A Science of Reasoning

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Overview of Talk

Understanding mathematical proofs

the role of logic.

the need for higher level explanations.

Proof plans

common structure in proofs.

tactics and methods.

A science of reasoning

the nature of the science.

criteria for assessing proof plans.

Relation to computation

the role of the computer.

automatic theorem proving.

Understanding Mathematical Proofs

- Alan Robinson:

  Proof = Guarantee + Explanation

- Logic provides 'guarantee' and low-level explanation.
- Need high-level explanation too.
- Provided by proof plans.

Evidence for Higher-Level Explanations

- Understanding whole proof *vs* understanding details.
- Common structure in proofs.
- Old proofs guide search for new ones.
- Interesting *vs* routine proof steps
- Intuition of theoremhood.
- Varying learning abilities.

Common Structure in Proofs 1: Rippling

Associativity of Addition

Induction Hypothesis:

$$x + (y + z) = (x + y) + z$$

Induction Conclusion:

$$(\boxed{x + 1}^{\uparrow}) + (y + z) = ((\boxed{x + 1}^{\uparrow}) + y) + z$$

$$\boxed{(x + (y + z)) + 1}^{\uparrow} = (\boxed{(x + y) + 1}^{\uparrow}) + z$$

$$\boxed{(x + (y + z)) + 1}^{\uparrow} = \boxed{((x + y) + z) + 1}^{\uparrow}$$

$$x + (y + z) = (x + y) + z$$

Wave Rules:

$$(\boxed{U + 1}^{\uparrow}) + V \Rightarrow \boxed{(U + V) + 1}^{\uparrow}$$

$$\boxed{U + 1}^{\uparrow} = \boxed{V + 1}^{\uparrow} \Rightarrow U = V$$

Common Structure in Proofs 2: Rippling

Additivity of Even Numbers

Induction Hypothesis:

$$even(x) \wedge even(y) \rightarrow even(x + y)$$

Induction Conclusion:

$$even(\boxed{((x + 1) + 1)}^{\uparrow}) \wedge even(y) \rightarrow even((\boxed{(x + 1) + 1}^{\uparrow}) + y)$$

$$even(x) \wedge even(y) \rightarrow even(\boxed{((\boxed{x + 1}^{\uparrow}) + y) + 1}^{\uparrow})$$

$$even(x) \wedge even(y) \rightarrow even(\boxed{((x + y) + 1) + 1}^{\uparrow})$$

$$even(x) \wedge even(y) \rightarrow even(x + y)$$

Wave Rules:

$$(\boxed{U + 1}^{\uparrow}) + V \Rightarrow \boxed{(U + V) + 1}^{\uparrow}$$

$$\boxed{U + 1}^{\uparrow} = \boxed{V + 1}^{\uparrow} \Rightarrow U = V$$

$$even(\boxed{(U + 1) + 1}^{\uparrow}) \Rightarrow even(U)$$

Common Structure in Proofs 3:

Equation Solving

4. $\log_x 2 + \log_2 x = 5$

homogenization

$$\frac{4}{\log_2 x} + \log_2 x = 5$$

change of unknown

$y = \log_2 x$

isolation

$x = 2^y$

$\frac{4}{y} + y = 5$

poly norm form

$y^2 - 5.y + 4 = 0$

quadratic

$y = 1 \vee y = 4$

$$\cos x + \sin^2 x = -1$$

homogenization

$$\cos x + 1 - \cos^2 x = -1$$

change of unknown

$y = \cos x$

isolation

$x = \cos^{-1} y$
$+2.\pi.n$

$y + 1 - y^2 = -1$

poly norm form

$y^2 - y - 2 = 0$

quadratic

$y = -1 \vee y = 2$

Proof Plans: What Are They?

- Attempt to capture common structure of family of proofs.
- Used to guide search for new proofs from same family.
- Three parts: tactic, method and critics.

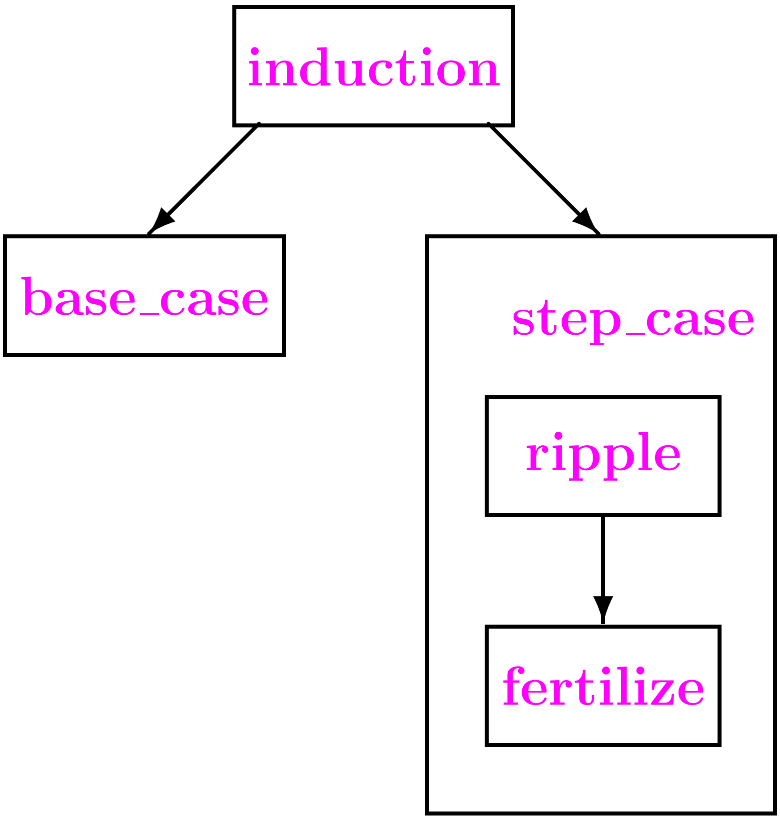
  Tactic is computer program for applying rules of inference.

  Method is meta-logical specification of tactic.

  Critic analyses failure and suggests patch.
- Use AI plan formation to construct special-purpose proof plan for conjecture using general-purpose sub-proof plans.
- Allows flexible application of heuristics.
- Understanding gained suggests extensions of heuristics.

General-Purpose Proof Plans

A Strategy for Inductive Proof:



Preconditions:

Declarative: Rippling must be possible in step cases.

Procedural: Look-ahead to choose induction rule that will permit rippling.

Special-Purpose Proof Plans

$ind\_strat(\boxed{\boldsymbol{x}} + 1^{\uparrow}, x)$

$ind\_strat(\boxed{\boldsymbol{x}} + 1^{\uparrow}, x)$ *then*

[$ind\_strat(\boxed{\boldsymbol{y}} + 1^{\uparrow}, y)$

$ind\_strat(\boxed{\boldsymbol{y}} + 1^{\uparrow}, y)$

]

Associativity of +

$$x + (y + z) = (x + y) + z$$

Commutativity of +

$$x + y = y + x$$

Critic: Lemma Speculation

- **Conjecture:**

$$even(N + N)$$

- **Wave-Rules:**

$$\boxed{s(\underline{X})}^{\uparrow} + Y \Rightarrow \boxed{s(\underline{X + Y})}^{\uparrow}$$

$$even(\boxed{s(s(\underline{X}))}^{\uparrow}) \Rightarrow even(X)$$

- **Induction Conclusion:**

$$even(\boxed{s(\underline{n})}^{\uparrow} + \boxed{s(\underline{n})}^{\uparrow})$$

$$\underbrace{even(\boxed{s(\underline{n + \boxed{s(\underline{n})}^{\uparrow}})}^{\uparrow})}_{\textbf{blocked}}$$

- **Pattern Sought:**

$$X + \boxed{s(\underline{Y})}^{\uparrow} \Rightarrow \boxed{F(\underline{X + Y})}^{\uparrow}$$

- **Lemma Discovered:**

$$X + \boxed{s(\underline{Y})}^{\uparrow} \Rightarrow \boxed{s(\underline{X + Y})}^{\uparrow}$$

Is this a Science?

- Study of the structure of proofs.

  by describing them with proof plans.

- Billions of proof plans problem.

  depends on state of mind.

- Problem common to all human sciences, *e.g.* Linguistics, Logic.

  adopt their solution.

  i.e. construct a few consensual grammars, logics, *etc.*

- Construct consensual proof plans: empirical, reflective, normative.

- Need criteria for assessing proof plans.

Criteria for Assessing Proof Plans

Correctness: Associated tactic will construct proof step.

Intuitiveness: Plan feels right.

Psychological Validity: Plan agrees with experiments on humans.

Expectancy: The more accurately success can be predicted the better.

Generality: The more proofs are accounted for by the plan the better.

Prescriptiveness: The less search the tactic generates the better.

Simplicity: The simpler the tactic the better.

Efficiency: The cheaper the tactic the better.

Parsimony: The fewer proof plans the better.

The Role of the Computer

- Automate testing of criteria.
- Automate statistics gathering.
- Ensure accuracy of proof plan.
- Disinterested checker of theory.

  source of inspiration.
- Application to automatic theorem proving.

Relation to Automatic Theorem Proving

- Conventional ATP methodology:

  heuristics suggested by shallow analysis,

  e.g. complexity measures.

  empirical success criterion.

- Proof plans alternative:

  proof plans suggested by deep analysis.

  proof plans must meet criteria.

- Slower initial progress, but no ultimate deadlock.

- Conventional ATP heuristics are valuable starting point.

Explanatory Role of Proof Plans

- Understanding whole proof *vs* understanding details.

  proof plan *vs* logical proof.

- Common structure in proofs.

  common proof plans.

- Old proofs guide search for new ones.

  use proof plan as guide.

- Interesting *vs* routine proof steps

  outside proof plan *vs* inside.

- Intuition of theoremhood.

  have proof plan but no logical proof.

- Varying learning abilities.

  have concepts to build proof plan.

Conclusion

- Science of reasoning:
  attachment of proof plans to proofs.
  provides multi-level understanding of proofs.
  normative, empirical and reflective.
- Proof plans consist of tactics, methods and critics.
  methods are meta-logical specification of tactics.
  critics patch failed proof attempts.
- Criteria for assessing proof plans.
- Application to ATP.
  advantages over conventional methodology.