

Triangulated Categories in Algebraic Geometry

Antony Maciocia

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Abelian Categories

- Introduced by Grothendieck in 1950s to unify homological algebra
- Associate abelian group type object to spaces via co-chain complexes:

$$\dots C^{i-1}(X) \xrightarrow{d^{i-1}} C^i(X) \xrightarrow{d^i} C^{i+1}(X) \dots,$$

where $d^i \circ d^{i-1} = 0$.

- The cohomology is then $H^i(X) = \frac{\ker d^i}{\text{Im } d^{i-1}}$, measures how inexact the complex is.
- This process is functorial.

Some constructions in abelian categories

A pair of composable maps $f : a \rightarrow b$ and $g : b \rightarrow c$ is exact at b if $\text{Im}(f) = \ker(g)$.

eg

$$0 \rightarrow \ker f \rightarrow a \xrightarrow{f} b \rightarrow \text{coker } f \rightarrow 0$$

is exact at all places.

We can define cohomology as before and it measures how a complex fails to be exact.

Let A, B and C be complexes and suppose $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact (we say **short exact**) then there is a canonical map $H^i(C) \rightarrow H^{i+1}(A)$ such that the resulting sequence

$$\dots \rightarrow H^i(A) \rightarrow H^i(B) \rightarrow H^i(C) \rightarrow H^{i+1}(A) \rightarrow H^{i+1}(B) \rightarrow \dots$$

is exact.

Extracting Geometrical Information

Example

Let X be a 2 dimensional (projective) variety. A curve on X can be viewed as the zero set of an algebraic map $s : \mathcal{O}_X \rightarrow L$, where L is a suitable (locally-free) rank 1 \mathcal{O}_X -module.

Given some points $Y \subset X$, we can consider the sheaf \mathcal{O}_Y as an object of $\text{Coh}(X)$.

Then the restriction map $L \rightarrow \mathcal{O}_Y$ is epic and its kernel K has the property that elements of $H^0(K)$ which map to s correspond precisely to incidence of Y on our curve.

Estimating the size of $H^0(K)$ is then useful to determine incidence properties.

Outline

Abelian Categories

Issues in Algebraic Geometry

Moduli Problems

Derived Categories

Triangulated Categories

And Back Again

Definition

An **abelian category** is a category \mathcal{A} satisfying

- (1) \mathcal{A} is additive: each set of morphisms is an abelian group (and composition is bilinear), there is a biproduct and a zero object.
- (2) \mathcal{A} has all equalisers and all monics are equalisers, and dually.

Notes

- (2) tells us that all maps have kernels, cokernels and images and that "monic+epic=isom".
- Some conditions are redundant.
- Functors between abelian categories are **exact** if they preserve the additive structure, kernels and cokernels.
- Examples: Ab , $R\text{-mod}$, Vect_k .
- AbCat is the (lax 2-)category of abelian categories with exact functors.
- Any abelian category \mathcal{A} admits an exact full embedding in Ab .

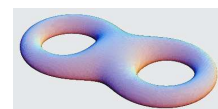
Issues in Algebraic Geometry

- Categories of objects of interest: Var , Sch , Sch/S , Sch/k .
- A scheme is a locally ringed space which is locally spec of a ring.
- Sch has terminal object $\text{spec } \mathbb{Z}$ and all pullbacks.
- Can also consider other morphisms: birational maps and associated equivalence.
- Well developed theory of dimension.
- Schemes come with structure sheaves \mathcal{O}_X . Can consider $\mathcal{O}_X\text{-mod}$. Restrict to finitely presented modules: $\text{Coh}(X)$.
- $\text{Coh}(X)$ is abelian.
- There are cohomology functors $\text{Coh}(X) \rightarrow \text{Ab}^{\mathbb{N}}$ with $H^0 = \Gamma$, the global sections functor.

Invariants

Example

Dimension 1 projective varieties can be classified into type according to a non-negative integer called its genus. Loosely, the genus corresponds to the number of holes in the space:



More generally, we can find a range of numerical invariants and, more generally, cohomology classes (**characteristic classes**) which allow us to crudely classify both schemes and sheaves on schemes.

Moduli Problems

- Consider an object \mathcal{M} in $\text{Set}^{\text{Sch}^{\text{op}}}$ of the form

$$\mathcal{M}(S) = \{\text{objects over } S \text{ under pullback}\}$$

- If such a functor is representable then there is a scheme M and natural isomorphism $\mathcal{M} \cong \text{Hom}(-, M)$.
- This means there is some object \mathbb{E} over M such that for all schemes S and $E \in \mathcal{M}(S)$, there is a map $f : S \rightarrow M$ such that $E = f^*\mathbb{E}$ and M is universal for such objects.
- eg. for a scheme X ,

$$\mathcal{M}_{X,c}(S) = \{\text{coherent sheaves with fixed char class } c \text{ on } S \times X\}.$$
- But $\mathcal{M}_{X,c}$ is not representable.
- Partly fix by tweaking the moduli functor (to make \mathcal{M} into a sheaf in a suitable subcanonical topology on Sch).

- The key for AG is the existence of a universal object. This is weaker than asking that \mathcal{M} be representable.
- For representability we have $\text{Hom}(\text{Hom}(-, S), \mathcal{M}) \cong \text{Hom}(S, M)$ for any scheme S .
- Replace with $\text{Hom}(\mathcal{M}, \text{Hom}(-, S)) \cong \text{Hom}(M, S)$. Then \mathcal{M} **corepresents** \mathcal{M} .
- This notion is strictly weaker than representability (we have a map $\mathcal{M} \rightarrow \text{Hom}(-, M)$ which need not be an isomorphism).
- Need to add that M represents \mathcal{M} when restricted to objects of the form $\text{spec } k$ for k algebraically closed.
- M is still universal called a **coarse moduli space**.
- This still doesn't exist for $\mathcal{M}_{X,c}$ in general.

Two solutions

The Problem: A large group of automorphisms acts rather badly.

Two possible solutions:

- Restrict the domain to objects where the action is better.
 - Extend the domain so that the action is better.
- Option 1 is the historical solution.
 - In practice we attempt to construct the representing object, eg using GIT.
 - This results in a condition to impose (usually called a **stability condition**).
 - eg for torsion-free coherent sheaves on a curve, we say that E is stable if $\frac{d(F)}{r(F)} < \frac{d(E)}{r(E)}$ for all proper subsheaves F .

But...

- While coarse moduli usually exist, M is not usually complete.
- This can usually be fixed by adding equiv classes of mildly unstable objects.
- The resulting moduli spaces often have some nasty properties.
- We sometimes fix this by resolving the spaces.
- Still doesn't capture all objects of interest.
- However, the set of spaces M are generally very useful.
- Now, stability conditions themselves have other applications (see later)

Option 2

- View \mathcal{M} instead as a 2-functor $\text{Sch} \rightarrow \text{Grpoid}$.
- \mathcal{M} is often a sheaf wrt suitable topologies. These are **stacks**.
- An object S of Sch can be viewed as the obvious stack $\text{Sch} \rightarrow \text{Grpoid}$ via Yoneda and the identity maps.
- In a very general setting, given a group G acting on a scheme S , we can view S/G as a stack. Then $\mathcal{M}_{X,c}$ can be viewed as a stack.
- Provides a better setting for generalities about geometric objects related to schemes and is now widely used.
- There are still many technical difficulties (eg need for **algebraic stacks**).

Derived Categories

Definition

- The correct setting for homological algebra (introduced by Grothendieck in the 1950s to unify a variety of homology theories).
- Start with an abelian category \mathcal{A} and first form $\mathcal{C}(\mathcal{A})$, the category of (co-)complexes of objects of \mathcal{A} .
- Define a morphism f of $\mathcal{C}(\mathcal{A})$ to be a **quasi-isomorphism (quis)** if $H^i(f)$ are isomorphisms for all i .
- Define the **derived category** $\mathcal{D}(\mathcal{A})$ of \mathcal{A} to be the localization $\mathcal{C}(\mathcal{A})/\text{quis}$.
- $\mathcal{D}(\mathcal{A})$ is additive but is not generally abelian.
- There is a fully faithful additive functor $\mathcal{A} \rightarrow \mathcal{D}(\mathcal{A})$ given by mapping and object a to the complex $\cdots \rightarrow 0 \rightarrow a \rightarrow 0 \rightarrow \cdots$ centred on 0.

- The localization functor factors through $\mathcal{K}(\mathcal{A})$, the category of maps up to homotopy.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^i & \xrightarrow{d_A^i} & A^{i+1} & \longrightarrow & \cdots \\ & \searrow^{h^i} & \downarrow f^i & \parallel g^i & \downarrow f^{i+1} & \parallel g^{i+1} & \\ \cdots & \longrightarrow & B^i & \xrightarrow{d_B^i} & B^{i+1} & \longrightarrow & \cdots \end{array}$$

$$d_B^{i-1}h^i + h^{i+1}d_A^i = f^i - g^i \text{ for all } i.$$

- In fact, the quis class is localizing in $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A}) \cong \mathcal{K}(\mathcal{A})/\text{quis}$.
- We can find a subcategory \mathcal{I} of \mathcal{A} such that $\mathcal{K}(\mathcal{I}) \cong \mathcal{D}(\mathcal{A})$.
- Useful to construct (derived) functors on $\mathcal{D}(\mathcal{A})$ and to explicitly compute their cohomology.

How far away is $\mathcal{D}(\mathcal{A})$ from being abelian?

- We certainly don't have all kernels and cokernels.
- But given a map $f : A \rightarrow B$ of complexes there is another complex C and a monic map $B \rightarrow C$ and epic map $C \rightarrow A[1]$, where $A[1]$ means "shift left by 1", given by $A[1] \oplus B$ and differentials $\begin{pmatrix} -d_A[1] & 0 \\ f & d_B \end{pmatrix}$.
- If f is the identity map $A \rightarrow A$ then (easy exercise) C is homotopic to 0.
- These induce an exact sequence

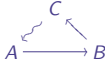
$$\cdots \rightarrow H^i(A) \rightarrow H^i(B) \rightarrow H^i(C) \rightarrow H^{i+1}(A) \rightarrow \cdots$$
- Applied to the cases $A = a$ and $B = b$ in $\mathcal{A} \subset \mathcal{D}(\mathcal{A})$, C is a twist of $\text{coker } f \oplus \text{ker } f[1]$.
- Giving $0 \rightarrow \text{ker } f \rightarrow a \xrightarrow{f} b \rightarrow \text{coker } f \rightarrow 0$.

Triangulated Categories

- The idea of triangulated categories is to abstract the categorical properties of the derived category
- Observe that $D(A)$ is additive with an automorphism $[1]$.
- There is a special set of diagrams of the form

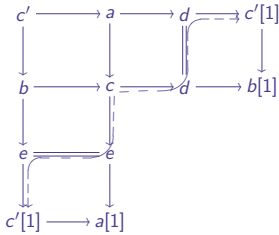
$$A \rightarrow B \rightarrow C \rightarrow A[1]$$

repeating with shifts in both directions. We call such diagrams **triangles**:



- A map of triangles is just a commuting diagram of maps in the obvious way.
- Observe that our special triangles have the property that if we are given maps $A \rightarrow A'$ and $B \rightarrow B'$ commuting with f and f' then we have a map $C \rightarrow C'$ which gives a map of triangles.

(6) Given two triangles in Δ with a common vertex



we can complete it to a commuting diagram with all rows and columns in Δ , for which the composites $c \rightarrow e \rightarrow c'[1]$ and $c \rightarrow d \rightarrow c'[1]$ agree. Called the **octahedral axiom**: "the bottom of such an octahedron can be completed to an octahedron".

Definition

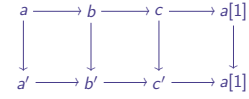
Suppose \mathcal{T} is a triangulated category and \mathcal{A} an abelian category. A functor $F : \mathcal{T} \rightarrow \mathcal{A}$ is **cohomological** if it is additive and for any triangle $a \rightarrow b \rightarrow c$ in Δ , $F(a) \rightarrow F(b) \rightarrow F(c)$ is exact.

- $H^0 : D(\mathcal{A}) \rightarrow \mathcal{A}$ is cohomological.
- $\text{Hom}(B, -)$ and $\text{Hom}(-, B)$ are cohomological.
- The functor category $Ab^{\mathcal{T}^{op}}$ is automatically abelian and the Yoneda functor is cohomological.
- Define the full subcategory $\underline{A}(\mathcal{T})$ of $Ab^{\mathcal{T}^{op}}$ to be those functors which are of the form $\text{coker Hom}(-, f)$. Then $\mathcal{T} \rightarrow \underline{A}(\mathcal{T})$ is the universal cohomological (contravariant) functor.

More generally:

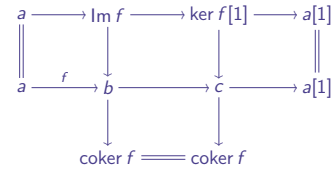
- Given an exact subcategory $\underline{D} \subset \mathcal{T}$ such that $\underline{D}[1] \subset \underline{D}$ and for each object a of \mathcal{T} there is a distinguished triangle $a' \rightarrow a \rightarrow a''$ with a' in \underline{D} and a'' in \underline{D}^\perp . We call this a **t-structure** on \mathcal{T} .
- Define $\underline{C} = \underline{D} \cap \underline{D}^\perp[1]$, the **core** or **heart** of the t-structure. Then \underline{C} is abelian.
- The maps $a \rightarrow a'$ and $a \rightarrow a''$ give well defined functors $\tau_{\leq 0} : \mathcal{T} \rightarrow \underline{D}$ and $\tau_{\geq 0} : \mathcal{T} \rightarrow \underline{D}^\perp[1]$ which are left (resp. right) adjoint to the inclusions.
- Then $H_D^0 : \mathcal{T} \rightarrow \underline{C}$ defined by $a \mapsto \tau_{\geq 0}\tau_{\leq 0}a$ is cohomological.
- \underline{C} gives us an abelian "viewport" into \mathcal{T} .
- For example, $\times 2 : \mathbb{Z} \rightarrow \mathbb{Z}$ injects in \underline{Ab} but there is a core in $\underline{D}(\underline{Ab})$ for which it does not inject but surjects with kernel $\mathbb{Z}_2[-1]$.

- A **triangulated category** is an additive category with automorphism $[1]$ which has a distinguished set Δ of triangles. These satisfy the following axioms
 - the triangle $a = a \rightarrow 0$ is in Δ .
 - any map $a \rightarrow b$ can be completed to a triangle in Δ
 - if $a \rightarrow b \rightarrow c \rightarrow a[1]$ is in Δ then so is $b \rightarrow c \rightarrow a[1] \rightarrow b[1]$, where one of the maps is minus the corresponding map in the original triangle.
 - any diagram
 - can be completed to a map of triangles.
 - any triangle isomorphic to a triangle in Δ is in Δ .



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- Triangulated categories form a category with **exact** functors (defined to preserve the distinguished triangles).
- Axiom (2) relates to existence of kernels and cokernels.
- Axiom (4) is what is left over of their universal property.
- Axiom (6) relates to the image/coimage of a map:



for a map $f : a \rightarrow b$ of $\underline{A} \subset D(\mathcal{A})$.

And Back Again

- The special subcategory \underline{A} of $\underline{D}(\mathcal{A})$ can be constructed as follows.
- Define functors $\tau_{\leq n}$ and $\tau_{\geq n} : \underline{D}(\mathcal{A}) \rightarrow \underline{D}(\mathcal{A})$ by truncating complexes at position n :

$$\dots \rightarrow A^{n-2} \rightarrow A^{n-1} \rightarrow \ker d^n \rightarrow 0 \rightarrow \dots$$

$$\dots \rightarrow 0 \rightarrow \text{coker } d^n \rightarrow A^{n+1} \rightarrow A^{n+2} \rightarrow \dots$$
- These have the property that for all objects a of $\underline{D}(\mathcal{A})$, there is a canonical distinguished triangle $\tau_{\leq n}a \rightarrow a \rightarrow \tau_{\geq n+1}a$.
- Then if $\underline{D}^{\leq n}$ is the full subcategory of $\underline{D}(\mathcal{A})$ of complexes with zero cohomology above n , $\tau_{\leq n}$ provides a right adjoint of the inclusion $\underline{D}^{\leq n} \subset \underline{D}(\mathcal{A})$.
- and $\underline{D}^{\geq n+1}$ is the left orthogonal of $\underline{D}^{\leq n}$
- Then $\underline{A} = \underline{D}^{\leq 0} \cap \underline{D}^{\geq 0}$.

Spaces from Triangulated categories

- Consider a linear triangulated category \mathcal{T} . We assume various finiteness conditions satisfied by $\underline{D}(\text{Coh}(X))$.
- We can define a topological (metric) space $\text{Stab}(\mathcal{T})$ as the set of pairs (Z, P) , where $Z : K_0(\mathcal{T}) \rightarrow \mathbb{C}$ is a linear map and $P : \mathbb{R} \rightarrow \text{sub } \mathcal{T}$ a path of full abelian subcategories of \mathcal{T} . These must satisfy:
 - For a in $P(\phi)$, $Z(a) = m(a)e^{i\pi\phi}$ and $E = 0$ iff $m(a) = 0$.
 - $\forall \phi, P(\phi + 1) = P(\phi)[1]$.
 - $\forall a \neq 0$ in \mathcal{T} , $\exists \phi_1 > \phi_2 > \dots > \phi_n$ and triangles $a_{i-1} \rightarrow a_i \rightarrow b_i$ with $a_0 = 0$, $a_n = a$ and b_i in $P(\phi_i)$.
 - for $\phi_1 > \phi_2$, and a_i in $P(\phi_i)$, we have $\text{Hom}(a_1, a_2) = 0$.
- These are called **Bridgeland stability conditions**.

- The a_i s are unique (up to isomorphism) and so ϕ_i are also unique. We let $\phi^+ = \phi_1$ and $\phi^- = \phi_n$.
- We can extend P to a map of interval (x, y) of reals via the full subcats of objects a s.t. $x < \phi^-(a) \leq \phi^+(a) < y$.
- $P((0, 1])$ is the core of a bounded t -structure on $\underline{\mathcal{T}}$.
- Conversely, given a bounded t -structure with core $\underline{\mathcal{C}}$ and a linear map $Z : K_0(\underline{\mathcal{C}}) \rightarrow \mathbb{C}$, such that $Z(c) = m(c)e^{i\pi\phi(c)}$ with $0 < \phi(c) \leq 1$ and $m(c) > 0$ if $c \neq 0$. We define c to be semistable if for all proper monics $a \rightarrow c$ in $\underline{\mathcal{C}}$ $\phi(a) \leq \phi(c)$. Then Z extends to (Z, P) is a stability condition, where $P(\phi) \subset \underline{\mathcal{C}}$ are the semistable objects of phase ϕ .
- We then have a continuously varying family of abelian subcategories of $\underline{\mathcal{T}}$.
- For example, on a curve we can choose $Z(a) = -d(a) + ir(a)$ and then the standard t -structure is $P((0, 1])$.

Concluding Remarks

- Recently category theory has provided essential tools for AG.
- Some tools already well developed (fibred categories, higher category theory, Grothedieck toposes, abelian categories).
- Some tools required additional development (triangulated categories, t -structures, stability conditions)
- The key additional feature is the need to measure objects (on a real or integral scale)
- Powerful deformation arguments common in AG may see applications in category theory.

THE END