The Locale of Random Sequences

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Random sequences are infinite binary sequences generated by tossing a fair coin \textit{ad infinitum}

A random sequence $\alpha \in 2^\omega$ (here $2 := \{0, 1\}$ with, say, 0 for heads and 1 for tails) exhibits:

- Local irregularity: e.g., cannot predict $\alpha_n$ from $\alpha_0 \ldots \alpha_{n-1}$

- Global regularity: e.g., asymptotic satisfaction of probabilistic laws such as law of large numbers

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \alpha_i = \frac{1}{2}
\]

Old question, cf. von Mises (1919): How does one characterise random sequences mathematically?
Naïve attempt

A random sequence is one that satisfies all probabilistic laws

So define set of random sequences using uniform/Lebesgue measure $\lambda$

$$ R := \bigcap \{ X \subseteq 2^\omega \mid X \text{ measurable with } \lambda(X) = 1 \} $$

Equivalently:

$$ \overline{R} := \bigcup \{ X \subseteq 2^\omega \mid X \text{ null} \} $$

Issues

Arguably too abstract. It’s not obvious that the notion of (Lebesgue) measurable subset is well-motivated here.

Definition yields $R = \emptyset$. 
Martin-Löf randomness (1966)

A random sequence is one that avoids all effective null sets.

A subset $X \subseteq 2^\omega$ is effectively null if there is a recursive sequence $(U_i)$ of effective open sets with $X \subseteq U_i$ and $\lambda(U_i) < 2^{-(i+1)}$.

Define:

$$\overline{R_{\text{ML}}} := \bigcup \{X \subseteq 2^\omega \mid X \text{ effectively null} \}$$

Motivation: For every $\alpha \not\in \overline{R_{\text{ML}}}$ and confidence level $\epsilon > 0$, there exists (recursively in $\epsilon$) an effective open $U_\epsilon \ni \alpha$ with measure $< \epsilon$. Thus the observation (see next slide) $\alpha \in U_\epsilon$ gives evidence that $\alpha$ is non-random with probability $> 1 - \epsilon$.

Slogan: Non-randomness can be statistically detected.
Kurtz randomness (1981)

A random sequence is one belonging to all measure 1 effective opens

Define:

\[ R_K := \bigcap \{ U \subseteq 2^\omega \mid U \text{ effective open}, \lambda(U) = 1 \} \]

Effective open subsets \( U \subseteq 2^\omega \) are characterised by the existence of a Type-2 Turing Machine that halts on \( \alpha \) iff \( \alpha \in U \). Thus we can observe when \( \alpha \in U \) holds (but not when \( \alpha \not\in U \))

Kurtz randomness is thus motivated by the following (empirical!): Effective postulate of randomness

Given an effective observable property of measure 1, any random sequence will eventually be observed to satisfy the property.
Algorithmic randomness more generally

**Fact** $\mathbf{R}_{\text{ML}} \subset \mathbf{R}_K$ (proper inclusion)

Other related (but different) notions have been proposed by Martin-Löf, Solovay, Schnorr, . . . .

Algorithmic randomness is a fascinating area of recursion theory, but does it properly model the empirical phenomenon of randomness?

**Criticisms**

- Does not provide one canonical notion of randomness
- While (arguably) the recursion-theoretic restrictions reflect our observational limitations, there is no reason to believe any recursion-theoretic dependencies to be inherent in the phenomenon of randomness itself
Are observations effective?

At time $i$, person $B$ receives a digit $\alpha_i \in 2$ from source $A$, and then tosses a fair coin to generate a digit $\beta_i \in 2$.

This process produces sequences $\alpha, \beta$ with $\beta$ random.

The property $\beta \in 2^\omega \setminus \{\alpha\}$, i.e., $\beta \neq \alpha$, is observable.

(But the property $\beta \in \{\alpha\}$, i.e., $\beta = \alpha$, is not observable.)

Because $\beta$ is random the observation $\beta \neq \alpha$ will eventually succeed.

Restricting $\alpha$ to be recursive would be making an additional and undesirable assumption.
Removing effectivity

Open sets $U \subseteq 2^\omega$ are characterised by the property that if $\alpha \in U$ then this fact can be “observed” by looking at a finite prefix of $\alpha$.

We take arbitrary opens sets as our observable properties.

Basic postulate of randomness

Given an observable property of measure 1, any random sequence will eventually be observed to satisfy the property.

We are apparently defining:

$$R := \bigcap \{U \subseteq 2^\omega \mid U \text{ open, } \lambda(U) = 1\}$$

This again gives $R = \emptyset$ (cf. slide 3)
Observable properties of random sequences

Our only access to a random sequence is via its finite prefixes

So natural to consider “space” $\mathbb{R}$ of random sequences as a sublocale of $2^\omega$.

For $U, V \subseteq 2^\omega$ open, define:

$$U \approx V \iff \lambda(U \cup V) = \lambda(U \cap V)$$

If $U \approx V$ then intuitively they represent the same observation on random sequences.

Define $\mathcal{O}(\mathbb{R}) := \mathcal{O}(2^\omega)/\approx$.

(Cf. the measure ($\sigma$-)algebra of $2^\omega$, equivalently of $[0, 1]$.)

**Proposition** The above defines $\mathbb{R}$ as a sublocale of $2^\omega$. 
Theorem \( R \) is the meet (in the lattice of sublocales) of all open sublocales \( U \subseteq 2^\omega \) with \( \lambda(U) = 1 \).

Thus the basic postulate of randomness (interpreted locally) determines \( R \).

Define the outer measure of a sublocale (not necessarily subset) \( Y \subseteq 2^\omega \) by:

\[
\lambda^*(Y) := \inf \{ \lambda(U) \mid U \in \mathcal{O}(2^\omega) \text{ and } Y \subseteq U \}
\]

Theorem The sublocale \( R \subseteq 2^\omega \) has outer measure 1. Moreover \( R \) is the smallest sublocale of \( 2^\omega \) of outer measure 1.

Thus \( R \) satisfies all probabilistic laws (cf. slide 3) . . . and more!

Above all, there is just one canonical locale of random sequences.
More generally

If we generalise $\lambda$ to other probability “measures” (technically, probability valuations), and $2^\omega$ to other locales, we still obtain canonical “random” sublocales.

A (continuous) probability valuation on a locale $X$ is a (necessarily monotone) function $\mu : \mathcal{O}(X) \to [0, 1]$ satisfying:

$$
\begin{align*}
\mu(\emptyset) &= 0, \\
\mu(U \cup V) &= \mu(U) + \mu(V) - \mu(U \cap V), \\
\mu(X) &= 1, \\
\mu(\bigcup_{U \in D}^\uparrow U) &= \sup_{U \in D} \mu(U) \quad (D \subseteq \mathcal{O}(X) \text{ directed})
\end{align*}
$$

Given $(X, \mu)$ define:

$$
U \approx_{\mu} V :\iff \mu(U \cup V) = \mu(U \cap V)
$$

$$
\mathcal{O}(\mathbb{R}(\mu)) := \mathcal{O}(X)/\approx_{\mu}
$$
Proposition \( R(\mu) \) is a sublocale of \( X \).

Define the outer value of a sublocale \( Y \subseteq X \) by:

\[
\mu^*(Y) := \inf\{\mu(U) \mid U \in \mathcal{O}(X) \text{ and } Y \subseteq U\}
\]

Theorem If the locale \( X \) is regular then:

1. \( R(\mu) \) is the meet of all open sublocales \( U \subseteq X \) with \( \mu(U) = 1 \)
2. \( R(\mu) \) is the meet of all sublocales \( Y \subseteq X \) with \( \mu^*(Y) = 1 \)
3. \( \mu^*(R(\mu)) = 1 \)

Obviously, \( \mu \) is also a probability valuation on \( R(\mu) \). Then \( (R(\mu), \mu) \) is random in the sense that:

\[
[U] \subset [V] \in \mathcal{O}(R(\mu)) \text{ (proper inclusion) } \Rightarrow \mu([U]) < \mu([V])
\]
Another characterisation of $\mathbb{R}$

The locale of random sequences, $\mathbb{R}$, is:

- countably based
- zero dimensional
- and has no points

**Theorem** If $X$ is a countably-based zero-dimensional locale with no points and $\mu$ is a probability valuation with $(X, \mu)$ random then $(X, \mu)$ and $(\mathbb{R}, \lambda)$ are isomorphic as valuation locales. (In particular, $X$ and $\mathbb{R}$ are homeomorphic.)

(Cf. the measure algebra isomorphism theorem in, e.g., Halmos’ “Measure Theory”)

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Further mathematical directions

- Establish characteristic properties of random sequences
  (invariance under von Mises’ place selections, van Lambalgen’s homogeneity, properties of independence)
- Drop regularity assumptions on locales
- Connections with measures and measure algebras
- Constructive version, whence computable version
- Random space?

Philosophical discussion point

The standard probabilistic notion of “almost surely” is vague

In contrast, the basic postulate of randomness (observations of probability 1 happen with certainty) is clear and empirically validated