Games for topological fixpoint logics

Clemens Kupke
University of Strathclyde, Glasgow
Scotland, UK & EU

joint work with

Nick Bezhanishvili
University of Amsterdam
The Netherlands

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Overview

- (topological) semantics of the modal μ-calculus

- fixpoint games

- main result: evaluation game for topological semantics

- bisimulations

- Disclaimer: topological semantics in our setting means
  - Kripke models based on topological spaces
  - semantics of a formula given by an "admissible" subset
  - different from "Spatial Logic of modal mu-calculus" by Goldblatt/Hodkinson (2016)

semantics of the modal μ-calculus

Modal μ-calculus

- adding least and greatest fixpoint operators to modal logic
- Goal: express properties about ongoing, (possibly) infinite behaviour

Formulas

$$\mathcal{L}_\mu \ni \varphi \quad ::= \quad p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \bot \mid \top \mid \Diamond\varphi \mid \Box\varphi \mid$$
$$\mu p.\varphi(p, q_1, \ldots, q_n) \mid \nu p.\varphi(p, q_1, \ldots, q_n)$$

where $p \in$ Prop and p occurs only positively in formulas of the form $\mu p.\varphi(p, q_1, \ldots, q_n)$ and $\nu p.\varphi(p, q_1, \ldots, q_n)$.

Formulas as operators

For a formula $\delta \in \mathcal{L}_\mu$, a Kripke frame (X, R) and a valuation $V : \mathrm{Prop} \to \mathcal{P}X$ let

$$\delta_p^V : \mathcal{P}X \to \mathcal{P}X$$
$$U \mapsto [\![\delta]\!]_{V[p \mapsto U]}^{(X,R)}$$

where

$$V[p \mapsto U](q) = \begin{cases} U & \text{if } p = q \\ V(q) & \text{otherwise.} \end{cases}$$

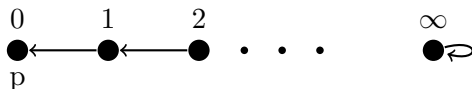and where $[\![\varphi]\!]_V^{(X,R)}$ denotes semantics of φ on a model (X, R, V).

Observation

If δ contains only positive occurrences of p, then δ_p^V is a monotone operator.

$\Rightarrow$ We define the semantics of $\mu p.\delta$ and $\nu p.\delta$ as least and greatest fixpoints of δ_p^V, respectively.

An example

Standard interpretation:

- R is the transition relation
- $\lozenge^* p = \mu q.p \vee \lozenge q$: p reachable via transitive closure of R
- $\mathbb{N}$ is the least fixed point of the map $U \mapsto V(p) \cup \lozenge U$, where

$$\lozenge U = \{x' \mid \exists x \in U.(x', x) \in R\}$$

Later with topology:

- basis of clopens: finite subsets of $\mathbb{N}$ and cofinite sets containing ∞
- least clopen fixpoint: $\mathbb{N} \cup \{\infty\}$
- p reachable via topological transitive closure of R

Topological semantics: Motivation

Denotations of formulas restricted to admissible subsets of a model - admissibility defined topologically.

Motivation:

- notion of $\boxed{\text{reachability in the limit}}$ in order to be able to specify systems with continuous behaviour
- apply techniques from $\boxed{\text{Stone duality}}$ to the μ-calculus
- modal μ-algebras give relatively easy $\boxed{\text{completeness proofs}}$ with respect to topological semantics (cf. work by Ambler/Bonsangue/Kwiatkowska)

Our goal

Characterise topological semantics using games, ultimately: automata!

Why games?

▶ consider the formula

$$\varphi = \nu p.\mu q.\Diamond((s \wedge p) \vee q)$$

▶ not immediate to check with the "denotational" semantics

▶ game semantics: outermost fixpoint that is unfolded infinitely often has to be ν

▶ consequently: φ expresses

"there is a path along which s holds infinitely often"

▶ ‖Problem:‖ Game operates on states of the model - how to ensure admissibilty?

Related Work

- ▶ S. Ambler, M. Kwiatkowska, N. Measor. Duality and the Completeness of the Modal mu-Calculus. (1995)

- ▶ M.M. Bonsangue, M. Kwiatkowska, Re-Interpreting the Modal mu-Calculus (1995)

- ▶ J. van Benthem, N. Bezhanishvili, I. M. Hodkinson. Sahlqvist Correspondence for Modal mu-calculus. (2012)

- ▶ N. Bezhanishvili, I. Hodkinson. Sahlqvist theorem for modal fixed point logic. (2012)

- ▶ W.Conradie, Y.Fomatati, A.Palmigiano, S.Sourabh, Algorithmic correspondence for intuitionistic modal mu-calculus.(2015)

- ▶ S. Enqvist, F. Seifan, Y. Venema, Completeness for the modal μ-calculus: separating the combinatorics from the dynamics. (2016)

Our topological setting

The spaces

- **extremally disconnected** Stone spaces: closure of any open set is open (and thus clopen)

- consequently: clopen sets form a complete lattice

- frames: descriptive frames $(\mathbb{X}, R)$ (ie, $\mathbb{V}$-coalgebras) based on an extremally disconnected Stone space $\mathbb{X} = (X, \tau)$

- model = frame + clopen valuation $V : \mathrm{Prop} \rightarrow \mathrm{Clp}(\mathbb{X})$

Formulas interpreted as clopen subsets

- for the "purely modal" formulas this is ensured by taking a clopen valuation and descriptive general frames (=Vietoris coalgebras)

- for the fixpoint operators we note that

    1. $\delta_p^V : \text{Clp}(\mathbb{X}) \to \text{Clp}(\mathbb{X})$ is a monotone operator
    2. $\text{Clp}(\mathbb{X})$ is a complete lattice

- we interpret $\mu p.\delta$ and $\nu p.\delta$ as the least and greatest clopen fixpoints, respectively

Fixpoint approximation on a complete lattice

$$
\begin{aligned}
F_0^\mu &= \bot, \\
F_{\alpha+1}^\mu &= F(F_{\alpha+1}^\mu) \\
F_\alpha^\mu &= \bigvee_{\beta<\alpha} F_\beta^\mu \quad \text{for } \alpha \text{ a limit ordinal.}
\end{aligned}
$$

$$
\begin{aligned}
F_0^\nu &= \top, \\
F_{\alpha+1}^\nu &= F(F_{\alpha+1}^\nu) \\
F_\alpha^\nu &= \bigwedge_{\beta<\alpha} F_\beta^\nu \quad \text{for } \alpha \text{ a limit ordinal.}
\end{aligned}
$$

Fact

For any monotone operator F there are ordinals α_μ and α_ν such that $F_{\alpha_\mu}^\mu = \mu F$ and $F_{\alpha_\nu}^\nu = \nu F$.

Extremally disconnected spaces & logical completeness

- For basic modal logic we have that

$$\mathrm{Log}(\mathrm{Coalg}(\mathbb{V})) = \mathrm{Log}(\mathrm{Coalg_{ed}}(\mathbb{V}))$$

  because

$$
\begin{array}{ccc}
GX & \xdashrightarrow{\ \exists e\ } & X \\
\exists \downarrow & & \downarrow \forall \\
\mathbb{V}GX & \xrightarrow{\ \mathbb{V}e\ } & \mathbb{V}X
\end{array}
$$

- (Problem: the above argument does not work for all of the μ-calculus...)

fixpoint games

A note about games

- all games in my talk are two-player graph games

- players are called $\exists$ and $\forall$ and board positions are partitioned into $\exists$'s and $\forall$'s positions

- plays can be infinite

- the winning condition on infinite games can be encoded using a parity condition

- this means that at any position of the game board either of the player has a historyfree/memoryless winning strategy

Tarski's fixpoint game on $\mathcal{P}X$

Here we have:

$$\bigwedge_{i \in I} U_i = \bigcap_{i \in I} U_i \qquad \text{and} \qquad \bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i$$

We define the game board of a two-player graph game:

Position	Player	Moves
$x \in X$	$\exists$	$\{C \subseteq X \mid x \in F(C)\}$
$C \in \mathcal{P}X$	$\forall$	C

Least fixpoint game: $\forall$ wins infinite plays.
Greatest fixpoint game: $\exists$ wins infinite plays.

Fact

$\exists$ has a winning strategy at position $x \in X$ in the least (greatest) fixpoint game iff $x \in \mu F$ ($x \in \nu F$).

Proof idea for least fixpoint game

Suppose $x \in \mu F$.

This means there exists an ordinal α such that $x \in F_\alpha^\mu$.

Suppose α is a limit ordinal, then

$$x \in \bigvee_{\beta < \alpha} F_\beta^\mu = \bigcup_{\beta < \alpha} F_\beta^\mu \subseteq F(\bigcup_{\beta < \alpha} F_\beta^\mu)$$

Therefore $\exists$ moves from x to $\bigcup_{\beta < \alpha} F_\beta^\mu$ and $\forall$ has to move to some $x' \in F_\beta^\mu$ with $\beta < \alpha$.

Similarly for successor ordinals.

Well-foundedness of the ordinals implies finiteness of the play.

Proof idea for least fixpoint game

For the converse assume that $x \notin \mu F$.

Then either $\exists$ gets stuck at x, or she moves to some $C \subseteq X$ with $x \in F(C)$.

By assumption we know that $C \not\subseteq \mu F$, ie., $\forall$ can continue the play by moving to some $x' \in C \setminus \mu F$ and the play continues ...

Either the play will be infinite or $\exists$ gets stuck which shows that $\forall$ has a winning strategy.

Fixpoints on an extremally disconnected Stone space

$$\bigwedge_{i \in I} U_i = \text{Int}(\bigcap_{i \in I} U_i) \qquad \text{and} \qquad \bigvee_{i \in I} U_i = \text{Cl}(\bigcup_{i \in I} U_i)$$

Possible consequences for the μ-game (informal!):

1. To show that $x \in \mu F$ it suffices for $\exists$ to provide a "suitable" set $C \subseteq X$ such that $x \in F(U)$ for all clopens $U \in \text{Clp}(\mathbb{X})$ with $C \subseteq U$.

2. If $\exists$ chooses some (open) set $O \subseteq X$ such that $x \in F(\text{Cl}(O))$, $\forall$ is only allowed to challenge her with elements of O.

Both observations lead to different (but equivalent) games.

Topological least fixpoint game I

Position	Player	Moves
$x \in X$	$\exists$	$\{C \subseteq X \mid x \in F(U)$ for all $U \in \mathrm{Clp}(\mathbb{X})$ with $C \subseteq U\}$
$C \in \mathcal{P}X$	$\forall$	C

Topological least fixpoint game II

Position	Pl.	Moves
$x \in X$	$\exists$	$\{(\forall, U) \in M \times \mathrm{Clp}(\mathbb{X}) \mid x \in F(U)\}$
$(\forall, U) \in M \times \mathrm{Clp}(\mathbb{X})$	$\forall$	$\{(\exists, U') \in M \times \mathrm{Clp}(\mathbb{X}) \mid U \cap U' \neq \emptyset\}$
$(\exists, U') \in M \times \mathrm{Clp}(\mathbb{X})$	$\exists$	U'

(Here $M = \{\forall, \exists\}$ is the collection of markers to keep sets of positions of $\exists$ and $\forall$ disjoint.)

Correctness of $\mathcal{G}^{II}$

Suppose that $x \in \mu F$ for some $x \in X$.

Then there is a least ordinal α such that $x \in F_\alpha^\mu$.

We will show that $\exists$ has a (winning) strategy that ensures that either $\forall$ gets stuck within the next round or that the play reaches a position $x' \in F_{\alpha'}^\mu$ with $\alpha' < \alpha$.

Correctness of $\mathcal{G}^{II}$

Case α is a limit ordinal

Then $\exists$'s strategy is to move from x to

$$(\forall, \bigvee_{\beta < \alpha} F_\beta^\mu) = (\forall, \mathrm{Cl}(\bigcup_{\beta < \alpha} F_\beta^\mu)).$$

Unless $\forall$ gets stuck, he will move to some position $(\exists, U')$ where $U' \in \mathrm{Clp}(\mathbb{X})$ with

$$U' \cap \bigvee_{\beta < \alpha} F_\beta^\mu = U' \cap \mathrm{Cl}(\bigcup_{\beta < \alpha} F_\beta^\mu) \neq \emptyset.$$

One can easily see that this implies $U' \cap \bigcup_{\beta < \alpha} F_\beta^\mu \neq \emptyset$.
Therefore $\exists$ can pick a suitable element $x' \in \bigcup_{\beta < \alpha} F_\beta^\mu$ and $x' \in F_\beta^\mu$ for some $\beta < \alpha$.

Why two games?

Pragmatic answer   Because this makes our arguments work.

Observation   There seems to be no direct transformation for $\exists$'s
winning strategies in the first game into winning
strategies in the second game.

Even 2 games are not enough!

Four Games: Two more for greatest fixpoints.

The two greatest fixpoint games

Game $\mathcal{G}_\nu^I$:

Position	Pl	Moves
$x \in X$	$\exists$	$\{C \subseteq X \mid x \in F(U) \text{ for all } U \in \text{Clp}(\mathbb{X}) \text{ with } \text{Int}(C) \subseteq U\}$
$C \subseteq X$	$\forall$	C

Game $\mathcal{G}_\nu^{II}$:

Position	Player	Moves
$x \in X$	$\exists$	$\{U \in \text{Clp}(\mathbb{X}) \mid x \in F(U)\}$
$U \in \text{Clp}(\mathbb{X})$	$\forall$	U

evaluation game

Evaluation game

Goal

For a formula $\varphi \in \mathcal{L}_\mu$ and a model $\mathbb{M} = (\mathbb{X}, R, V)$ define a game s.t. $\exists$ has a winning strategy at position (φ, x) iff $x \in [\![\varphi]\!]_V^{\mathbb{M}}$.

Evaluation game $\mathcal{E}(\varphi, \mathbb{M})$: Standard formulation

Given a formula $\varphi \in \mathcal{L}_\mu$ we define its evaluation game as follows:

Position	Condition	Player	Possible Moves
(p, x)	$p \in FV(\varphi), x \notin V(p)$	$\exists$	$\emptyset$
(p, x)	$p \in FV(\varphi), x \in V(p)$	$\forall$	$\emptyset$
$(\neg p, x)$	$p \in FV(\varphi), x \notin V(p)$	$\forall$	$\emptyset$
$(\neg p, x)$	$p \in FV(\varphi), x \in V(p)$	$\exists$	$\emptyset$
$(\psi_1 \wedge \psi_2, x)$		$\forall$	$\{(\psi_1, x), (\psi_2, x)\}$
$(\psi_1 \vee \psi_2, x)$		$\exists$	$\{(\psi_1, x), (\psi_2, x)\}$
$(\Diamond\psi, x)$		$\exists$	$\{(\psi, x') \mid x' \in R[x]\}$
$(\Box\psi, x)$		$\forall$	$\{(\psi, x') \mid x' \in R[x]\}$
$(\eta p.\psi, x)$	$\eta \in \{\mu, \nu\}$	$\exists/\forall$	(ψ, x)
(p, x)	$p \in BV(\varphi), \varphi@p = \mu p.\psi$	$\exists/\forall$	(ψ, x)

Plus some suitable winning condition on infinite plays.

Evaluation game $\mathcal{E}(\varphi, \mathbb{M})$: our modification

Given a formula $\varphi \in \mathcal{L}_\mu$ we remove the last rule and add the following rules on fixpoint unfolding:

Position	Condition	Player	Possible Moves
$(\eta p.\psi, x)$	$\eta \in \{\mu, \nu\}$	$\exists/\forall$	(ψ, x)
(p, x)	$\eta = \mu$	$\forall$	$\{(p, U) \mid U \in \mathrm{Clp}(\mathbb{X}),\ x \in U\}$
(p, x)	$\eta = \nu$	$\exists$	$\{(p, U) \mid U \in \mathrm{Clp}(\mathbb{X}),\ x \in U\}$
(p, U)	$\eta = \mu$	$\exists$	$\{(\psi, x') \mid x' \in U\}$
(p, U)	$\eta = \nu$	$\forall$	$\{(\psi, x') \mid x' \in U\}$

where $p \in \mathrm{BV}(\varphi)$, $\varphi @ p = \eta p.\psi$ with $\eta \in \{\mu, \nu\}$, $U \in \mathrm{Clp}(\mathbb{X})$.

Intuition

Reachability becomes easier to prove; safety more difficult.

Bisimulations

Let $\mathbb{M}_1 = (\mathbb{X}_1, R_1, V)$ and $\mathbb{M}_2 = (\mathbb{X}_2, R_2, V)$ be extremally disconnected Kripke models.

A relation $Z \subseteq X_1 \times X_2$ is called a clopen bisimulation iff

- $Z \subseteq X_1 \times X_2$ is a (standard) Kripke bisimulation and

- for any clopen subsets $U_1 \in \mathrm{Clp}(\mathbb{X}_1)$ and $U_2 \in \mathrm{Clp}(\mathbb{X}_2)$:

$$Z[U_1] = \{x' \in X_2 \mid \exists x \in U_1.(x, x') \in Z\} \in \mathrm{Clp}(\mathbb{X}_2)$$
$$Z^{-1}[U_2] = \{x \in X_1 \mid \exists x' \in U_2.(x, x') \in Z\} \in \mathrm{Clp}(\mathbb{X}_1).$$

Bisimulation invariance

Let Z be a clopen bisimulation between extremally disconnected Kripke models $\mathbb{M}_1 = (\mathbb{X}_1, R_1, V)$ and $\mathbb{M}_2 = (\mathbb{X}_2, R_2, V)$.

Proposition

For any formula $\varphi \in \mathcal{L}_\mu$ and states $x \in X_1$ and $x' \in X_2$ such that $(x, x') \in Z$, we have $x \in [\![\varphi]\!]$ iff $x' \in [\![\varphi]\!]$.

Proof sketch

Suppose that $(x, x') \in Z$ and that $x \in [\![\varphi]\!]$ for some formula φ.

$\Rightarrow (\varphi, x) \in \mathrm{Win}_\exists(\mathcal{E}(\varphi, \mathbb{M}_1))$.

$\Rightarrow$ Transform $\exists$'s winning strategy in $\mathcal{G}_1 = \mathcal{E}(\varphi, \mathbb{M}_1)$ at position (φ, x) into a winning strategy for $\exists$ in $\mathcal{G}_2 = \mathcal{E}(\varphi, \mathbb{M}_2)$ at position (φ, x').

$\Rightarrow x' \in [\![\varphi]\!]$

Future & Ongoing Work

- other topological spaces
- other notions of admissible subset
- automata & complexity?
- shedding light on completeness proof of fixpoint logics (modal μ-calculus)
- if (some of) this works out for Kripke frames, we could look into coalgebraic variants (game logic)

Thanks!