Infinite–Dimensional Categorical Quantum Mechanics, Spectra, and Contextuality

Kevin Dunne

University of Strathclyde

April 2017

Categorical Quantum Mechanics

Monoidal Approach $\hookrightarrow$ Topos Approach

Categorical Quantum Mechanics

Monoidal Approach $\hookrightarrow$ Topos Approach

Internal vs. External

Categorical Quantum Mechanics

Monoidal Approach $\hookrightarrow$ Topos Approach

Internal vs. External

1. what is the "topos approach"?
2. what is "$\hookrightarrow$"?
3. what's the point?

Symmetric Monoidal Categories

Definition

A strict symmetric monoidal category $(\mathcal{C}, \otimes, I)$ consists of

- Objects $A, B, C, \ldots$
- Morphisms $f : A \to B$
- Monoidal product $\otimes$

$$f : A \to B \qquad\qquad g : C \to D$$

$$f \otimes g : A \otimes B \to C \otimes D$$

Symmetric Monoidal Categories

A *dagger* on $(\mathcal{C}, \otimes, I)$ consists of an involutive symmetric monoidal functor

$$\dagger : \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$$

i.e. every morphism has an adjoint

$$A \xrightarrow{\quad f \quad} B \qquad\qquad B \xrightarrow{\quad f^{\dagger} \quad} A$$

$$f^{\dagger\dagger} = f$$

†-Biproducts

A †-category *has finite †-biproducts* if it has:

- a zero object 0
- for each X and Y, an object $X \oplus Y$, which is a product and coproduct such that $\pi = \kappa^\dagger$

Example

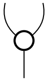
The category **Hilb** has:

- Objects: Hilbert spaces
- Morphisms $f : H_1 \to H_2$ bounded linear maps
- Monoidal product: tensor product of Hilbert spaces
- Monoidal unit : $\mathbb{C}$
- Dagger: Hermitian adjoint
- Biproducts: direct sum of Hilbert spaces

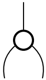
Monoidal/Internal Algebra Approach

Definition

An *algebra* in $(\mathcal{A}, \otimes, I)$ consists of: an object $A \in \mathcal{A}$ and

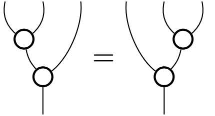
$$\mu : A \otimes A \to A, \qquad \mu =$$ 

A *coalgebra* in $(\mathcal{A}, \otimes, I)$ consists of: an object $A \in \mathcal{A}$ and

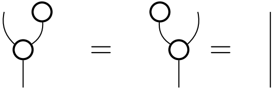
$$\delta : A \to A \otimes A, \qquad \delta =$$ 

A †–*algebra* consists of an algebra coalgebra pair $(A, \mu, \mu^{\dagger})$

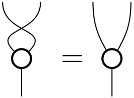
Monoidal/Internal Algebra Approach



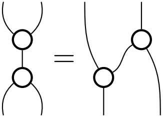
(A)



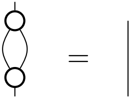
(U)



(C)

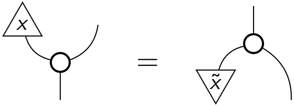


(F)



(S)

for each x there exists unique $\tilde{x}$ s.t



(H)

Monoidal/Internal Algebra Approach

Definition

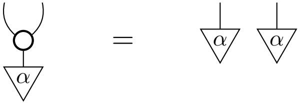
A *Frobenius algebra* is a †–algebra satisfying (A), (C), (S), (F) and (U).

Definition

An *H^*–algebra* is a †–algebra satisfying (A), (C), (S) and (H).

Definition

α is a *set–like* element for μ if



The Topos/External Algebra Approach

For H in **Hilb**, the set $\mathrm{Hom}(H, H)$ is a C^*–algebra

Definition

Let **Hilb-Alg**(H) be the poset of commutative subalgebras

$$\mathbf{A} \subset \mathrm{Hom}(H, H)$$

considered a category.

Let $\mathbf{Hilb\text{-}Alg}_{\mathrm{vN}}(H) \hookrightarrow \mathbf{Hilb\text{-}Alg}(H)$ be the subcategory of commutative von Neumann algebras

The Topos/External Algebra Approach

Classical Physics

Physics lab	$\rightarrow$	Commutative unital k–algebra A
Measuring device	$\rightarrow$	Element of the algebra $m \in A$
State of the observed system	$\rightarrow$	Algebra homomorphism $h : A \rightarrow k$
Output on measuring device	$\rightarrow$	The value $h(m)$

Quantum Physics

Physical system represented by non–commutative $\mathrm{Hom}(H, H)$

Hilb-Alg(H) is the collection of classical subsystems

The Topos/External Algebra Approach

The topos of the "Topos Approach" is the category of presheaves

$$\mathbf{Hilb\text{-}Alg}(H)^{\mathrm{op}} \xrightarrow{\ F\ } \mathbf{Set}$$

For example, the *Gelfand spectrum*

$$\mathbf{Hilb\text{-}Alg}(H)^{\mathrm{op}} \xrightarrow{\ \mathrm{GSpec}\ } \mathbf{Set}$$

$$\mathrm{GSpec}(\mathbf{A}) = \{\ \rho : \mathbf{A} \to \mathbb{C} \mid \rho \ \text{ a } \ C^*\text{–algebra homomorphism }\}$$

or the *prime Spectrum*

$$\mathrm{PSpec}(\mathbf{A}) = \{\ J \subset \mathbf{A} \mid J \ \text{ a prime ideal }\ \}$$

The Topos/External Algebra Approach

In the monoidal approach we consider $(\mathcal{A}, \otimes, I, \oplus, \dagger)$.

Consider $\mathrm{Hom}(I, I)$, we have:

- addition via *biproduct convolution*

  For $s, t : I \to I$ define the $s + t : I \to I$

  $$I \xrightarrow{\ \Delta\ } I \oplus I \xrightarrow{\ s \oplus t\ } I \oplus I \xrightarrow{\ \nabla\ } I$$

- commutative multiplication via morphism composition

$S = \mathrm{Hom}(I, I)$ is a commutative semiring

The Topos/External Algebra Approach

Consider $\mathrm{Hom}(X, Y)$, we have:

- addition via *biproduct convolution*

  For $f, g : X \to Y$ define the $f + g : X \to Y$

  $$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y$$

- scalar multiplication

  $$X \xrightarrow{\sim} X \otimes I \xrightarrow{f \otimes s} Y \otimes I \xrightarrow{\sim} Y$$

$\mathrm{Hom}(X, Y)$ is an S–semimodule

The Topos/External Algebra Approach

Consider $\text{Hom}(X, X)$, we have:

- addition
- scalar multiplication
- multiplication via morphism composition
- an involution given by $\dagger$

$\text{Hom}(X, X)$ is an involutive S–semialgebra

The Topos/External Algebra Approach

For X in $\mathcal{A}$, the set $\mathrm{Hom}(X, X)$ is a involutive S–semialgebra

Definition

Let $\mathcal{A}$-$\mathbf{Alg}(X)$ be the poset of commutative S–subsemialgebras

$$\mathbf{A} \subset \mathrm{Hom}(X, X)$$

considered a category.

The Topos/External Algebra Approach

Let $\mathcal{A}\text{-}\mathbf{Alg}_{\mathsf{vN}}(X) \hookrightarrow \mathcal{A}\text{-}\mathbf{Alg}(X)$ be the subcategory of commutative von Neumann semialgebras

$$\mathbf{A} = \mathbf{A}''$$

Where for any $B \subset \mathrm{Hom}(X, X)$ the *commutant* B' is defined

$$B' = \{ \; f : X \to X \mid f \circ g = g \circ f \text{ for all } g \in B \; \}$$

The Topos/External Algebra Approach

The topos of the “Topos Approach” is the topos is presheaves

$$\mathcal{A}\text{-}\mathbf{Alg}(X)^{\mathrm{op}} \xrightarrow{\quad F \quad} \mathbf{Set}$$

For example, the *generalised Gelfand spectrum*

$$\mathcal{A}\text{-}\mathbf{Alg}(X)^{\mathrm{op}} \xrightarrow{\quad \mathrm{GSpec} \quad} \mathbf{Set}$$

$$\mathrm{GSpec}(\mathbf{A}) = \{\ \rho : \mathbf{A} \to S \mid \rho \ \text{ a } \ S\text{–semialgebra homomorphism}\ \}$$

Or the *prime spectrum*

$$\mathrm{PSpec}(\mathbf{A}) = \{\ K \subset \mathbf{A} |\ K \text{ a subtractive prime ideal}\ \}$$

Recap

Monoidal Approach $\hookrightarrow$ Topos Approach

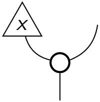
Internal vs. External

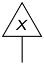
1. what is the "topos approach"?
2. what is "$\hookrightarrow$"?
3. what's the point?

From Internal to External

Theorem

Let $\mathcal{A}$ be a †–symmetric monoidal category with finite †–biproducts, and let (X, μ) be an H^*–algebra in $\mathcal{A}$. Consider the regular representation

$R(\mu) = \{$ 

$\mid$ *for all points* 

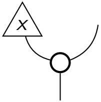
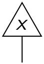
$\}$

The commutant $R(\mu)'$ is a maximal von Neumann endomorphism semialgebra.

From Internal to External

Theorem

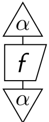
Let $\mathcal{A}$ be a †–symmetric monoidal category with finite †–biproducts, and let (X, μ) be an H^*–algebra in $\mathcal{A}$. Consider the regular representation

$R(\mu) = \{$  $\mid$ *for all points*  $\}$

The commutant $R(\mu)'$ is a maximal von Neumann endomorphism semialgebra.

Theorem

Each set–like element α of (X, μ) determines an S^*–semialgebra homomorphism $\rho_\alpha : \mathbf{A} \to S$

 $\mapsto$ 

What's the Point?

Why move to the external point of view?

- In order to study the internal algebras
- For its own sake

A Structure Theorem for H^*–algebras

Definition

$\mathcal{A}$ has †–*kernels* if for each $f : X \to Y$ the equalizer

$$K \xrightarrow{k} A \underset{0}{\overset{f}{\rightrightarrows}} B$$

exists and $k^\dagger \circ k = \mathrm{id}_K$.

Definition

$\mathcal{A}$ has *complemented* †–kernels if $\mathcal{A}$ has finite †–biproducts and for each †–kernel $k : K \to A$ there is $\overline{K}$ such that

$$A \cong K \oplus \overline{K}$$

A Structure Theorem for H^*–algebras

For $\mathcal{A}$ is a complemented †–kernel category we can prove a version of Ambrose's Theorem

$$\begin{array}{lccc}
\text{External} & \mathbf{X} = R(\mu)' & \longrightarrow & \mathbf{X} \hookrightarrow \prod_i e_i \, \mathbf{X} \\
 & \uparrow & & \downarrow \\
\text{Internal} & \mu : X \otimes X \to X & & \widehat{\bigoplus_i} \mu_i : X_i \otimes X_i \to X_i
\end{array}$$

Main idea: find $\{e_i\}$ self–adjoint idempotents s.t. $\sum_i e_i = \mathrm{id}_X$

Kochen–Specker Contextuality

A presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ has a terminal object $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$

Definition

A *global section* (or *global element*) of a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is a natural transformation $x : T \to F$.

Theorem

The Kochen–Specker Theorem is equivalent to the statement that for H with dimension ≥ 3

$$\mathbf{Hilb\text{-}Alg}(H)^{\mathrm{op}} \xrightarrow{\mathrm{GSpec}} \mathbf{Set}$$

has no global sections.

Kochen–Specker Contextuality

Definition

An object X in $\mathcal{A}$ is *Kochen–Specker contextual* if the spectral presheaf

$$\mathcal{A}\text{-}\mathbf{Alg}_{\mathsf{vN}}(X)^{\mathrm{op}} \xrightarrow{\mathsf{GSpec}} \mathbf{Set}$$

has no global sections.

Kochen–Specker Contextuality

A *commutative quantale* Q consists of a complete semilattice with commutative monoid operation $\cdot : Q \times Q \to Q$ such that

$$x \cdot (\bigvee y) = \bigvee (x \cdot y)$$

Q is *zero–divisor free* if $x \cdot y = \bot$ implies $x = \bot$ or $y = \bot$.

Definition

Let Q be a commutative ZDF quantale. $\mathbf{Rel}_Q$ has sets for objects and Q–valued matrices for morphisms.

Kochen–Specker Contextuality

Theorem

For each set X the spectral presheaf

$$\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\mathsf{vN}}(X)^{\mathrm{op}} \xrightarrow{\mathsf{GSpec}} \mathbf{Set}$$

has global sections.

Every set in $\mathbf{Rel}_Q$ *is non–contextual.*

Topologising the State Space

The Spectrum of a C^*–algebra

$$\mathbf{Hilb\text{-}Alg}_{\mathsf{vN}}(X)^{\mathrm{op}} \xrightarrow{\mathsf{GSpec}} \mathbf{Top}$$

Theorem

GSpec($\mathbf{A}$) *is a compact* T_2 *(Hausdorff) space.*

Topologising the State Space

These Spectra can be equipped with the Zariski topology

$$\mathcal{A}\text{-}\mathbf{Alg}_{\mathsf{vN}}(X)^{\mathrm{op}} \xrightarrow{\mathsf{GSpec}} \mathbf{Top}$$

$$\mathcal{A}\text{-}\mathbf{Alg}_{\mathsf{vN}}(X)^{\mathrm{op}} \xrightarrow{\mathsf{PSpec}} \mathbf{Top}$$

Theorem

$\mathsf{GSpec}(\mathbf{A})$ *is a compact space.*

$\mathsf{PSpec}(\mathbf{A})$ *is a compact T_0 space.*

Theorem

Let (X, μ) be a H^*–algebra and let $\mathbf{X} = R(\mu)'$. The set–like elements form a compact T_1 subspace of $\mathsf{GSpec}(\mathbf{X})$.