe in which sets are the only type of mathematical object in existence; but the assumption that all objects are sets has no basis in mathematical practice.

The second law is simply the axiom of extensionality for sets. Extensionality is an essential component of the concept of set as it is actually used in mathematics.

The third law is obviously wide open to interpretation. In the sequel, we make the third law precise in a number of different ways, leading to several set theories of differing strength. Not all these set theories will be acceptable to every constructive mathematician. Indeed, it is in the choice of interpretation of the third law that the constructive mathematician nails his or her individual beliefs to the door.

3 Axioms for constructive set theory

The goal of this section is to present axioms for working with sets in constructive mathematics. These axioms are motivated by the the three laws of sets that we identified above. The formal setting for the axiomatization is intuitionistic first-order logic with equality, where variables $x, y, z, \dots$, and hence quantifiers, are to be understood as ranging over the entire (open-ended) universe.

In order to emphasise the pragmatic side, we shall, as far as possible, present the axioms in the informal style in which they would actually be used when doing

[6]Indeed, for some applications, it is convenient to explicitly contravene this assumption and consider non-well-founded sets [3].

[7]We prefer the term "non-set" to "atom" or "urelement", as, according to our view of constructive mathematics, there is nothing to preclude non-sets from being composite (and hence non-atomic) structures.

constructive mathematics. To give a formal basis to such informal formulations, it is convenient to introduce notation for classes.

We write $\phi[x]$ for a formula ϕ with the free variable x distinguished; and having distinguished x in ϕ in this way, we write $\phi[y]$ for $\phi[y/x]$ Note that ϕ may contain free variables other than x; also x is not required to actually occur in ϕ. A *class* is defined by a formula $\phi[x]$, representing the class of all x satisfying ϕ, for which we use the notation:

$$\{x \mid \phi\}.$$

We use $X, Y, \ldots$ as meta-variables for classes, If X is the class defined by $\phi[x]$, then we write $y \in X$ to mean $\phi[y]$. Given a class X, a formula $\phi[x, y]$ (with two distinguished free variables x, y) can be seen as defining an X-indexed family of classes $\{Y_x\}_{x \in X}$, where Y_x is the class $\{y \mid \phi\}$, for any $x \in X$. The union of such a family of classes is defined as the class:

$$\bigcup_{x \in X} Y_x \ = \ \{y \mid \exists x.\, x \in X \ \wedge \ \phi\},$$

To incorporate sets into the theory, we include two predicates in the language: a unary predicate $\mathcal{S}(x)$, stating that x is a set; and a binary predicate $x \in y$, stating that x is a member of the set y.[8] These linguistic constructs reflect the first law of sets in Section 2. The natural reading of membership above, in which y is forced to be a set, is imposed by an axiom:[9]

Membership If $x \in y$ then y is a set.

The second law of sets in Section 2 is implemented by the expected extensionality axiom, formulated in such a way as to be appropriate for a universe possibly containing non-sets.

Extensionality If A, B are sets and, for all z, $z \in A$ iff $z \in B$ then $A = B$.

Here, and henceforth, we use $A, B, \ldots$ as variables specifically ranging over the class $\{x \mid \mathcal{S}(x)\}$ of all sets.[10]

The third law of sets from Section 2 is addressed by the four (in the first instance) axioms below, which state basic closure properties of the class of sets. These closure properties are all intuitively reasonable if one takes the notion of set to be given by the following informal conception: a *set* is a class that is *absolute* in the sense that its membership is fixed once and for all, independently of any future extension that may be made to the universe. We call this idea the *absolute conception of set.*

Emptyset The empty class $\{x \mid \bot\}$ is a set. We write $\emptyset$ for this set.

[8]Note that we are using $\in$ as a predicate in $x \in y$ and as meta-notation in $x \in X$. This will cause no problems. The only situation in which there is any ambiguity is when the class X is a set, in which case the two interpretations coincide.

[9]As is usual, all axioms are implicitly universally quantified over their free variables.

[10]$A, B, \ldots$ are first-order variables, not meta-variables. Nevertheless, we shall also allow set variables $A, B, \ldots$ to be treated as classes in the obvious way.

Pairing The class $\{x \mid x = y \vee x = z\}$ is a set. We write $\{y, z\}$ for this set.

Equality The class $\{x \mid x = y \wedge x = z\}$ is a set. We write δ_{yz} for this set (the Kronecker delta notation seems appropriate here).

Indexed Union If A is a set and $\{B_x\}_{x \in A}$ is an A-indexed family of sets then $\bigcup_{x \in A} B_x$ is a set.

Although the informal readings of these axioms are clear, it is worth spelling out their formal incarnations. To this end, it is useful to define a quantifier $Ƨx.\, \phi$ to be read as "the class $\{x \mid \phi\}$ is a set" or as "there are set-many x satisfying ϕ". Formally, $Ƨx.\, \phi$, abbreviates:

$$\exists A.\, \mathcal{S}(A) \;\wedge\; \forall x.\, (x \in A \;\leftrightarrow\; \phi),$$

where A does not occur free in ϕ. Note that the variable x is bound in the statement "$\{x \mid \phi\}$ is a set", thus, for example, the implicit universal quantification in the Pairing and Equality axioms is just over y and z. The formal versions of the Emptyset, Pairing and Equality axioms are now obvious. The Indexed Union axiom expands to[11]

$$\mathcal{S}(A) \;\wedge\; (\forall x \in A.\, Ƨy.\, \phi) \;\rightarrow\; Ƨy.\, \exists x \in A.\, \phi.$$

In the presence of the other axioms, Indexed Union is equivalent to a combination of the two axioms below, which, although standard, are less mathematically appealing.

Union $(\mathcal{S}(A) \wedge \forall x \in A.\, \mathcal{S}(x)) \;\rightarrow\; Ƨy.\, \exists x \in A.\, y \in x.$

Replacement $\mathcal{S}(A) \wedge (\forall x \in A.\, \exists! y.\, \phi) \;\rightarrow$
$\exists B.\, \mathcal{S}(B) \;\wedge\; (\forall x \in A.\, \exists y \in B.\, \phi) \;\wedge\; (\forall y \in B.\, \exists x \in A.\, \phi),$

where $\exists!$ is the "there exists a unique" quantifier. The (straightforward) proof of this equivalence can be found in [4], where, in its recognition, the Indexed Union axiom is called Union-Replacement.

We call the set theory axiomatized above BCST$^-$, which stands for Basic Constructive Set Theory without Infinity.[12] This will be the base theory upon which all our other theories will be built. In spite of its apparent weakness, a surprising number of set-theoretic constructions can be squeezed out of the axioms of BCST$^-$, as we now proceed to show.

The most conspicuous omission from BCST$^-$ is any form of the axiom of separation. Given a set A and formula $\phi[x]$, one can form the subclass $\{x \in A \mid \phi\}$ of A. An *instance of separation* is the assertion, for a particular $\phi[x]$ and set A, that the class $\{x \in A \mid \phi\}$ is itself a set. We shall see that a useful range of instances of separation are derivable in BCST$^-$.

To prepare for what follows, observe that propositions ϕ are in correspondence with subclasses of the singleton set $\{\emptyset\}$. On the one hand, any proposition

[11] The relative quantifiers $\forall x \in A$ and $\exists x \in A$ have the obvious meanings.
[12] This name was suggested by Michael Warren.

ϕ determines the subclass $\{w \in \{\emptyset\} \mid \phi\}$, where w is a variable that does not occur free in ϕ. Conversely, any subclass X of $\{\emptyset\}$ determines the proposition $\emptyset \in X$. We say that the proposition ϕ is *restricted* if the class $\{w \in \{\emptyset\} \mid \phi\}$ is a set, and we write $!\phi$ for the statement that ϕ is restricted.[13] The notion of restricted formula entirely determines the valid instances of separation.

Proposition 3.1 *In* BCST$^-$*, the following are equivalent, for any set A and property $\phi[x]$:*

1. *The class $\{x \in A \mid \phi\}$ is a set.*
2. *For all $x \in A$ it holds that $!\phi$.*

To help familiarize the reader with the kind of reasoning needed to reduce ordinary set-theoretic constructions to the axioms of BCST$^-$, we give, as the only proof included in the paper, the short proof of this proposition.

Proof. First we observe that sets in BCST$^-$ are closed under binary intersection. This follows from the Indexed Union and Equality axioms because $A \cap B = \bigcup_{x \in A} \bigcup_{y \in B} \delta_{xy}$.

Now assume that $\{x \in A \mid \phi[x]\}$ is a set. Then, for any $x \in A$, the class $\{x\} \cap \{x \in A \mid \phi[x]\}$ is a set, i.e. $\{w \in \{x\} \mid \phi[x]\}$ is a set. By Replacement, the class $\{w \in \{\emptyset\} \mid \phi[x]\}$ is also a set, i.e. $!\phi$.

Conversely, suppose that $!\phi$, for all $x \in A$. Then, by Replacement, for every $x \in A$, it holds that $\{w \in \{x\} \mid \phi\}$ is a set. By Indexed Union, $\bigcup_{x \in A}\{w \in \{x\} \mid \phi[x]\}$ is a set, i.e. $\{x \in A \mid \phi[x]\}$ is a set, as required. □

The proposition below establishes useful closure conditions on restricted properties, and hence, by the result above, yields rules for deriving valid instances of separation in BCST$^-$.

Proposition 3.2 (cf. [4]) *The following all hold in* BCST$^-$.

1. $!(x = y)$.
2. *If* $\mathcal{S}(A)$ *then* $!(x \in A)$.
3. *If* $!\phi$ *and* $!\psi$ *then* $!(\phi \wedge \psi)$, $!(\phi \vee \psi)$, $!(\phi \rightarrow \psi)$ *and* $!(\neg\phi)$.
4. *If* $\mathcal{S}(A)$ *and* $\forall x \in A.\, !\phi$ *then* $!(\forall x \in A.\, \phi)$ *and* $!(\exists x \in A.\, \phi)$.
5. *If* $\phi \vee \neg\phi$ *then* $!\phi$.

Here, statement 5 generalises the standard fact that, in classical set theory, full separation follows from Replacement. Using intuitionistic logic, full separation is not a consequence of Replacement, but separation does remain derivable for so-called decidable formulas (those formulas ϕ for which $\phi \vee \neg\phi$ holds).

Statements 1–4 above can be used to build up restricted formulas from atomic formulas $x = y$ and $x \in A$ (assuming A is a set), by closing under

[13]Formally, $!\phi$ is thus an abbreviation for $\text{Ɂ}w.\, w = \emptyset \wedge \phi$, where w is not free in ϕ.

the propositional connectives and under quantification in which quantifiers are bounded by sets. Note that an atomic formula $\mathcal{S}(x)$ need not be restricted. One could remedy this by adding a further axiom to BCST^- (see [5]), but such an axiom makes a somewhat arbitrary assumption about sets and does not seem useful for the purpose of carrying out constructive mathematics.

What is indisputably useful for doing mathematics is the construction of various standard derived sets and classes. We next consider to what extent BCST^- is able to cope with the most basic of these: cartesian products, exponentials (i.e. function spaces), powersets and quotient sets.

Given classes X, Y, a product class $X \times Y$ can be defined by

$$X \times Y \;=\; \{p \mid \exists x \in X.\, \exists y \in Y.\, p = (x, y)\},$$

where (x, y) can be taken to be the usual Kuratowski pairing $(x, y) = \{\{x\}, \{x, y\}\}$ (see [4] for a proof that this is a sensible intuitionistic notion of pairing).

Proposition 3.3 *In* BCST^-*, if* A, B *are sets then so is* $A \times B$*.*

Given a set A and a class X, we can define a class X^A of all functions from A to X by

$$X^A \;=\; \{f \mid \mathcal{S}(f) \;\wedge\; (\forall z \in f.\, z \in A \times X) \;\wedge\; (\forall x \in A.\, \exists! y.\, (x, y) \in f)\}.$$

The assumption that A is a set is important because, although the above definition also makes sense if A is a class, by Replacement, X^A can be inhabited only when A is a set. BCST^- is too weak to prove that B^A is a set, for every set B.

Given a class X the "power*class*" $\mathcal{P}(X)$ of all sub*sets* of X is defined by

$$\mathcal{P}(X) \;=\; \{A \mid \mathcal{S}(A) \;\wedge\; \forall x \in A.\, x \in X\}.$$

In BCST^-, it cannot be shown that $\mathcal{P}(A)$ is a set, for every set A.

Finally, we turn to quotients. Suppose X is a set and $\sim$ is a subclass of $X \times X$ determining an equivalence relation on X. Under the condition that $\sim$ is a set, i.e. that $\sim \in \mathcal{P}(X \times X)$, we can form the quotient class

$$X/{\sim} \;=\; \{B \in \mathcal{P}(X) \mid \exists x.\, x \in B \;\wedge\; \forall y \in X.\, x \sim y \;\leftrightarrow\; y \in B\}.$$

Proposition 3.4 *In* BCST^-*, if* A *is a set and* $\sim \in \mathcal{P}(A \times A)$ *is an equivalence relation then* $A/{\sim}$ *is a set.*

The development above highlights both strengths and weaknesses of BCST^-. To address the weaknesses, we consider the following list of further axioms (and schema) that will be used to extend BCST^-, in order to obtain stronger theories.

Exponentiation (Exp) If A, B are sets then so is B^A.

Powerset (Pow) If A is a set then so is $\mathcal{P}(A)$.

Separation (Sep) For any property $\phi[x]$ and set A, the class $\{x \in A \mid \phi\}$ is a set.

We briefly examine the credentials of each of these axioms.

Exp The Exponentiation axiom is standardly taken as one of the main pillars of the predicative constructive set theories, as found, e.g., in [23, 1, 4]. The axiom may be seen to be in accordance with the absolute conception of set (see Section 3). However, such an understanding relies on viewing the notion of function between two sets as being fixed, even though one might, in future, add new entities to the universe (for example, very large sets) that would enable one to prove the existence of functions whose existence could not otherwise be established.

Pow It is straightforward to show that the Powerset axiom implies Exponentiation in BCST^-. The converse is provable using classical logic, but not intuitionistic logic. Indeed, in contrast to Exponentiation, Powerset is unequivocally impredicative. Using it, one may define new subsets of a set by quantifying over the collection (which is now a set) of all subsets of the set, a collection that includes the subset being defined. One can, nevertheless, still make a case that this axiom also conforms with the absolute conception of set: it seems a reasonable enough assumption that the collection of subsets (i.e. absolute subclasses) of a given set is fixed once and for all as soon as the latter set is defined.

Sep The Separation axiom is again impredicative. One may use Separation to define a subset using an unbounded quantifier ranging over all sets, including the set being defined. Further, Separation is not compatible with the absolute conception of set. Using Separation, one may define sets using unbounded quantification over the universe, and such quantifications may readily be affected by future extensions to the universe. However, Separation does agree with a rather looser idea of sets corresponding to those classes that are in some (slightly vague) sense “small”.

Because Powerset implies Exponentiation, BCST^- has (including itself) six different extensions using combinations of the above axioms, all of which can indeed be shown to be distinct. We shall discuss several such extensions in Section 5; but first we consider the, thus far omitted, axiom of Infinity.

4 Infinity and induction

The axiom of Infinity is most naturally incorporated in BCST^-, by extending the first-order language with three new constants $N, 0, s$.[14] In the context of weak set theories such as BCST^-, one has to be careful in the formulation of

[14] Using such an extended language is inessential, because the axiom of Infinity could instead be prefixed by “$\exists N.\, \exists 0.\, \exists s.$”; in which case, derived consequences of Infinity also need to be prefixed by corresponding existential statements.

Infinity, as induction over the natural numbers does not appear strong enough to derive definition by primitive recursion.[15] Because of this, we follow Lawvere and turn primitive recursion into the defining property of the natural numbers:[16]

Infinity (Inf) N is a set, $0 \in N$, $s \in N^N$ and, for every set A, element $z \in A$, and function $f \in A^A$, there exists a unique function $h \in A^N$ satisfying: $h(0) = z$ and $h(s(x)) = f(h(x))$ for all $x \in N$.

We write BCST for the theory BCST$^-$+ Inf.

It is not hard to show that Infinity implies that $0 \neq s(x)$ for all $x \in N$, and that $s(x) = s(y)$ implies $x = y$ for all $x, y \in N$. Also, it is straightforward to derive an induction principle for restricted formulas:

!-Ind For any $\phi[x]$ such that $\forall x \in N.\, !\phi$, if $\phi[0]$ and $\forall x \in N.\, \phi \;\rightarrow\; \phi[s(x)]$ then $\forall x \in N.\, \phi$.

However, the full induction schema below is not derivable, even in the theory BCST + Pow.[17]

Ind For any $\phi[x]$, if $\phi[0]$ and $\forall x \in N.\, \phi \;\rightarrow\; \phi[s(x)]$ then $\forall x \in N.\, \phi$.

Indeed, one can show (cf. [5]) that the statement:

$$\exists I.\ \emptyset \in I \ \wedge\ \forall x \in I.\, \mathcal{S}(x) \ \wedge\ x \cup \{x\} \in I,$$

which asserts the existence of an infinite set containing the von Neumann numerals, is derivable in BCST + Ind but not in BCST + Pow.[18]

It is obvious that the full induction schema, Ind, is derivable in the theory BCST+Sep. However, in contrast to Separation, there is nothing impredicative about full induction. Indeed full induction is a standard feature of (predicative) constructive set theories. In Myhill's CST [23], Ind is assumed as a basic axiom. In Aczel's CZF [1, 4], full induction is a consequence of a "set induction" principle, which is a constructively acceptable way of stating the well-foundedness of sets.

The distinction between !-Ind and Ind is part of a wider phenomenon. For induction in general, whether it be ordinary induction over the natural numbers, or any given variety of transfinite induction, one may distinguish two types of induction principle. *Restricted induction principles*, for example !-Ind above, hold for restricted formulas only, or equivalently for sub*sets* of the domain of induction. Such principles can be expressed by single formulas. In contrast, *full induction principles*, for example Ind, hold for arbitrary formulas, or equivalently for arbitrary sub*classes* of the domain. Such principles are expressed

[15] I am grateful to Peter Aczel for pointing this out to me.

[16] Various simpler formulations of Infinity are possible in the presence of the Exp, Pow or Sep axioms.

[17] Because it is a schema, in order to formulate full induction in a language without the constants $N, 0, s$ it suffices to use one of the standard set-theoretic encodings of the numerals. This avenue is not available for formulating Inf in the absence of Ind, see footnote 18 below.

[18] Thus it would not be equivalent to formulate the Infinity axiom of BCST using a set-theoretic encoding of the natural numbers.

by schemata. In general, full induction principles are stronger than the corresponding restricted ones. However, in the presence of full separation, the two forms of induction principle coincide.

In this paper, we shall only be concerned with induction over the natural numbers. We shall assume the Infinity axiom as basic, and we shall consider Ind as a possible additional axiom (schema) to add to theories.

5 Constructive set theories

We take BCST as our basic constructive set theory, on top of which we shall add combinations of the axioms Exp, Pow, Sep and Ind.

According to the foregoing analysis, the strongest predicative set theory that can be obtained as a combination of the above axioms is:

$$\mathrm{BCST} + \mathrm{Exp} + \mathrm{Ind}.$$

This theory corresponds very closely to Myhill's theory CST [23].[19]

It is a widely held (though not universal) view that a theory has to be predicative in order to be constructive. In the opinion of the author, this view represents an overly narrow demarcation of constructivism. The hallmarks of constructive mathematics are surely that mathematical definitions should be interpretable concretely as constructions of mathematical objects, each with some inherent numerical (or at least computational) meaning, and that mathematical proofs should provide associated computational information, as required by the constructive interpretation of mathematical statements.

To the predicative constructivist, impredicative definitions are, because of their intrinsic cyclicity, not sufficiently definite to unambiguously furnish the requisite mathematical construction together with its associated computational information. Accordingly, such a constructivist doubts the very meaningfulness of impredicative forms of definition. Nevertheless, to the many that do not share such (necessarily subjective) doubts, there are impredicative type theories, such as the Calculus of Constructions and its extensions [11, 20], which provide very plausible computational readings of impredicative methods of definition. In the view of the author, such calculi give perfectly constructive interpretations to (at least some forms of) impredicative definition.

In the light of the above, we should not *a priori* banish the impredicative axioms Pow and Sep from appearing in so-called constructive set theories. In view of its arguable adherence to the absolute conception of set, the theory

$$\mathrm{BCST} + \mathrm{Pow} + \mathrm{Ind}$$

seems of particular interest as a candidate impredicative constructive set theory. However, the theory

$$\mathrm{BCST} + \mathrm{Pow} + \mathrm{Sep},$$

[19] There are technical differences in formulation, but the theories are intertranslatable.

which is the strongest theory obtainable as a combination of the above axioms, also suggests itself as being mathematically natural. Indeed this latter theory is exactly the theory IST from [27], and thus equivalent in strength to the well-known intuitionistic set theory IZF (in its variant with Replacement rather than Collection).

We have not yet provided any justification that the above impredicative set theories do merit the appellation "constructive", where this is to be understood in the sense described above. Such justification might be given by way of a translation into an impredicative type theory (along the lines of [1, 29, 14]), or, more directly, by defining notions of construction and computation designed specifically for the purpose of justifying the set theories. We leave these tasks as a challenge for future work. In any case, the existence of realizability models of BCST + Pow + Sep (any realizability model of IZF is one such) means that, from a classical point of view, BCST + Pow + Sep is compatible with the idea that proofs should have computational meaning. Thus the possibility of obtaining a constructive (in our sense) justification seems plausible enough.

We end this section by discusing two further axioms. Collection is a strengthening of Replacement:[20]

Collection (Coll) $\mathcal{S}(A) \wedge (\forall x \in A. \exists y. \phi) \rightarrow$
$\exists B. \mathcal{S}(B) \wedge (\forall x \in A. \exists y \in B. \phi) \wedge (\forall y \in B. \exists x \in A. \phi)$,

Collection has played an important rôle in the metamathematics of constructive set theories, see [28], and is a basic axiom of Aczel's CZF [1, 4]. Our reason for postponing Collection to this point, is that we are not convinced that it is a genuinely useful axiom for applications in constructive mathematics (as opposed to metamathematics). Moreover, it has a different character from the other axioms in asserting the existence of a set that is not uniquely characterized by the properties it is required to satisfy.

Finally, for completeness, we consider the manifestly non-constructive Law of the Excluded Middle:

LEM $\phi \vee \neg\phi$.

The addition of LEM collapses most of the distinctions between the axioms previously discussed. Indeed, we have:

$$\mathrm{BCST} + \mathrm{Exp} + \mathrm{LEM} \;=\; \mathrm{BCST} + \mathrm{Pow} + \mathrm{Sep} + \mathrm{Ind} + \mathrm{LEM}$$

and this theory is intertranslatable with classical Zermelo-Fraenkel set theory. What is remarkable here is that the theory BCST + Exp is equiconsistent with first-order arithmetic, and thus of very low proof-theoretic strength. So, as is well known, from a proof-theoretic point of view, the Law of the Excluded Middle is a far from harmless addition to a constructive set theory.

[20] This axiom is often called Strong Collection, because the $\forall y.\exists x.$ clause is absent in the traditional formulation of Collection. However, in the presence of Separation, both stronger and weaker versions are equivalent, and, in the absence of Separation, the weaker is a less natural axiom. Thus it seems reasonable to call the stronger one Collection.

6 Categories of classes

The crucial idea for obtaining category-theoretic models of set theory is that, rather than axiomatizing the structure of the category of sets on its own, one should instead axiomatize the structure of the category of sets together with that of its surrounding supercategory of classes. This approach was first followed by Joyal and Moerdijk in their pioneering book on *algebraic set theory* [17], and has since been further developed in [27, 22, 10, 5, 24, 6, 7]. The goal of this section is to give a uniform and accessible presentation of the approaches taken in [27, 5, 7]. Our exposition is aimed at a reader with a basic knowledge of category theory only. Accordingly, we do not give all technical details. For these, the reader must refer to the original sources.

We first axiomatize the structure we require of a category $\mathcal{C}$ to act like a category of classes, and then below we axiomatize the properties of its distinguised full subcategory $\mathcal{S}$ of *small* objects, which will be the sets amongst the classes. For intuition, the reader should simply think of $\mathcal{C}$ as actually being the category of all classes, with class functions as morphisms, with $\mathcal{S}$ as the full subcategory of sets, and with sets and classes satisfying the properties of one of the constructive set theories presented earlier (e.g. BCST$^-$).

The category $\mathcal{C}$ is assumed to be a *positive Heyting category* in the sense of [16, A1.4.4]. For the purposes of this paper, all that the reader needs to know about this structure is the following. The category $\mathcal{C}$ has: a terminal object $\mathbf{1}$, binary (hence finite) products $X \times Y$, an equalizer $m\colon X \rightarrowtail Y$ for every parallel pair $f, g\colon Y \longrightarrow Z$, an initial object $\mathbf{0}$, and binary (hence finite) coproducts $X + Y$. Every morphism $f\colon X \longrightarrow Y$ factors as

$$f \;=\; X \xrightarrow{e_f} \mathrm{Img}(f) \overset{m_f}{\rightarrowtail} Y,$$

where e_f is a regular epimorphism[21] and m_f is a monomorphism. Intuitively, regular epis are to be thought of as surjective maps, so $\mathrm{Img}(f)$ represents the image of the map f, and m_f exhibits the image as a subobject of Y. In general, subobjects $P \rightarrowtail X$ can be understood as representing predicates over X, and intuitionistic first-order logic with equality can be used to reason about such predicates. For example, given a predicate $P \rightarrowtail X \times Y$, one can form the following two subobjects of X,

$$\{x \in X \mid \exists y \in Y.\, P(x,y)\} \rightarrowtail X \qquad \{x \in X \mid \forall y \in Y.\, P(x,y)\} \rightarrowtail X,$$

and these obey the expected logical laws. This logic of subobjects is called the *internal logic* of $\mathcal{C}$.

For an object I of $\mathcal{C}$, the *slice category* $\mathcal{C}/I$ is a simple category-theoretic construction whose objects intuitively represent I-indexed families of classes. Formally, $\mathcal{C}/I$ is defined as the category whose objects are mophisms $X \longrightarrow I$ in $\mathcal{C}$, and whose morphisms from $X \longrightarrow I$ to $Y \longrightarrow I$ are maps $X \longrightarrow Y$ in $\mathcal{C}$

[21] A *regular epi(morphism)* is a morphism that can be obtained as a coequalizer. Dually, a *regular mono(morphism)* is a morphism that can be obtained as an equalizer.

making the triangle commute. Intuitively, one thinks of the object $f\colon X \longrightarrow I$ in $\mathcal{C}/I$ as representing the family $\{f^{-1}(i)\}_{i\in I}$. Following this intuition, we use the notation $\{X_i\}_{i\in I}$ for an arbitrary object of $\mathcal{C}/I$. Given a map $h \in I \longrightarrow J$ in $\mathcal{C}$, one obtains two functors: a *reindexing* functor $h^*\colon \mathcal{C}/J \to \mathcal{C}/I$ that maps each $g\colon Y \longrightarrow J$ to the pullback of g along h; and a functor $\Sigma_h\colon \mathcal{C}/I \to \mathcal{C}/J$ that maps $f\colon X \longrightarrow I$ to $h\circ f\colon X \longrightarrow J$ (with the latter functor left adjoint to the former). Using the indexed class notation above, the action of these functors is more intuitively described as follows:

$$
\begin{aligned}
h^*(\{Y_j\}_{j\in J}) &= \{Y_{h(i)}\}_{i\in I} \\
\Sigma_h(\{X_i\}_{i\in I}) &= \{\textstyle\sum_{i\in h^{-1}(j)} X_i\}_{j\in J}\,.
\end{aligned}
$$

(The actions of the functors on morphisms are obvious.) Importantly, the assumed structure on $\mathcal{C}$ is preserved under slicing, i.e., for every I, the slice category $\mathcal{C}/I$ is a positive Heyting category and, for every $h\colon J \longrightarrow I$ the functor h^* preserves the positive Heyting category structure.

We now turn to the axioms needed for the notion of smallness. We shall assume that we have a collection $\mathcal{S}$ of objects of $\mathcal{C}$, designated as *small*, and satisfying the axioms expressing properties of smallness listed below. Furthermore, we shall assume that all this structure is preserved under slicing. In other words, for each slice category $\mathcal{C}/I$, we require an associated collection $\mathcal{S}_I$ of objects of $\mathcal{C}/I$, which are again designated as small, and which: satisfy the same axioms on smallness listed below, and are preserved by reindexing functors. The non-category-theoretically-minded reader is encouraged to ignore the issue of slicing entirely, and simply think of the above data using the intuition that $\mathcal{S}$ is the collection of those classes that are sets, and $\mathcal{S}_I$ is the collection of families $\{A_i\}_{i\in I}$ in which A_i is a set for each $i \in I$. Indeed, we shall refer to a small object $\{A_i\}_{i\in I}$ in $\mathcal{C}/I$ as a *family of small objects*.

We impose five basic axioms on small objects, each named in accordance with the set-theoretic principle it corresponds to. The first four axioms express closure properties of smallness, and thus address the third law of sets from Section 2.

Restricted separation If B is small and $A \rightarrowtail B$ is a regular mono then A is small.

Replacement If A is small and $A \twoheadrightarrow B$ is regular epi then B is small.

Disjoint union If A is small and $\{B_x\}_{x\in A}$ is a family of small objects then $\sum_{x\in A} B_x$ is small.

Pairing The object $\mathbf{1}+\mathbf{1}$ is small.

The fifth axiom addresses the first and second laws of sets from Section 2. It ensures that small classes (i.e. sets) themselves appear as elements of certain (power)classes, and that these elements behave extensionally. In order to formulate this, we write *(I-indexed) family of subobjects of* X to mean a subobject

$\{X_i\}_{i \in I}$ of the constant I-indexed family $\{X\}_{i \in I}$ (this constant family is the object of $\mathcal{C}/I$ given by the projection $X \times I \longrightarrow I$). Similarly, a *family of small subobjects of* X is a family $\{X_i\}_{i \in I}$ of subobjects of X that is itself a family of small objects.

Powerclass For every X there exists an object $\mathcal{P}(X)$ and family $\{M_A\}_{A \in \mathcal{P}(X)}$ of small subobjects of X such that, for every family $\{X_i\}_{i \in I}$ of small subobjects of X, there exists a unique $f \colon I \longrightarrow \mathcal{P}(X)$ such that $\{X_i\}_{i \in I}$ and $\{M_{f(i)}\}_{i \in I}$ are equal as families of subobjects of X.

For intuition here, think of $\mathcal{P}(X)$ as the class of all subsets of X, and think of M_A as the set of elements of the element $A \in X$ (i.e. as the set A itself). The family $\{M_A\}_{A \in \mathcal{P}(X)}$ is given by an object $\in_X \longrightarrow \mathcal{P}(X)$ of $\mathcal{C}/\mathcal{P}(X)$ which, as it is a family of subobjects of $\mathcal{P}(X)$, comes as a subobject $\in_X \rightarrowtail X \times \mathcal{P}(X)$ in $\mathcal{C}$. This subobject defines the membership relation on $\mathcal{P}(X)$ in $\mathcal{C}$.

Definition 6.1 A *category with basic class structure* is given by a positive Heyting category together with a (stable under slicing) collection $\mathcal{S}$ of small objects satisfying the above axioms.

The notion of basic class structure roughly embodies the closure properties on sets of the theory BCST$^-$ (precise results are given below).

We shall need two consequences of basic class structure below. The first is that every small object A of $\mathcal{C}$ is exponentiable in the sense that there is a natural function space object X^A in $\mathcal{C}$.[22] The second consequence is that the mapping $X \mapsto \mathcal{P}(X)$ on objects extends to an endofunctor on $\mathcal{C}$ whose action on morphisms takes $f \colon X \longrightarrow Y$ to the function $\mathcal{P}(f) \colon \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ that maps a subset A of X to its image under f, which is a subset of Y.

The following additional axioms on class structure exactly mirror the axioms we considered earlier as possible additions to BCST$^-$.[23]

Exponentiation (Exp) If A, B are small then so is B^A.

Powerset (Pow) If A is small then so is $\mathcal{P}(A)$.

Separation (Sep) If A is small and $B \rightarrowtail A$ is mono then B is small.

Infinity (Inf) The category $\mathcal{S}$, of small objects, has a natural numbers object.

Induction (Ind) $\mathcal{C}$ has a small natural numbers object.

Collection (Coll) The functor $\mathcal{P}(\cdot) \colon \mathcal{C} \to \mathcal{C}$ preserves regular epis.

There is one aspect of sets and classes, as they are used in (constructive) set theory, which we have not implemented in the axioms we have so far considered for $\mathcal{C}$ and $\mathcal{S}$. In set theory, there is a distinguished largest class, namely the universe itself. This can be implemented in a category of classes by asking for $\mathcal{C}$ to have a universal object in the sense of [27].

[22] Technically, A is *exponentiable* if the functor $A \times (\cdot) \colon \mathcal{C} \to \mathcal{C}$ has a right adjoint.

[23] Again, these axioms are to be imposed on all slice categories $\mathcal{C}/I$ as well as $\mathcal{C}$ itself.

Definition 6.2 A *universal object* in $\mathcal{C}$ is an object U such that, for every object X, there is a distinguished mono $X \rightarrowtail U$.

Given a category with basic class structure and universal object U, it is possible to interpret the first-order language of BCST^- internally in $\mathcal{C}$ as expressing properties of U. In general, a formula ϕ with k free variables will be interpreted as a subobject $\phi \rightarrowtail U^k$ (where U^k is the k-fold product of U with itself). The unary predicate $\mathcal{S}(x)$ is interpreted as the distinguished mono $u\colon \mathcal{P}(U) \rightarrowtail U$ and the binary predicate $x \in y$ is interpreted as the composite

$$\in_U \rightarrowtail U \times \mathcal{P}(U) \xrightarrow{\mathsf{id}\times u} U \times U\,.$$

The connectives, quantifiers and equality predicate are simply interpreted by their own internal selves. We write $(C, U) \models \phi$ to mean that the subobject $\phi \rightarrowtail U^k$ is the whole of U^k (i.e. its representing mono is an isomorphism).

The theorem below shows that the assumed structure on our category-theoretic models $\mathcal{C}$ corresponds exactly to the constructive set theories considered in Sections 3–5.

Theorem 6.3 (Soundness and Completeness) *The following are equivalent for any ϕ.*[24]

1. *ϕ is a theorem of* BCST^- *(plus any combination of* Exp*,* Pow*,* Sep*,* Inf*,* Ind *and* Coll*).*
2. *For all $\mathcal{C}$ with basic class structure (plus the same combination of* Exp*,* Pow*,* Sep*,* Inf*,* Ind *and* Coll *as in 1) and universal object U, it holds that $(C, U) \models \phi$.*

Particular (but fully illustrative) cases of this result are proved in [27, 5, 7].

7 Categories of sets

It is pleasing that all the constructive set theories we have considered have sound and complete classes of category-theoretic models that are easily axiomatized. The goal of this section is to show that, for certain constructive set theories, there are interesting examples of models that occur naturally in mathematics. Such natural models provide a source of possible applications of the associated constructive set theory, and thus potentially enable a mathematician to appreciate the value of the theory for non-ideological reasons.

Our approach to finding such models is to analyse the category-theoretic structure of categories of sets independently of the containing category of classes. First we state properties of the category of sets $\mathcal{S}$ which fall out easily from the assumed structure on $\mathcal{C}$.

Proposition 7.1 *Suppose $\mathcal{C}$ has basic class structure.*

[24] We have omitted spelling out how the additional constants $N, 0, s$ are interpreted in U in the case that the Inf and Ind axioms are considered. This is routine.

1. *$\mathcal{S}$ is a Heyting pretopos.*[25]

2. *If* Exp *holds then $\mathcal{S}$ is locally cartesian closed.*

3. *If* Pow *holds then $\mathcal{S}$ is an elementary topos.*

Surprisingly, the above properties completely characterise the possible categories of sets in each case. The technology for demonstrating this has been developed in [5, 6, 7]. To a small category $\mathcal{E}$, one associates a category $\mathrm{Idl}(\mathcal{E})$ of "ideals" over $\mathcal{E}$. This construction enjoys the following properties.[26]

Theorem 7.2 ([5, 6, 7]) *Suppose that $\mathcal{E}$ is a Heyting pretopos.*

1. $\mathrm{Idl}(\mathcal{E})$ *has basic class structure with $\mathcal{E}$ as its full subcategory of small objects, it has a universal object and it satisfies* Coll.

2. *If $\mathcal{E}$ is locally cartesian closed then* $\mathrm{Idl}(\mathcal{E})$ *satisfies* Exp

3. *If $\mathcal{E}$ is an elementary topos then* $\mathrm{Idl}(\mathcal{E})$ *satisfies* Pow.

The proof of this theorem is distributed over the three references [5, 6, 7]. First, in Awodey *et al* [5], $\mathrm{Idl}(\mathcal{E})$ was defined for an elementary topos $\mathcal{E}$ using a "system of inclusions" on $\mathcal{E}$, and it was shown that $\mathrm{Idl}(\mathcal{E})$ has basic class structure and satisfies Pow and Coll. Later, in [6] (see also [24]), this construction was improved to avoid the system of inclusions, by defining $\mathrm{Idl}(\mathcal{E})$ to be the full subcategory of the presheaf category $[\mathcal{E}^{\mathrm{op}}, \mathbf{Set}]$ whose objects are directed colimits of diagrams of monos between representables.[27] That the improved construction also adapts to models of predicative set theories was realised by Awodey, Warren and the present author, who proved statement 2 of Theorem 7.2, after which Awodey and Warren extended the approach to obtain statement 1 [7].

Let us spell out the ramifications for constructive set theories of the results above. By Theorem 6.3, models of the impredicative theory

$$\mathrm{BCST} + \mathrm{Pow}\ (+\ \mathrm{Coll})$$

are given by categories with basic class structure satisfying Inf, Pow (and Coll). By Proposition 7.1 and Theorem 7.2, the categories of sets arising in such categories of classes are exactly the elementary toposes with natural numbers object (nno). Thus, the categories of sets that model BCST + Pow (+ Coll) are exactly the elementary toposes with nno. Elementary toposes are abundant and natural mathematical structures, hence the theory BCST + Pow (+ Coll) is of independent mathematical interest in being strongly tied to such categories. Indeed, this is the viewpoint developed in [5], where the theory BCST + Pow and its models are studied in detail.

[25] A *Heyting pretopos* is a positive Heyting category in which all equivalence relations have effective quotients.

[26] For discussion on the appropriate meta-theory for proving the results in this section, see [5].

[27] In order to obtain a universal object, one needs to define a "universe" U in $\mathrm{Idl}(\mathcal{E})$ and then restrict to the full subcategory of subobjects of U.

An analogous story holds for the predicative theory

$$\mathrm{BCST} + \mathrm{Exp}\,(+\,\mathrm{Coll})\,.$$

For this theory, the associated categories of sets are exactly the locally cartesian closed (lcc) pretoposes with nno.[28] There are many interesting examples of lcc pretoposes that are not elementary toposes. For example, if one performs the usual construction that builds a realizability topos from a partial combinatory algebra (pca) [15], but does so using a *typed pca* in the sense of Longley then the resulting category is an lcc pretopos, but not necessarily a topos, see [19]. A particularly natural mathematical example of an lcc pretopos that is not an elementary topos is the "exact completion" of the category of topological spaces [8]. The possibility of using BCST + Exp (+ Coll) to reason within such categories gives non-ideological mathematical motivation for the consideration of such predicative constructive set theories.

It should be noted that the above results characterizing categories of sets apply only to theories not involving the Ind and Sep axiom schemata. For the theories that do include these schemata, it is not clear whether it is possible to characterize the properties of the category of sets without involving the containing category of classes in the characterization. The difficulty is that the axiom schemata themselves relate properties of sets and classes, rather than stating closure properties of sets alone. Nevertheless, for the strongest theory

$$\mathrm{BCST}\ +\ \mathrm{Pow}\ +\ \mathrm{Sep}\ +\ \mathrm{Coll}\,,$$

we can at least demonstrate that the theory has a rich collection of interesting mathematical models.

Theorem 7.3 ([5]) *If $\mathcal{E}$ is a cocomplete topos or a realizability topos then $\mathcal{E}$ fully and faithfully embeds as the subcategory of small objects in a category with basic class structure satisfying* Pow*,* Sep *(hence* Ind*) and* Coll*.*

We end the paper with a challenge to find an analogue of Theorem 7.3 for the predicative theory BCST + Exp + Ind + Coll. In particular, we would like to see a naturally occurring mathematical example of a category of sets for this theory which is truly predicative in the sense that, even when considered classically, it cannot be used to model Pow or Sep. Even more interesting would be to find naturally occurring truly predicative model of constructive set theories that include general mechanisms for inductive definition (including their associated full induction principles), of the sort considered in e.g. [2, 4, 22].

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