A sequent calculus for a semi-associative law[1]

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Catgories, Logic, and Physics
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[1] Based on a paper: `http://noamz.org/papers/tamari-fscd.pdf`

Directed semigroups

Imagine a preordered set equipped with a "multiplication" operation $A * B$ which is monotonic in each argument

$$\frac{A_1 \leq A_2 \quad B_1 \leq B_2}{A_1 * B_1 \leq A_2 * B_2}$$

and obeys a **semi-associative law**:

$$(A * B) * C \leq A * (B * C)$$

(e.g., finite lists of $\mathbb{N}$ ordered pointwise, with $\alpha * \beta \stackrel{\text{def}}{=} \alpha, |\beta|, \beta$)

Directed semigroups

Some other interesting examples:

- Let $(P, \leq, \multimap)$ be any *imploid*, that is, a preordered set equipped with an operation $A \multimap B$ which is contravariant in A, covariant in B, and obeys a **composition law**:

  $$B \multimap C \leq (A \multimap B) \multimap (A \multimap C)$$

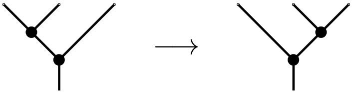
  Then consider upwards closed subsets of P, ordered by $\supseteq$, with

  $$R * S \stackrel{\text{def}}{=} \{\, C \mid \exists B.\, B \multimap C \in R \ \wedge \ B \in S \,\}$$

- The free example...

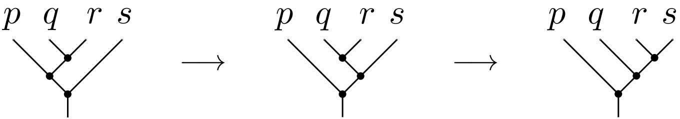
Directed semigroups

Concretely, semi-associativity may be visualized as *right rotation*



acting on the inner nodes of a rooted binary tree.

Example: $(p * (q * r)) * s \leq p * (q * (r * s))$



The Tamari order and Tamari lattices

The free directed semigroup on one generator was first studied in:

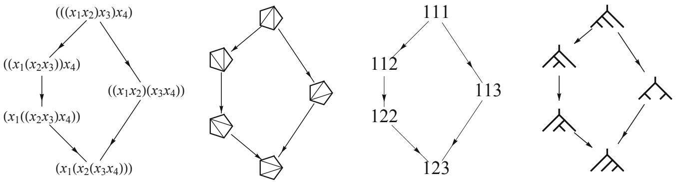
- D. Tamari, "Monoïdes préordonnés et chaînes de Malcev," PhD Thesis, Université de Paris, 1951.

For each $n \in \mathbb{N}$, the $C_n = \binom{2n}{n}/(n+1)$ binary trees with n internal nodes actually form a *lattice* Y_n under right (or left) rotation. This was already stated in Tamari's thesis, and proved in:

- D. Tamari, "Sur quelques problèmes d'associativité," *Ann. sci. de Univ. de Clermont-Ferrand 2, Sér. Math.*, vol. 24, 1964.
- H. Friedman and D. Tamari, "Problèmes d'associativité: une structure de treillis finis induite par une loi demi-associative," *J. Combinatorial Theory*, vol. 2, 1967.
- S. Huang and D. Tamari, "Problems of associativity: A simple proof for the lattice property of systems ordered by a semi-associative law," *J. Combin. Theory Ser. A*, vol. 13, no. 1, 1972.

The Tamari order and Tamari lattices

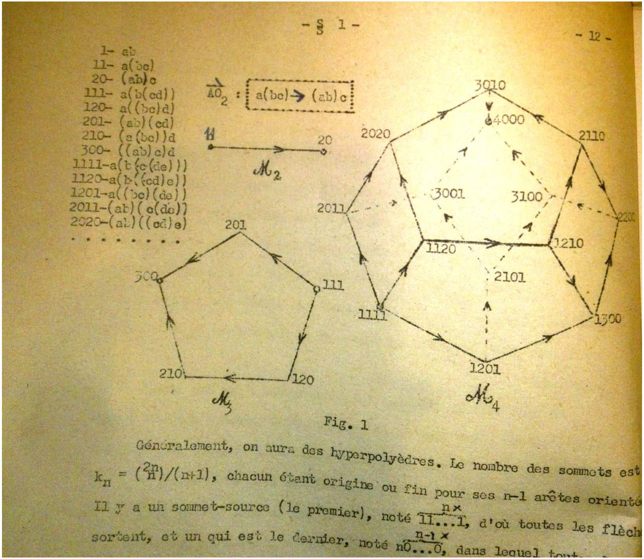
Since the Catalan numbers C_n also count many isomorphic families of objects, there are many equivalent descriptions[2] of Y_n...



The Hasse diagrams of the Y_n form the 1-skeleta of a family of $(n-1)$-dimensional polytopes known as the **associahedra** (independently discovered by Jim Stasheff).

[2] Figure taken from F. Müller-Hoissen and H.-O. Walther, Eds., *Associahedra, Tamari Lattices and Related Structures: Tamari Memorial Festschrift*, Birkhauser, 2012.

The Tamari order and Tamari lattices



- § 1 -

- 12 -

1- ab
11- a(bc)
20- (ab)c
111- a(b(cd))
120- a((bc)d)
201- (ab)(cd)
210- (a(bc))d
300- ((ab)c)d
1111-a(b(c(de)))
1120-a(b((cd)e))
1201-a((bc)(de))
2011-(ab)(c(de))
2020-(ab)((cd)e)
. . . .

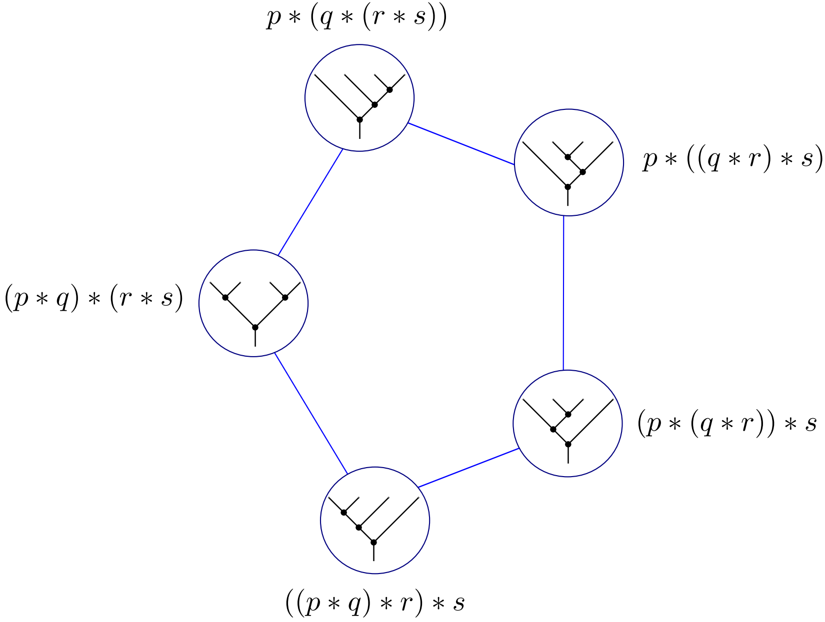
$\overrightarrow{AO}_2$: $\boxed{a(bc) \rightarrow (ab)c}$

Fig. 1

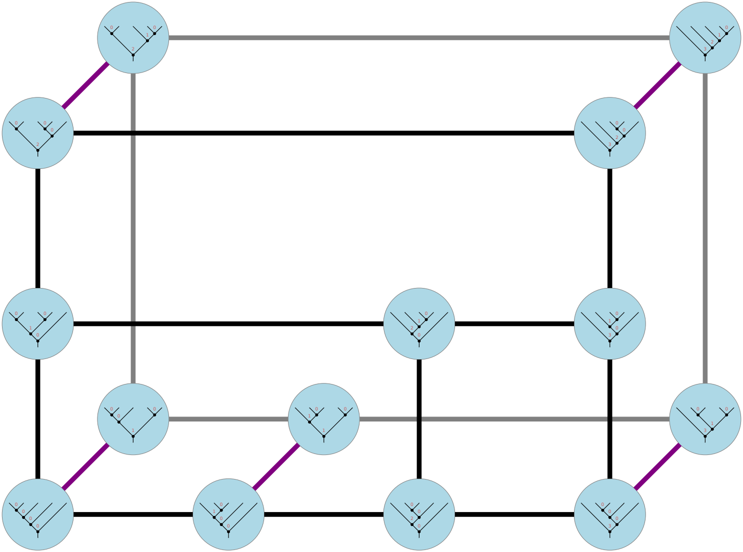
Généralement, on aura des hyperpolyèdres. Le nombre des sommets est $k_n = \binom{2n}{n}/(n+1)$, chacun étant origine ou fin pour ses n-1 arêtes orienté
Il y a un sommet-source (le premier), noté $\overbrace{11\ldots1}^{n\times}$, d'où toutes les flèch
sortent, et un qui est le dernier, noté $n\overbrace{0\ldots0}^{n-1\times}$, dans lequel tout

(from Tamari's 1951 thesis)

The Tamari lattice Y_3



The Tamari lattice Y_4



(in a geometric realization due to Don Knuth)

The FSCD paper

Summary of contributions:

- A surprisingly simple presentation of the Tamari order as a **sequent calculus** in the style of Lambek
- A proof of **focusing completeness** (a strong form of cut-elimination) together with a **coherence theorem**
- An **application to combinatorics**: a new proof of Chapoton's theorem on the number of *intervals* in Y_n

The sequent calculus

Four rules for deriving sequents of the form $A_0, \ldots, A_n \longrightarrow B$

$$\frac{}{A \longrightarrow A}\ id \qquad \frac{\Theta \longrightarrow A \quad \Gamma, A, \Delta \longrightarrow B}{\Gamma, \Theta, \Delta \longrightarrow B}\ cut$$

$$\frac{A, B, \Delta \longrightarrow C}{A * B, \Delta \longrightarrow C}\ {*L} \qquad \frac{\Gamma \longrightarrow A \quad \Delta \longrightarrow B}{\Gamma, \Delta \longrightarrow A * B}\ {*R}$$

where "," denotes concatenation (a strictly associative operation)

(Note: no weakening, contraction, or exchange rules.)

The sequent calculus

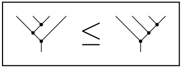
These rules are *almost* straight from Lambek[3]...

$$\frac{A, B, \Delta \longrightarrow C}{A * B, \Delta \longrightarrow C} *L \qquad \textit{versus} \qquad \frac{\Gamma, A, B, \Delta \longrightarrow C}{\Gamma, A * B, \Delta \longrightarrow C} *L^{\mathrm{amb}}$$

This simple restriction makes all the difference!

[3] J. Lambek, "The mathematics of sentence structure," *The American Mathematical Monthly*, vol. 65, no. 3, pp. 154–170, 1958.

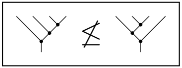
The sequent calculus



Example: $(p*(q*r))*s \overset{\text{Tam}}{\leq} p*(q*(r*s))$

$$
\dfrac{
  \dfrac{
    \overline{p \longrightarrow p}
    \qquad
    \dfrac{
      \dfrac{
        \overline{q \longrightarrow q}
        \qquad
        \dfrac{\overline{r \longrightarrow r} \quad \overline{s \longrightarrow s}}{r, s \longrightarrow r*s}\,R
      }{q, r, s \longrightarrow q*(r*s)}\,R
    }{q*r, s \longrightarrow q*(r*s)}\,L
  }{
    \dfrac{p, q*r, s \longrightarrow p*(q*(r*s))}{p*(q*r), s \longrightarrow p*(q*(r*s))}\,L
  }\,R
}{(p*(q*r))*s \longrightarrow p*(q*(r*s))}\,L
$$

The sequent calculus



Counterexample: $p * (q * (r * s)) \overset{\text{Tam}}{\not\leq} (p * (q * r)) * s$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$p \longrightarrow p$}
\AxiomC{}
\UnaryInfC{$q \longrightarrow q$}
\AxiomC{}
\UnaryInfC{$r \longrightarrow r$}
\RightLabel{R}
\BinaryInfC{$q, r \longrightarrow q * r$}
\RightLabel{R}
\BinaryInfC{$p, q, r \longrightarrow p * (q * r)$}
\AxiomC{}
\UnaryInfC{$s \longrightarrow s$}
\RightLabel{R}
\BinaryInfC{$p, q, r, s \longrightarrow (p * (q * r)) * s$}
\RightLabel{$\color{red}{L^{\text{amb}}}$}
\UnaryInfC{$p, q, r * s \longrightarrow (p * (q * r)) * s$}
\RightLabel{$\color{red}{L^{\text{amb}}}$}
\UnaryInfC{$p, q * (r * s) \longrightarrow (p * (q * r)) * s$}
\RightLabel{L}
\UnaryInfC{$p * (q * (r * s)) \longrightarrow (p * (q * r)) * s$}
\end{prooftree}
$$

The sequent calculus

$$\frac{}{A \longrightarrow A}\ id \qquad \frac{\Theta \longrightarrow A \quad \Gamma, A, \Delta \longrightarrow B}{\Gamma, \Theta, \Delta \longrightarrow B}\ cut$$

$$\frac{A, B, \Delta \longrightarrow C}{A * B, \Delta \longrightarrow C}\ {*L} \qquad \frac{\Gamma \longrightarrow A \quad \Delta \longrightarrow B}{\Gamma, \Delta \longrightarrow A * B}\ {*R}$$

Theorem (Completeness)

If $A \overset{\mathrm{Tam}}{\leq} B$ *then* $A \longrightarrow B$.

Theorem (Soundness)

If $\Gamma \longrightarrow B$ *then* $\phi[\Gamma] \overset{\mathrm{Tam}}{\leq} B$, *where*
$\phi[A_0, \ldots, A_n] = ((A_0 * A_1) \cdots) * A_n$ *is the left-associated product*

Proof of completeness (easy)

Reflexivity + transitivity: immediate by *id* and *cut*.

Monotonicity:

$$\dfrac{\dfrac{A \longrightarrow A' \quad B \longrightarrow B'}{A, B \longrightarrow A' * B'}\ R}{A * B \longrightarrow A' * B'}\ L$$

Semi-associativity:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \longrightarrow A} \quad \dfrac{\dfrac{}{B \longrightarrow B} \quad \dfrac{}{C \longrightarrow C}}{B, C \longrightarrow B * C}\ R}{A, B, C \longrightarrow A * (B * C)}\ R}{A * B, C \longrightarrow A * (B * C)}\ L}{(A * B) * C \longrightarrow A * (B * C)}\ L$$

Proof of soundness (mildly satisfying)

Key lemmas about $\phi[-]$:

- "colaxity": $\phi[\Gamma, \Delta] \leq \phi[\Gamma] * \phi[\Delta]$
- $\phi[\Gamma, \Delta] = \phi[\Gamma] \circledast \Delta$, where the (monotonic) right action $- \circledast \Delta$ is defined by $A \circledast (B_1, \ldots, B_n) = ((A * B_1) \cdots) * B_n$

Soundness follows by induction on derivations...

(Case *id*): by reflexivity.

(Case $*L$): $\phi[A * B, \Gamma] = \phi[A, B, \Gamma] \leq C$

(Case $*R$): $\phi[\Gamma, \Delta] \leq \phi[\Gamma] * \phi[\Delta] \leq A * B$

(Case *cut*): $\phi[\Gamma, \Theta, \Delta] = \phi[\Gamma, \Theta] \circledast \Delta \leq (\phi[\Gamma] * \phi[\Theta]) \circledast \Delta \leq (\phi[\Gamma] * A) \circledast \Delta = \phi[\Gamma, A, \Delta] \leq B$

Focusing completeness

Definition

A context Γ is said to be **reducible** if its leftmost formula is compound, and **irreducible** otherwise. A sequent $\Gamma \longrightarrow A$ is

- **left-inverting** if Γ is reducible;
- **right-focusing** if Γ is irreducible and A is compound;
- **atomic** if Γ is irreducible and A is atomic.

Definition

A closed derivation $\mathcal{D}$ is said to be **focused** if left-inverting sequents only appear as the conclusions of $*L$, right-focusing sequents only as the conclusions of $*R$, and atomic sequents only as the conclusions of *id*.

Focusing completeness

Proposition

A closed derivation is focused iff it is constructed using only $*L$ and the following restricted forms of $*R$ and id (and no cut):

$$\frac{A, B, \Delta \longrightarrow C}{A * B, \Delta \longrightarrow C}\ *L \qquad \frac{\Gamma^{\mathrm{irr}} \longrightarrow A \quad \Delta \longrightarrow B}{\Gamma^{\mathrm{irr}}, \Delta \longrightarrow A * B}\ *R^{\mathrm{foc}} \qquad \frac{}{p \longrightarrow p}\ id^{\mathrm{atm}}$$

Theorem (Focusing completeness)

Every derivable sequent has a focused derivation.

Proof not hard. (Show admissibility of *cut*, $*R$, and *id* by standard inductions – no surprises other than that it works!)

The coherence theorem

Lemma

For any context Γ and formula A, there is at most one focused derivation of $\Gamma \longrightarrow A$.

Corollary (Coherence)

Every derivable sequent has exactly one focused derivation.

One application: a simple decision procedure for $A \overset{\mathrm{Tam}}{\leq} B$

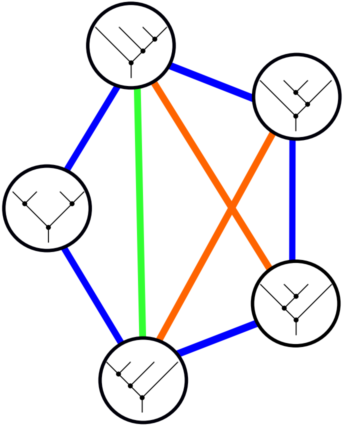
More exciting application: counting intervals!

Counting intervals in Tamari lattices

Theorem (Chapoton 2006)

Let $\mathcal{I}_n = \{ (A, B) \in \mathsf{Y}_n \times \mathsf{Y}_n \mid A \overset{\mathrm{Tam}}{\leq} B \}$. *Then* $|\mathcal{I}_n| = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$.

For example, Y_3 contains 13 intervals:



5 + 5 + 2 + 1 = 13

Counting intervals in Tamari lattices

The original proof is in:

- F. Chapoton, "Sur le nombre d'intervalles dans les treillis de Tamari," *Sém. Lothar. Combin.*, no. B55f, 2006

Chapoton mentions that he found the formula through the OEIS (see oeis.org/A000260) before he was able to prove it.

The formula itself was first derived over half a century ago by Bill Tutte, but for a completely different family of objects!

- W. T. Tutte, "A census of planar triangulations," *Canad. J. Math.*, vol. 14, pp. 21–38, 1962

Tutte proved that $\frac{2(4n+1)!}{(n+1)!(3n+2)!}$ is the number of rooted 3-connected **triangulations of the sphere** with $3(n+1)$ edges.

Counting intervals in Tamari lattices

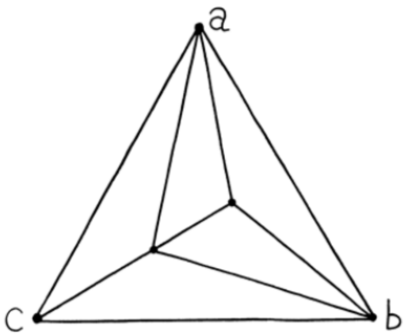


FIGURE 2

We shall prove that

$$(1.5) \qquad \psi_{n,0} = \frac{2}{(n+1)!}(3n+3)(3n+4)\ldots(4n+1)$$

when $n \geqslant 2$. Our main objective in this paper is the complete evaluation of the function $\psi_{n,m}$ (§ 5).

Counting intervals in Tamari lattices

Chapoton's observation sparked combinatorialists to look for (and find) bijective explanations (and extensions) of these connections between *planar maps* and Tamari intervals, see e.g.:

- O. Bernardi and N. Bonichon, "Intervals in Catalan lattices and realizers of triangulations," *J. Combin. Theory Ser. A*, vol. 116, no. 1, pp. 55–75, 2009.
- F. Chapoton, G. Châtel, and V. Pons, "Two bijections on Tamari intervals," In *Proceedings of the 26th International Conference on Formal Power Series and Algebraic Combinatorics*, pp. 241–252, 2014.
- W. Fang, "Planar triangulations, bridgeless planar maps and Tamari intervals," arXiv:1611.07922, 2016.

Counting intervals in Tamari lattices

Outline of our proof of Chapoton's theorem ($|\mathcal{I}_n| = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$):

1. Observe # intervals = # focused derivations (**by coherence**)
2. Consider generating functions $L(z,x)$ and $R(z,x)$ counting focused derivations of $\Gamma \to A$ (resp. $\Gamma^{\mathrm{irr}} \to A$) by size($A$) and length($\Gamma$). The following eqns are essentially immediate:

$$\begin{aligned} L(z,x) &= (L(z,x) - xL_1(z))/x + R(z,x) \\ &= x\frac{R(z,x) - R(z,1)}{x-1} \qquad (1) \\ R(z,x) &= zR(z,x)L(z,x) + x \qquad (2) \end{aligned}$$

3. Use "off-the-shelf" algebraic combinatorics to solve (1) & (2), obtaining the Tutte–Chapoton formula for coeff. of z^n in

$$R(z,1) = 1 + z + 3z^2 + 13z^3 + 68z^4 + 399z^5 + \dots$$

Aside: the surprising combinatorics of linear lambda calculus

My original motivation for this work was wanting to better understand an apparent link between the Tamari order and lambda calculus, inferred indirectly via their *mutual* connection to the combinatorics of embedded graphs.[4]

[4] I'll be speaking about this a lot more tomorrow at the LFCS seminar!

Aside: the surprising combinatorics of linear lambda calculus

family of lambda terms	family of rooted maps	OEIS
linear terms[1,4]	trivalent maps	A062980
planar terms[4]	planar trivalent maps	A002005
unit-free linear[4]	bridgeless trivalent	A267827
unit-free planar[4]	bridgeless planar trivalent	A000309
normal linear terms/$\sim$[3]	(combinatorial) maps	A000698
normal planar terms[2]	planar maps	A000168
normal unit-free linear/$\sim$[5]	bridgeless maps	A000699
normal unit-free planar	bridgeless planar	A000260

1. Olivier Bodini, Danièle Gardy, and Alice Jacquot, *TCS* 502, 2013.

2. Z, Alain Giorgetti, *LMCS* 11(3:22), 2015.

3. Z, arXiv:1509.07596, 2015.

4. Z, *JFP* 26(e21), 2016.

5. Julien Courtiel, Karen Yeats, and Z, arXiv:1611.04611, 2017.

Aside: the surprising combinatorics of linear lambda calculus

An explicit (albeit somewhat roundabout) bijection between unit-free normal planar terms and Tamari intervals was given in an earlier, longer version of the paper (arXiv:1701.02917).

Conceptually, this link seems closely related to the duality between **skew-monoidal categories** and **skew-closed categories**.

- Kornél Szlachányi. Skew-monoidal categories and bialgebroids. *Advances in Math.*, 231(3–4):1694–1730, 2012.
- Ross Street. Skew-closed categories. *J. Pure and Appl. Alg.*, 217(6):973–988, 2013.

Conclusions and questions

We have a natural encoding of semi-associativity in sequent calculus, with a surprising application to combinatorics.

The simplicity of the solution suggests natural questions and directions for research:

- Is the SC helpful for understanding lattice structure of Y_n?
- Interaction with other connectives.[5]
- Formalizing categorical coherence theorems (cf. Uustalu 2014)
- Linguistic motivations? (cf. Lambek '58 & '61.) Applications to (LL/LR) parsing? (cf. Thielecke '12 & '13.)
- Other bridges between proof theory and combinatorics?

[5] Cf. Jason Reed's "Queue logic: An undisplayable logic?" (unpublished manuscript, April 2009), jcreed.org/papers/queuelogic.pdf.

Postscript: a missed connection

From "Sur quelques problèmes d'associativité" (Tamari, 1964):

> *En 1951, après sa thèse [21] et après la publication de [12],[6] l'auteur a proposé à LAMBEK un travail commun, pour mettre en évidence le rôle prépondérant joué par l'associativité générale. Malheureusement, par suite de circonstances extérieures, ce travail n'a jamais été écrit.*
>
> *In 1951, after his thesis [21] and after the publication of [12], the author proposed to Lambek joint work, to highlight the important role played by general associativity. Unfortunately, due to external circumstances, this work has never been written.*

[6] [12] = J. Lambek, "The immersibility of a semigroup into a group", *Canad. J. of Math.*, vol. 3, pp. 34–43, 1951.