“Spaces of Valuations”
Revisited

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Valuations

A (continuous) valuation on a topological space \( X \) is a function \( \nu: \mathcal{O}(X) \to [0, \infty] \) (where \( \mathcal{O}(X) \) is lattice of opens) satisfying:

1. \( \nu(\emptyset) = 0 \)

2. \( \nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V) \) (modularity)

3. \( \nu(\bigcup \mathcal{U}) = \sup_{U \in \mathcal{U}} \nu(U) \), for any directed \( \mathcal{U} \subseteq \mathcal{O}(X) \) (continuity)

N.B., 3 is just continuity w.r.t. Scott topologies on \( \mathcal{O}(X) \) and \([0, \infty]\). (Scott topology on \([0, \infty]\) a.k.a. topology of lower semicontinuity.)

\( \nu \) is finite if \( \nu(X) < \infty \).

\( \nu \) is a probability valuation if \( \nu(X) = 1 \).
Why valuations?

Valuations are closely related to Borel measures, and there are precise results equating the two for wide classes of spaces.

In contrast to measures, for any topological space $X$, directed sets of valuations have suprema. This motivates the use of valuations to define the probabilistic powerdomain (Jones and Plotkin 1989).

Valuations can be used to define computational approaches to integration (Edalat). Plausibly, valuations offer a foundation for constructive/computable measure/probability theory.

Valuations support a theory of integration of lower semicontinuous functions, which sits between integration of continuous functions in analysis and integration of measurable functions in measure theory. This is the subject of positive analysis (cf. Lawson).
Topology of lower semicontinuity

In this talk, we always consider $[0, \infty]$ with the topology of lower semicontinuity, whose nontrivial open sets are:

$$(r, \infty]$$

for any $0 \leq r < \infty$. Likewise for $[0, \infty)$.

Thus the continuous functions from a topological space $X$ to $[0, \infty]$ (or to $[0, \infty)$) are the lower semicontinuous functions: those $f: X \to [0, \infty]$ satisfying, for all $0 \leq r < \infty$,

$$f^{-1}(r, \infty] \text{ is open}.$$ 

Although, given the assumed topology, we could simply call such functions continuous, we choose to emphasise the topology by explicitly referring to them as lower semicontinuous.
Step functions

A step function is a lower semicontinuous function $f : X \to [0, \infty)$ with finite image.

Example  If $U \subseteq X$ is open then $\chi_U$ is a step function:

$$\chi_U(x) := \begin{cases} 
1 & \text{if } x \in U \\
0 & \text{otherwise}
\end{cases}$$

Proposition  Every step function is of the form:

$$\sum_{i=1}^{n} d_i \chi_{U_i}$$

where $n \geq 0$, $d_1, \ldots, d_n \in (0, \infty)$ and $U_1, \ldots, U_n \subseteq X$ are open.

Proposition  Every lower semicontinuous $f : X \to [0, \infty]$ is the directed supremum of the step functions below it.
Integration (Jones, Kirch)

For a step function $s = \sum_{i=1}^{n} d_i \chi_{U_i} : [0, \infty)$ and $\nu \in \mathcal{V}(X)$, define:

$$\int s \, d\nu := \sum_{i=1}^{n} d_i \nu(U_i) .$$

For a lower semicontinuous $f : X \to [0, \infty]$ and $\nu \in \mathcal{V}(X)$, define:

$$\int f \, d\nu := \sup_{s \leq f} \int s \, d\nu .$$

where $s$ ranges over step functions.

Proposition  Integration is well-defined.
Properties of integration

Write $\mathcal{L}(X)$ for set of lower semicontinuous functions $X \to [0, \infty]$, and $\mathcal{V}(X)$ for set of valuations. Then:

$$\int : \mathcal{L}(X) \times \mathcal{V}(X) \to [0, \infty]$$

This is bilinear w.r.t. the natural cone structures on $\mathcal{L}(X)$ and $\mathcal{V}(X)$

Topologize $\mathcal{L}(X)$ and $\mathcal{V}(X)$ with coarsest topologies that make integration separately lower semicontinuous. This gives subbasic sets for $\mathcal{L}(X)$ and $\mathcal{V}(X)$ respectively:

$$\{ f' \mid \int f' \, d\nu > r \}$$

$$\{ \nu' \mid \int f \, d\nu' > r \} ,$$

generated by $0 \leq r < \infty$, valuations $\nu$, and lower semicontinuous $f$. 
Topological cones

These topologies turn $\mathcal{L}(X)$ and $\mathcal{V}(X)$ into topological cones.

A topological cone is a topological space $X$ with together with distinguished $0 \in X$ and jointly continuous:

$$r.x: \quad [0, \infty) \times X \rightarrow X \quad \text{(scalar multiplication)}$$

$$x + y: \quad X \times X \rightarrow X \quad \text{(addition)}$$

where $(X, 0, +)$ is a commutative monoid and:

$$1.x = x \quad 0.x = 0 \quad r.0 = 0$$

$$(r.s).x = r.(s.x) \quad (r + s).x = (r.x) + (s.x) \quad r.(x + y) = r.x + r.y$$

A continuous function $f: X \rightarrow Y$, between topological cones, is said to be linear if it preserves $0$, scalar multiplication and addition.
Spaces of valuations

A simpler subbasis for the topology on $\mathcal{V}(X)$ is given by sets:

$$\langle U > r \rangle := \{ \nu | \nu(U) > r \},$$

generated by opens $U \subseteq X$ and $0 \leq r < \infty$

Write $\mathcal{V}_{<\infty}(X)$ for subspace of finite valuations.

Write $\mathcal{V}_1(X)$ for subspace of probability valuations.

Obviously $\mathcal{V}_1(X) \subseteq \mathcal{V}_{<\infty}(X) \subseteq \mathcal{V}(X)$.

Proposition (Heckmann 1996) The spaces $\mathcal{V}(X)$ and $\mathcal{V}_1(X)$ are sober.
Why these topologies?

The topology on $\mathcal{V}(X)$ agrees with the Scott topology when $X$ is a continuous dcpo.

The topology on $\mathcal{V}_1(X)$ agrees with the vague topology on Radon probability measures when $X$ is a (locally?) compact Hausdorff space.

Since the topologies are the coarsest such that $\nu \mapsto \int f \, d\nu$ is continuous for every lower semicontinuous $f$, they are weak topologies. They are also weak* topologies (see later).

The weak topology on $\mathcal{V}_1(X)$ is (for countably based spaces) the correct topology on the probabilistic powerdomain in a general topological framework for the semantics of computation: topological domain theory (cf. Battenfeld, Schröder and S.)

They are given by universal properties (Schröder and S., MFPS 2005).
Simple valuations

For \( x \in X \), define the Dirac valuation \( \delta_x \in \mathcal{V}_1(X) \) by

\[
\delta_x(U) = \begin{cases} 
1 & \text{if } x \in U \\
0 & \text{otherwise}
\end{cases}
\]

The function \( \delta : X \to \mathcal{V}_1(X) \) is continuous.

A simple valuation is one of the form:

\[
\sum_{i=1}^{n} r_i \cdot \delta_{x_i}
\]

for some \( n \geq 0 \), \( r_1, \ldots, r_n \in (0, \infty) \) and \( x_1, \ldots, x_n \in X \).

**Proposition**  The simple valuations are dense in \( \mathcal{V}(X) \).
Integration as an extension of $f$

For any lower semicontinuous $f : X \to [0, \infty]$, the function

$$\nu \mapsto \int f \, d\nu : \mathcal{V}(X) \to [0, \infty]$$

enjoys the following properties:

1. It is lower semicontinuous.
2. It is linear.
3. It extends $f$, i.e.

$$\int f \, d\delta_x = f(x) .$$
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Problem 1 (Heckmann 1996) Do the three properties above characterise the function $\nu \mapsto \int f \, d\nu$?
Theorem (Schröder and S.) For any topological space $X$, and lower semicontinuous $f : X \to [0, \infty]$:

1. If $h : \mathcal{V}_{<\infty}(X) \to [0, \infty]$ is lower semicontinuous, linear and extends $f$ then $h(\nu) = \int f \, d\nu$, for all $\nu \in \mathcal{V}_{<\infty}(X)$.

2. If $h : \mathcal{V}(X) \to [0, \infty]$ is lower semicontinuous, linear and extends $f$ then $h(\nu) = \int f \, d\nu$, for all $\nu \in \mathcal{V}(X)$.

3. If $g : \mathcal{V}_1(X) \to [0, \infty]$ is lower semicontinuous, affine and extends $f$ then $g(\nu) = \int f \, d\nu$, for all $\nu \in \mathcal{V}_1(X)$.

Statements 2 answers Heckmann’s question affirmatively. This and statement 3 were announced at MFPS 2005 (also in Birmingham).

Goal of this talk is to present our proof of the theorem. First, however, consider one consequence of the theorem for positive functional analysis.

Remark Cohen, Escardó and Keimel have obtained a different proof of statements 1 and 2 for the special case of stably compact $X$ (TAMC 2006).
Some positive functional analysis

Given a topological cone $A$, write $A^*$ for the dual cone

$$\{ f : A \to [0, \infty] \mid f \text{ lower semicontinuous and linear} \}$$

topologized with the weak* topology whose subbasic opens are:

$$\{ f \mid f(a) > r \} ,$$
genenerated by $0 \leq r < \infty$ and $a \in A$.

By a known theorem (Kirch, Tix, Heckmann) (analogous to the Riesz representation theorem of functional analysis):

$$\mathcal{V}(X) \cong \mathcal{L}(X)^* ,$$

By our Theorem (2), we have dually

$$\mathcal{L}(X) \cong \mathcal{V}(X)^* .$$
Call a topological cone reflexive if the natural linear map $A \to A^{**}$ is a homeomorphism (the inverse is automatically linear).

By the previous slide, $\mathcal{V}(X)$ and $\mathcal{L}(X)$ are reflexive.

Open question Characterise the reflexive topological cones.

The remainder of the talk will be the proof of the theorem.
Integration as a maximum extension

The first step is to obtain an alternative characterisation of integration. Henceforth fix a topological space $X$ and lower semicontinuous $f: X \to [0, \infty]$.

For any simple valuation $q \in \mathcal{V}(X)$, define $f^s(q) \in [0, \infty]$ by:

$$f^s\left(\sum_{i=1}^{n} r_i \cdot \delta_{x_i}\right) = \sum_{i=1}^{n} r_i \cdot f(x_i).$$

$f^s$ is extended to a lower semicontinuous function $f^\#: \mathcal{V}(X) \to [0, \infty]$ by

$$f^\#(\nu) = \sup_{O \supseteq \nu} \inf_{q \in O} f^s(q),$$

where $O \subseteq \mathcal{V}(X)$ ranges over open sets, and $q$ ranges over simple valuations.
Proposition (ad Scott) If \( h: \mathcal{V}(X) \rightarrow [0, \infty] \) is a lower semicontinuous function extending \( f^s \), then \( h \leq f^\# \). (Similarly, for every lower semicontinuous \( h: \mathcal{V}_{<\infty}(X) \rightarrow [0, \infty] \) extending \( f^s \), we have \( h \leq f^\# \).)

Since \( \nu \mapsto \int f \, d\nu \) is a lower semicontinuous function extending \( f^s \), we have, in particular, that \( \int f \, d\nu \leq f^\#(\nu) \).

The first main proof effort is to prove the converse inequality:

**Proposition 1** \( f^\#(\nu) \leq \int f \, d\nu \), for all \( \nu \in \mathcal{V}(X) \).

**Corollary** \( \int f \, d\nu = f^\#(\nu) \), for all \( \nu \in \mathcal{V}(X) \).

Thus integration is determined as the greatest lower semicontinuous extension of its (easily defined) restriction to simple valuations. This result is perhaps of interest in its own right.
Proof of Proposition 1

Recall, we have lower semicontinuous $f : X \to [0, \infty]$.

**Lemma** $f^\#(\nu) \leq \int f \, d\nu$, for all $\nu \in \mathcal{V}_{<\infty}(X)$.

Note the restriction to finite valuations.

**Proof** Consider any $\nu \in \mathcal{V}_{<\infty}(X)$ and $0 < t < 1$. We show that $t^3 \cdot f^\#(\nu) \leq \int f \, d\nu$.

By def. of $f^\#$, there exists a basic open $O \ni \nu$, 

$$O := \langle U_1 > r_1 \rangle \cap \cdots \cap \langle U_n > r_n \rangle,$$

such that, for all simple $q \in O$, $f^s(q) > t \cdot f^\#(\nu)$.

W.l.o.g., $U_1, \ldots, U_n$ is closed under $\nu$-positive intersections (i.e., if $U$ is a finite intersection of some of $U_1, \ldots, U_n$, and $\nu(U) > 0$ then $U = U_i$ for some $1 \leq i \leq n$), and each $U_i$ is enumerated exactly once.
If \( f \) is constantly \( \infty \) on \( U_i \), for some \( 1 \leq i \leq n \), then it is easy to show that \( \int f \, d\nu = \infty \), hence required inequality holds.

Otherwise, suppose, for each \( 1 \leq i \leq n \), there exists \( c_i \in U_i \) with \( f(c_i) < \infty \). We choose these so \( U_i \subseteq U_j \) implies \( f(c_i) \geq f(c_j) \).

Define \( i \prec j \) if \( U_i \subset U_j \) (strict containment). For each \( i \) define:

\[
C_i := U_i \setminus \bigcup_{j \prec i} U_j.
\]

W.l.o.g., let \( C_1, \ldots, C_l \) (where \( 0 \leq l \leq n \)) be the nonempty \( C_i \). Then \( C_1, \ldots, C_l \) form a pairwise disjoint collection of crescent sets, and for any finite union \( U \) of sets from \( U_1, \ldots, U_n \) we have \( U = \bigcup \{C_i \mid C_i \subseteq U\} \). Also

\[
\nu(U) = \sum \{\nu(C_i) \mid C_i \subseteq U\},
\]

where \( \nu(C_i) := \nu(U_i) - \nu(\bigcup_{j \prec i} U_j) \).
By lower semicontinuity of $f$, for any $x \in C_i$, we have open $U_x \subseteq U_i$ with $x \in U_x$ such that $f(y) > t. \min(f(c_i), f(x))$, for all $y \in U_x$.

By Scott continuity of $\nu$, we can find finite $X_i \subseteq C_i$ satisfying $c_i \in X_i$, and, when $\nu(C_i) > 0$, $\nu(\bigcup_{x \in X_i} U_x) - \nu(\bigcup_{j \prec i} U_j) > t.\nu(C_i)$.

For $1 \leq i \leq l$, define $U'_i := \bigcup\{U_x \mid C_j \subseteq U_i \text{ and } x \in X_j\}$.

Choose $m_i \in \{x \mid C_j \subseteq U_i \text{ and } x \in X_j\} \cup \{c_i\}$ such that $f(x)$ is minimal.

Define two step functions

$$s(x) := \sup_{U_i \ni x} f(m_i)$$

$$s'(x) := \sup_{U_i \ni x} t.f(m_i).\chi_{U'_i}(x).$$

Obviously $s' \leq s$. It is easily shown that also $s' \leq f$.

By construction of $s$, $s'$ one calculates that $\int s' \, d\nu \geq t^2 \int s \, d\nu$. 
Define the simple valuation:

\[ q := \sum_{i=1}^{l} \nu(C_i) \cdot \delta_{m_i}. \]

It is easily seen that \( q \in O \).

By construction of \( s \) and \( q \), we have \( \int s \, d\nu = \int f \, dq \). Then:

\[
\int f \, d\nu \geq \int s' \, d\nu \quad (s' \leq f)
\]

\[
\geq t^2 \int s \, d\nu \quad \text{(previous slide)}
\]

\[
= t^2 \int f \, dq \quad \text{(as above)}
\]

\[
= t^2 f^s(q) \quad (q \text{ simple})
\]

\[
\geq t^3 \cdot f^\#(\nu) \quad (\nu \in O) .
\]

This completes the proof of the lemma.
Proposition (Heckmann 1996). For any $\nu \in \mathcal{V}(X)$ there exists directed $\mathcal{F}_\nu \subseteq \mathcal{V}_{<\infty}(X)$ with $\nu = \sup_{\mu \in \mathcal{F}_\nu} \mu$.

Proposition 1 $f^\#(\nu) \leq \int f \, d\nu$, for all $\nu \in \mathcal{V}(X)$.

Proof

\[
f^\#(\nu) = f^\#(\sup_{\mu \in \mathcal{F}_\nu} \mu)
= \sup_{\mu \in \mathcal{F}_\nu} f^\#(\mu) \quad \text{(continuity of $f^\#$)}
\leq \sup_{\mu \in \mathcal{F}_\nu} \int f \, d\mu \quad \text{(by lemma)}
= \int f \, d\nu \quad \text{(continuity of $\int f$)}
\]

This completes the characterisation of integration as a maximum extension.
Proof of main theorem: Statement 1

Suppose $h: \mathcal{V}_{<\infty}(X) \to [0, \infty]$ is a lower semicontinuous linear function extending $f$ to finite valuations. We must show that $h(\nu) = \int f \, d\nu$, for all $\nu \in \mathcal{V}_{<\infty}(X)$.

Clearly $h$ extends $f^s$ to $\mathcal{V}_{<\infty}(X)$. Therefore, by Scott’s Proposition on slide 19, $h(\nu) \leq f^\#(\nu)$, and thus, by Proposition 1, $h(\nu) \leq \int f \, d\nu$, for all $\nu \in \mathcal{V}_{<\infty}(X)$.

It remains to prove:

**Lemma**  \( \int f \, d\nu \leq h(\nu) \), for all $\nu \in \mathcal{V}_{<\infty}(X)$. 
Proof of lemma  Consider any $\nu \in \mathcal{V}_{<\infty}(\nu)$.

Since $\int f \ d\nu = \sup_{s \leq f} \int s \ d\nu$, where $s$ ranges over step functions, it suffices to prove that $\int s \ d\nu \leq h(\nu)$, for all step functions $s \leq f$.

Consider any step function $s \leq f$,

$$s := \sum_{i=1}^{n} d_i \chi_{U_i},$$

where, w.l.o.g.,

$$X = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_n \supseteq U_{n+1} = \emptyset.$$ 

Take any $t$ with $0 < t < 1$. We shall prove:

$$h(\nu) \geq t. \int s \ d\nu. \quad (A)$$
Define

$$\mathcal{Z} := \{ U \in \mathcal{O}(X) \mid \exists i. U_i \supseteq U \supseteq U_{i+1} \text{ and } h(\nu[U]) \geq t. \int s[U] \, d\nu \}.$$ 

Obviously $\emptyset \in \mathcal{Z}$.

Suppose $\mathcal{U} \subseteq \mathcal{Z}$ is a chain (under inclusion). We show that: $\bigcup \mathcal{U} \in \mathcal{Z}$.

— The required $i$ is $\min \{ i \mid U \in \mathcal{U} \text{ and } U_i \supseteq U \supseteq U_{i+1} \}$.

— For the inequality:

$$h(\nu[\bigcup \mathcal{U}]) = h(\bigcup_{U \in \mathcal{U}} \nu[U])$$

$$= \sup_{U \in \mathcal{U}} h(\nu[U]) \quad \text{(continuity of } h)$$

$$\geq \sup_{U \in \mathcal{U}} t. \int s[U] \, d\nu \quad \text{(def. } \mathcal{Z})$$

$$= t. \int s[\bigcup \mathcal{U}] \, d\nu \quad \text{(Scott continuity of } \int \text{ in function arg)}$$
Since $\mathcal{Z}$ is inhabited and chain complete it has, by Zorn’s Lemma, a maximal element $U'$.

We show that $U' = X$, and this establishes (A) since $X \in \mathcal{Z}$.

Suppose $U' \neq X$. Then $\exists! i$ with $U_i \supseteq U' \supseteq U_{i+1}$ and $x \in U_i \setminus U'$.

By assumption, $h(\delta_x) = f(x) \geq s(x)$. Thus, by continuity of $h$, there exists a basic open neighbourhood $O \ni \delta_x$,

$$O := \langle V_1 > r_1 \rangle \cap \cdots \cap \langle V_k > r_k \rangle,$$

such that $h(\mu) \geq t.s(x)$, for all $\mu \in O$.

Since $\delta_x \in O$, we have $x \in U_x := V_1 \cap \cdots \cap V_k \cap U_i$. Define $r_x := \max\{r_1, \ldots, r_k\}$. Since $\delta_x \in O$, we have $r_x < 1$.

Then $\delta_x \in \langle U_x > r_x \rangle \subseteq O$. Hence $h(\mu) \geq t.s(x)$, for all $\mu \in \langle U_x > r_x \rangle$. (B)
Define $\mu := \nu_{\Delta U'} (= \nu - \nu_{U'}).$

Claim: $h(\mu_{U_x}) \geq t.s(x).\mu(U_x)$ \hspace{1cm} (C)

Immediate if $\mu(U_x) = 0$. Otherwise $0 < \mu(U_x) < \infty$, so:

$$\frac{1}{\mu(U_x)}.\mu_{U_x}(U_x) = 1 > r_x .$$

Thus $\frac{1}{\mu(U_x)}.\mu_{U_x} \in \langle U_x > r_x \rangle$.

So $h(\frac{1}{\mu(U_x)}.\mu_{U_x}) \geq t.s(x)$, by (B).

I.e., $h(\mu_{U_x}) \geq t.s(x).\mu(U_x)$, by linearity (actually homogeneity) of $h$.

Thus claim holds.
Claim: \( h(\nu_{U' \cup U_x}) \geq t. \int s_{U' \cup U_x} \, d\nu \). \hfill (D)

Indeed:

\[
\begin{align*}
    h(\nu_{U' \cup U_x}) &= h(\nu_{U'} + \mu_{U_x}) \\
                        &= h(\nu_{U'}) + h(\mu_{U_x}) \\
                        &\geq (t. \int s_{U'} \, d\nu) + t.s(x).\mu(U_x) \\
                        &= t.\left[ (d_1 + \cdots + d_i).\nu(U') + d_{i+1}.\nu(U_{i+1}) + \cdots + d_n.\nu(U_n) \\
                          &\quad + (d_1 + \cdots + d_i).\nu(U_x) - \nu(U' \cap U_x) \right] \\
                        &= t.\left[ (d_1 + \cdots + d_i).\nu(U' \cup U_x) + d_{i+1}.\nu(U_{i+1}) + \cdots + d_n.\nu(U_n) \right] \\
                        &= t. \int s_{U' \cup U_x} \, d\nu, \tag{6}
\end{align*}
\]

where: (1) is because \( \mu = \nu_{X \setminus U'} \); (2) is by linearity of \( h \); (3) is because \( U' \in \mathcal{Z} \) and by (C); (4) is because \( s(x) = d_1 + \cdots + d_i \); (5) is by modularity of \( \nu \); (6) is because \( U_i \supseteq U' \supseteq U_{i+1} \) and \( U_x \subseteq U_i \).
We have: \( h(\nu_{U' \cup U_x}) \geq t \int s_{U' \cup U_x} \, d\nu. \) (D)

Recall:

\[ Z = \{ U \in \mathcal{O}(X) \mid \exists i. U_i \supseteq U \supseteq U_{i+1} \text{ and } h(\nu_{U}) \geq t. \int s_{U} \, d\nu \}. \]

Thus, since \( U_i \supseteq U' \cup U_x \supseteq U_{i+1} \), and (D) holds, we have \( U' \cup U_x \in Z \).

But \( x \in U' \cup U_x \setminus U' \), which contradicts the maximality of \( U' \).

This completes the proof of the lemma. \( \square \)

Proof of main theorem: Statement 2

Suppose \( h : \mathcal{V}(X) \to [0, \infty] \) is a lower semicontinuous linear function extending \( f \) to all valuations. We must show that \( h(\nu) = \int f \, d\nu \), for all \( \nu \in \mathcal{V}(X) \).
Recall from slide 24:

Proposition (Heckmann 1996) For any $\nu \in \mathcal{V}(X)$ there exists directed $\mathcal{F}_\nu \subseteq \mathcal{V}_{<\infty}(X)$ with $\nu = \sup_{\mu \in \mathcal{F}_\nu} \mu$.

Proof of Statement 2

\[
h(\nu) = h\left( \sup_{\mu \in \mathcal{F}_\nu} \mu \right) \\
= \sup_{\mu \in \mathcal{F}_\nu} h(\mu) \quad \text{(continuity of } h) \\
= \sup_{\mu \in \mathcal{F}_\nu} \int f \, d\mu \quad \text{(Statement 1)} \\
= \int f \, d\nu \quad \text{(continuity of } \int f \text{)}
\]

$\square$
Proof of main theorem: Statement 3

Suppose \( g : \mathcal{V}_1(X) \to [0, \infty] \) is a lower semicontinuous affine function extending \( f \) to all valuations. We must show that \( g(\nu) = \int f \, d\nu \), for all \( \nu \in \mathcal{V}_1(X) \).

Recall, an affine function is one that preserves convex combinations:

\[
\sum_{i=1}^{n} r_i \cdot x_i ,
\]

where \( \sum_{i=1}^{n} r_i = 1 \).
Define \( h: \mathcal{V}_{<\infty}(X) \to [0, \infty] \) by:

\[
h(\nu) = \begin{cases} 
|\mu| \cdot g\left(\frac{\mu}{|\mu|}\right) & |\mu| > 0, \text{ where } |\mu| := \mu(X) \\
0 & |\mu| = 0
\end{cases}
\]

Clearly \( h(\nu) = g(\nu) \), for all \( \nu \in \mathcal{V}_1(X) \).

That \( h \) is linear follows from \( g \) being affine, by an easy calculation.

We prove in the sequel that \( h \) is lower semicontinuous.

Given this, for all \( \nu \in \mathcal{V}_1(X) \), we have:

\[
g(\nu) = h(\nu) = \int f \, d\nu \quad \text{(by Statement 1)}
\]

which is what we are required to prove.
It remains to prove:

**Lemma** \( h: \mathcal{V}_{<\infty} \to [0, \infty] \) is lower semicontinuous.

**Proof** It is enough to show that \( h \) is lower semicontinuous at \( \mu \) when \( |\mu| = 1 \).

Consider any \( s < h(\mu) = g(\mu) \). We shall find open \( O' \ni \mu \) such that \( h(\nu) > s \), for all \( \nu \in \mathcal{V}_{<\infty} \cap O' \).

Consider any \( t \) with \( s < t < g(\mu) \). By lower semicontinuity of \( g \), we have \( \mu \in \mathcal{V}_1(X) \cap O \),

\[
O := \langle U_1 > r_1 \rangle \cap \cdots \cap \langle U_k > r_k \rangle,
\]

such that \( g(\nu) > t \), for all \( \nu \in \mathcal{V}_1(X) \cap O \). \hspace{1cm} (A)

W.l.o.g., assume that \( U_1, \ldots, U_k \) are closed under finite unions and \( \mu \)-positive finite intersections.
There exists $r$ with $\frac{s}{t} < r < 1$ such that

$$r.\mu(U_1) > r_1 \text{ and } \ldots \text{ and } r.\mu(U_k) > r_k.$$ 

So

$$\mu \in O' := \langle U_1 > r.\mu(U_1) \rangle \cap \cdots \cap \langle U_k > r.\mu(U_k) \rangle \subseteq O.$$

Consider any $\nu \in V_{<\infty} \cap O'$. We must show that $h(\nu) > s$.

We have $r.\mu(U) \leq \nu(U)$ for any $U$ obtained as an intersection of sets from $U_1, \ldots, U_k$ (since such sets are either in $U_1, \ldots, U_k$, or satisfy $\mu(U) = 0$). Moreover, such sets are closed under finite unions and intersections.

By a decomposition lemma (see below), it follows that $\nu = \nu_1 + \nu_2$, where $\nu_1(X) = r.\mu(X) = r$ and $\nu_1(U_i) \geq r.\mu(U_i)$, for all $1 \leq i \leq k$.

Thus $\frac{1}{r}.\nu_1$ is a probability valuation and $\frac{1}{r}.\nu_1 \in \langle U_1 > r_1 \rangle \cap \cdots \cap \langle U_k > r_k \rangle = O$. 
We have:

\[ h(\nu) = h(\nu_1 + \nu_2) \]
\[ = h(\nu_1) + h(\nu_2) \quad \text{(} h \text{ linear}) \]
\[ = r.h\left(\frac{1}{r}.\nu_1\right) + h(\nu_2) \quad \text{(} h \text{ homogeneous}) \]
\[ = r.g\left(\frac{1}{r}.\nu_1\right) + h(\nu_2) \quad \left( |\frac{1}{r}.\nu_1| = 1 \right) \]
\[ > r.t + h(\nu_2) \quad \left( \frac{1}{r}.\nu_1 \in O \right) \]
\[ > s \quad \left( \frac{s}{t} < r \right). \]

This completes the proof of the lemma, modulo the promised decomposition lemma. □
A decomposition lemma

Lemma (decomposition)  Suppose $\mu, \nu \in \mathcal{V}<\infty(X)$. Suppose $U_1, \ldots, U_k \subseteq X$ are open sets closed under finite union and intersection (so $\emptyset, X$ are included) and satisfy $\mu(U_i) \leq \nu(U_i)$, for all $1 \leq i \leq k$. Then we can find a decomposition $\nu = \nu_1 + \nu_2$ satisfying:

\[
\nu_1(X) = \mu(X)
\]
\[
\nu_1(U_i) \geq \mu(U_i), \text{ for all } 1 \leq i \leq k .
\]

This lemma has a (quite involved) combinatorial proof using a version of the splitting lemma (Jones 1990).