

# The Locale of Random Sequences

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**Random sequences** are infinite binary sequences generated by tossing a fair coin *ad infinitum*

A random sequence  $\alpha \in \mathbf{2}^\omega$  (here  $\mathbf{2} := \{0, 1\}$  with, say, 0 for heads and 1 for tails) exhibits:

- Local irregularity: e.g., cannot predict  $\alpha_n$  from  $\alpha_0 \dots \alpha_{n-1}$
- Global regularity: e.g., asymptotic satisfaction of probabilistic laws such as **law of large numbers**

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \alpha_i = \frac{1}{2}$$

Old question, cf. von Mises (1919): What are the characteristic mathematical properties of random sequences?

## Observable properties

At any time, we only see a finite prefix of a random sequence, and this might be *any* finite sequence of digits.

A subset  $U \subseteq \mathbf{2}^\omega$  is **open** (in the Cantor/product topology) if it satisfies:

$$\alpha \in U \implies \exists n. \forall \beta \in \mathbf{2}^\omega. (\beta \upharpoonright_n = \alpha \upharpoonright_n \implies \beta \in U)$$

Open subsets correspond to **observable properties**, cf. Smyth, Abramsky.

For example,  $\{\alpha \mid \alpha \neq 0^\omega\}$  is observable. But  $\{0^\omega\}$  is not observable.

## Empirical properties of randomness

**An empirical fact:** If  $\alpha$  is a random sequence, then we will sooner or later observe that  $\alpha \neq 0^\omega$ , i.e., that  $\alpha \in \{\beta \in \mathbf{2}^\omega \mid \beta \neq 0^\omega\}$ .

Note that, for any  $n$ , the probability that  $\alpha \upharpoonright_n = 0^n$  is  $2^{-n}$ . Thus the probability that the prefix  $\alpha \upharpoonright_n$  fails to show that  $\alpha \neq 0^\omega$  is  $2^{-n}$ .

We soon have to have been *very* unlucky not to have observed that  $\alpha \neq 0^\omega$ .

The empirical fact is explained by a general principle that the observer is not infinitely unlucky.

We turn this principle into a (first) postulate of randomness.

## Postulates of randomness

We postulate properties of the desired set  $\mathbf{R} \subseteq \mathbf{2}^\omega$  of random sequences. Let  $\lambda$  be the uniform/Lebesgue measure on  $\mathbf{2}^\omega$ .

### First postulate of randomness

For every open  $U \subseteq \mathbf{2}^\omega$ , if  $\lambda(U) = 1$  then  $\mathbf{R} \subseteq U$ .

Informally: if  $\alpha$  is a random sequence, and  $U$  is an almost sure observable property then we will (eventually) observe that  $\alpha \in U$ .

### Second postulate of randomness

For every open  $U \subseteq \mathbf{2}^\omega$ , if  $\mathbf{R} \subseteq U$  then  $\lambda(U) = 1$ .

This is a nontriviality postulate. Informally: if  $U$  is an observable property with  $\lambda(U) < 1$  then there has to be a random sequence outside  $U$ .

## Inconsistency

The two postulates of randomness are inconsistent.

For any sequence  $\beta$ , the open set  $\{\alpha \mid \alpha \neq \beta\}$  has measure 1.

Therefore, by the first postulate of randomness:

$$\mathbf{R} \subseteq \{\alpha \mid \alpha \neq \beta\} ,$$

i.e.,  $\beta \notin \mathbf{R}$ .

Thus  $\mathbf{R} = \emptyset$ , but this contradicts the second postulate of randomness.

The postulates, as formulated, seem too naïve.

## Algorithmic and logical notions of randomness

Consistency is achieved by defining:

$$\mathbf{R}_{\mathcal{F}} := \bigcap \mathcal{F} ,$$

where  $\mathcal{F}$  is a **countable** family of measure 1 subsets of  $2^\omega$ .

Because  $\mathcal{F}$  is countable, we have  $\lambda(\mathbf{R}_{\mathcal{F}}) = 1$ .

Thus the first postulate of randomness is weakened, but the second holds as stated.

Typically the family  $\mathcal{F}$  is given using either recursion theoretic or logical notions of definability.

## Example notions of algorithmically random sequence

### Kurtz randomness (1981)

$$\mathbf{R}_K := \bigcap \{U \subseteq \mathbf{2}^\omega \mid U \text{ effective open, } \lambda(U) = 1\}$$

Motivation: given an effective observable property of measure 1, any random sequence will eventually be observed to satisfy the property. This is an effective version of our first postulate of randomness.

N.B., effective open subsets  $U \subseteq \mathbf{2}^\omega$  are characterised by the existence of a Type-2 Turing Machine that halts on  $\alpha$  iff  $\alpha \in U$ . Thus, effective opens are observable in the strong sense that there exists a Type-2 Turing Machine that will perform the observation.

## Martin-Löf randomness (1966)

A subset  $X \subseteq 2^\omega$  is **effectively null** if there is a recursive sequence  $(U_i)$  of effective open sets with  $X \subseteq U_i$  and  $\lambda(U_i) < 2^{-i}$ .

$$\overline{\mathbf{R}_{\text{ML}}} := \bigcup \{X \subseteq 2^\omega \mid X \text{ effectively null}\}$$

Motivation: For every  $\alpha \notin \mathbf{R}_{\text{ML}}$  and confidence level  $\epsilon > 0$ , there exists (recursively in  $\epsilon$ ) an effective open  $U_\epsilon \ni \alpha$  with measure  $< \epsilon$ . Thus the observation  $\alpha \in U_\epsilon$  gives evidence that  $\alpha$  is non-random with probability  $> 1 - \epsilon$ .

**Slogan:** Non-randomness can be statistically detected.

N.B., In this case,  $\mathbf{R}_{\text{ML}} = \bigcap \mathcal{F}$  where  $\mathcal{F}$  is a countable family of  $F_\sigma$  sets, not just of open sets.

Algorithmic (and logical) randomness more generally

**Fact**  $\mathbf{R}_{ML} \subset \mathbf{R}_K$  (proper inclusion)

Other related (but different) notions have been proposed by Martin-Löf (again), Solovay, Schnorr, . . . .

Algorithmic randomness is an important area of recursion theory, but does it properly model the empirical phenomenon of randomness?

## Criticisms

- No one canonical notion of randomness
- While (arguably) the recursion-theoretic restrictions reflect our observational limitations, there is no reason to believe any recursion-theoretic dependencies to be inherent in the empirical phenomenon of randomness itself

## Returning to the naïve viewpoint

Our two postulates of randomness are couched in terms of observable properties (open sets).

Thus it is natural to define randomness in a setting in which observable properties play a fundamental role.

Further, we never experience a completed infinite random sequence in its entirety — only its finite prefixes.

Thus it is natural to define randomness in a setting in which completed entities (points) are not the basic ingredient.

All this points to using **locale theory**.

## Locales

A **locale**  $X$  is given by a **frame**  $\mathcal{O}(X)$ , that is,  $\mathcal{O}(X)$  is a partially-ordered set with:

- arbitrary joins  $\bigcup$  (including the empty join  $\emptyset$ ),
- (hence) finite meets  $\cap$  (including the empty meet  $X$ ),
- satisfying the distributive law:

$$U \cap \left( \bigcup_{i \in I} V_i \right) = \bigcup_{i \in I} U \cap V_i .$$

Motivating example:  $\mathcal{O}(X)$  is the lattice of opens of a topological space, joins are unions and finite meets are intersections.

## Locales versus spaces

Locale theory abstracts away from topological spaces. The frame  $\mathcal{O}(X)$  does not need to arise as a family of sets under union and intersection. Indeed there need not be any underlying set of points.

The generality can be motivated by considering frames as theories in a natural **logic of observable properties**. (Reference: Vickers' textbook: "Topology via Logic".)

Topological spaces give rise to locales with special properties, the **spatial** locales.

There is an exact agreement between spatial locales and **sober** spaces. (Sobriety is a topological property lying between  $T_0$  and  $T_2$ , but incomparable with  $T_1$ .)

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## Maps of locales

A (continuous) map  $f: X \rightarrow Y$ , between locales  $X, Y$ , is given by a function  $f^{-1}: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$  that preserves arbitrary joins and finite meets.

**Motivation:** if  $f: X \rightarrow Y$  is a continuous function between topological spaces then  $f^{-1}$  preserves arbitrary unions and finite intersections of opens.

Moreover, if  $X, Y$  are sober spaces then any function from  $\mathcal{O}(Y)$  to  $\mathcal{O}(X)$ , that preserves arbitrary unions and finite intersections of opens, arises as  $f^{-1}$  for a unique continuous function  $f$ .

## Sublocales

A **map**  $f: X \rightarrow Y$ , between locales  $X, Y$ , is said to be an **embedding** if the function  $f^{-1}: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$  is surjective.

**Motivation:** If  $X, Y$  are topological spaces and  $f: X \rightarrow Y$  is a topological embedding then  $f^{-1}$  is surjective.

Moreover, if  $X, Y$  are sober spaces and  $f^{-1}$  is surjective then  $f: X \rightarrow Y$  is a topological embedding.

The embeddings are exactly the regular monos in the category of locales.

The embeddings determine the notion of **sublocale**.

## The two postulates interpreted for locales

$\mathbf{2}^\omega$  is a topological space, hence a locale. The open subsets  $U \subseteq \mathbf{2}^\omega$  are the elements of  $\mathcal{O}(\mathbf{2}^\omega)$ , they determine the **open sublocales** of  $\mathbf{2}^\omega$ .

We interpret the postulates as expressing properties of a desired sublocale  $\mathbf{R} \subseteq \mathbf{2}^\omega$ .

1. For every open  $U \subseteq \mathbf{2}^\omega$ , if  $\lambda(U) = 1$  then  $\mathbf{R} \subseteq U$  (i.e.,  $\mathbf{R}$  is a sublocale of  $U$ ).
2. For every open  $U \subseteq \mathbf{2}^\omega$ , if  $\mathbf{R} \subseteq U$  then  $\lambda(U) = 1$ .

**Theorem** *The above properties determine a unique sublocale  $\mathbf{R} \subseteq \mathbf{2}^\omega$ . By property 2, it has outer measure 1. Indeed,  $\mathbf{R}$  is characterised as the smallest sublocale of  $\mathbf{2}^\omega$  of outer measure 1.*

## Concrete description of $\mathbf{R}$

For  $U, V \subseteq \mathbf{2}^\omega$  open, define:

$$U \approx V \quad :\Leftrightarrow \quad \lambda(U) = \lambda(U \cap V) = \lambda(V)$$

If  $U \approx V$  then intuitively they represent the same observation on random sequences.

Define  $\mathcal{O}(\mathbf{R}) := \mathcal{O}(\mathbf{2}^\omega) / \approx$ .

(Cf. the measure ( $\sigma$ -)algebra of  $\mathbf{2}^\omega$ , equivalently of  $[0, 1]$ .)

**Proposition** *The above defines  $\mathbf{R}$  as a sublocale of  $\mathbf{2}^\omega$ . This is the sublocale characterised by the previous theorem.*

## What are the random sequences in $\mathbf{R}$ ?

The locale  $\mathbf{R}$  has no points, that is there are no maps from the terminal locale to  $\mathbf{R}$ .

Thus, according to this theory, there is no such thing as a completed infinite random sequence.

This is in keeping with the set-theoretical inconsistency of the postulates of randomness.

Nonetheless  $\mathbf{R}$  is a nontrivial “space” of *all* random sequences.

## What are the maps from $\mathbf{R}$ to $\mathbf{R}$ ?

Intuitively, these correspond to continuous transformations on random sequences.

**Theorem** *The maps  $\mathbf{R} \rightarrow \mathbf{R}$  are in one-to-one correspondence with the continuous nonsingular maps from  $\mathbf{2}^\omega$  to  $\mathbf{2}^\omega$  modulo almost everywhere equivalence.*

A **continuous nonsingular map** from  $\mathbf{2}^\omega$  to  $\mathbf{2}^\omega$  is a continuous function from a measure 1 subspace  $D \subseteq \mathbf{2}^\omega$  to  $\mathbf{2}^\omega$ , satisfying: for every null  $Z \subseteq \mathbf{2}^\omega$ , it holds that  $f^{-1}(Z)$  is null.

## More generally

If we generalise  $\lambda$  to other probability “measures” (technically, probability valuations), and  $\mathbf{2}^\omega$  to other locales, we still obtain canonical “random” sublocales.

A (continuous) probability valuation on a locale  $X$  is a (necessarily monotone) function  $\mu: \mathcal{O}(X) \rightarrow [0, 1]$  satisfying:

$$\begin{aligned} \mu(\emptyset) &= 0 & \mu(U \cup V) &= \mu(U) + \mu(V) - \mu(U \cap V) \\ \mu(X) &= 1 & \mu\left(\bigcup_{U \in \mathcal{D}}^{\uparrow} U\right) &= \sup_{U \in \mathcal{D}} \mu(U) \quad (\mathcal{D} \subseteq \mathcal{O}(X) \text{ directed}) \end{aligned}$$

Given  $(X, \mu)$  define:

$$\begin{aligned} U \approx_{\mu} V &:\Leftrightarrow \mu(U \cup V) = \mu(U \cap V) \\ \mathcal{O}(\mathbf{R}(\mu)) &:= \mathcal{O}(X) / \approx_{\mu} \end{aligned}$$

**Proposition**  $\mathbf{R}(\mu)$  is a sublocale of  $X$ .

Define the **outer value** of a sublocale  $Y \subseteq X$  by:

$$\mu^*(Y) := \inf\{\mu(U) \mid U \in \mathcal{O}(X) \text{ and } Y \subseteq U\}$$

**Theorem** If the locale  $X$  is regular then:

1.  $\mathbf{R}(\mu)$  is the meet of all open sublocales  $U \subseteq X$  with  $\mu(U) = 1$
2.  $\mathbf{R}(\mu)$  is the meet of all sublocales  $Y \subseteq X$  with  $\mu^*(Y) = 1$
3.  $\mu^*(\mathbf{R}(\mu)) = 1$

Obviously,  $\mu$  is also a probability valuation on  $\mathbf{R}(\mu)$ . Then  $(\mathbf{R}(\mu), \mu)$  is **random** in the sense that:

$$[U] \subset [V] \in \mathcal{O}(\mathbf{R}(\mu)) \text{ (proper inclusion)} \Rightarrow \mu([U]) < \mu([V])$$

When does one measure subsume another?

For  $\mathbf{2}^\omega$ , valuations are in one-one correspondence with (Borel) measures.

**Theorem** *Given two probability measures  $\mu, \nu$  on  $\mathbf{2}^\omega$ , the following are equivalent:*

- $\mathbf{R}(\mu) \subseteq \mathbf{R}(\nu)$  (as sublocales)
- $\mu \ll \nu$  (i.e.,  $\mu$  is absolutely continuous relative to  $\nu$ )

Recall,  $\mu$  is **absolutely continuous** relative to  $\nu$  if every  $\nu$ -null (Borel) set is also  $\mu$ -null.

## Another characterisation of $\mathbf{R}$

The locale of random sequences,  $\mathbf{R}$ , is:

- countably based
- zero dimensional
- and has no points

**Theorem** *If  $X$  is a countably-based zero-dimensional locale with no points and  $\mu$  is a probability valuation with  $(X, \mu)$  random then  $(X, \mu)$  and  $(\mathbf{R}, \lambda)$  are isomorphic as valuation locales. (In particular,  $X$  and  $\mathbf{R}$  are homeomorphic.)*

(Cf. the measure algebra isomorphism theorem of measure theory.)

## Further directions

- Establish characteristic properties of random sequences (invariance under von Mises' place selections, van Lambalgen's homogeneity, properties of independence)
- Drop regularity assumptions on locales
- Connections with measures and measure algebras
- Constructive version, whence computable version
- Random space?
- Philosophy of probability?

(The precise empirical meaning of the probabilistic notion of “almost surely” is unclear. In contrast, the first postulates of randomness, that observations of probability 1 happen with certainty, is clear and empirically validated.)