H*-algebras and nonunital Frobenius algebras: first steps in infinite-dimensional categorical quantum mechanics

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Abstract

A certain class of Frobenius algebras has been used to characterize orthonormal bases and observables on finite-dimensional Hilbert spaces. The presence of units in these algebras means that they can only be realized finite-dimensionally. We seek a suitable generalization, which will allow arbitrary bases and observables to be described within categorical axiomatizations of quantum mechanics. We develop a definition of H*-algebra that can be interpreted in any symmetric monoidal dagger category, reduces to the classical notion from functional analysis in the category of (possibly infinite-dimensional) Hilbert spaces, and hence provides a categorical way to speak about orthonormal bases and quantum observables in arbitrary dimension. Moreover, these algebras reduce to the usual notion of Frobenius algebra in compact categories. We then investigate the relations between nonunital Frobenius algebras and H*-algebras. We give a number of equivalent conditions to characterize when they coincide in the category of Hilbert spaces. We also show that they always coincide in categories of generalized relations and positive matrices.

1 Introduction

The context for this paper comes from the ongoing work on *categorical quantum mechanics* [5, 6]. This work has shown how large parts of quantum mechanics can be axiomatized in terms of monoidal dagger categories and structures definable within them. This axiomatization can be used to perform high-level reasoning and calculations relating to quantum information, using diagrammatic methods [35]; and also as a basis for exploring foundational issues in quantum

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mechanics and quantum computation. In particular, a form of Frobenius algebras has been used to give an algebraic axiomatization of *orthonormal bases* and *observables* [18, 19].

The structures used so far (*e.g.* compact closure, Frobenius algebras) have only finite-dimensional realizations in Hilbert spaces. This raises some interesting questions and challenges:

- Find a good general notion of Frobenius structure which works in the infinite-dimensional case in **Hilb**.
- Use this to characterize general bases and observables.
- Similarly extend the analysis for other categories.
- Clarify the mathematics, and relate it to the wider literature.

As we shall see, an intriguing problem remains open, but much of this program of work has been accomplished.

The further contents of the paper are as follows. Section 2 recalls some background on monoidal dagger categories and Frobenius algebras, and poses the problem. Section 3 introduces the key notion of H*-algebra, in the general setting of symmetric monoidal dagger categories. In Section 4, we prove our results relating to **Hilb**, the category of Hilbert spaces (of unrestricted dimension). We show how H*-algebras provide exactly the right algebraic notion to characterize orthonormal bases in arbitrary dimension. We give several equivalent characterizations of when H*-algebras and nonunital Frobenius algebras coincide in the category of Hilbert spaces. Section 5 studies H*-algebras in categories of generalized relations and positive matrices. We show that in these settings, where no phenomena of 'destructive interference' arise, H*-algebras and nonunital Frobenius algebras always coincide. Finally, Section 6 provides an outlook for future work.

2 Background

The basic setting is that of *dagger symmetric monoidal categories.* We briefly recall the definitions, referring to [6] for further details and motivation.

A *dagger category* is a category $\mathbf{D}$ equipped with an identity-on-objects, contravariant, strictly involutive functor. Concretely, for each arrow $f\colon A \to B$, there is an arrow $f^\dagger\colon B \to A$, and this assignment satisfies

$$\mathrm{id}^\dagger = \mathrm{id}, \qquad (g \circ f)^\dagger = f^\dagger \circ g^\dagger, \qquad f^{\dagger\dagger} = f\,.$$

An arrow $f\colon A \to B$ is *dagger monic* when $f^\dagger \circ f = \mathrm{id}_A$, and a *dagger iso(morphism)* if both f and $f^\dagger$ are dagger monics.

A *symmetric monoidal dagger category* is a dagger category with a symmetric monoidal structure $(\mathbf{D}, \otimes, I, \lambda, \rho, \alpha, \sigma)$ such that

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$$

and moreover the natural isomorphisms λ, ρ, α, σ are componentwise dagger isomorphisms.

Examples

- The category **Hilb** of Hilbert spaces and continuous linear maps, and its (full) subcategory **fHilb** of finite-dimensional Hilbert spaces. Here the dagger is the adjoint, and the tensor product has its standard interpretation for Hilbert spaces. Dagger isomorphisms are *unitaries*, and dagger monics are *isometries*.

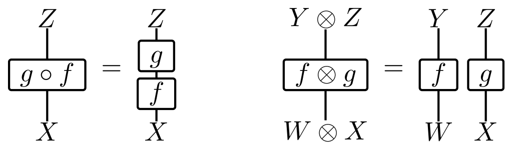
- The category **Rel** of sets and relations. Here the dagger is relational converse, while the monoidal structure is given by the cartesian product. This generalizes to relations valued in a commutative quantale [34], and to the category of relations of any regular category [15]. This has a full sub-category **fRel**, of finite sets and relations.

- The category **lbfRel**, of *locally bifinite relations*. This is the subcategory of **Rel** comprising those relations which are image-finite, meaning that each element in the domain is related to only finitely many elements in the codomain, and whose converses are also image-finite. This forms a monoidal dagger subcategory of **Rel**. It serves as a kind of qualitative approximation of the passage from finite- to infinite-dimensional Hilbert spaces. For example, a set carries a compact structure in **lbfRel** if and only if it is finite.

- A common generalization of **fHilb** and **fRel** is obtained by forming the category $\mathbf{Mat}(S)$, where S is a commutative semiring with involution [22]. The objects of $\mathbf{Mat}(S)$ are finite sets, and morphisms are maps $X \times Y \to S$, which we think of as 'X times Y matrices'. Composition is by matrix multiplication, while the dagger is conjugate transpose, where conjugation of a matrix means elementwise application of the involution on S. The tensor product of X and Y is given by $X \times Y$, with the action on matrices given by componentwise multiplication, corresponding to the 'Kronecker product' of matrices. If we take $S = \mathbb{C}$, this yields a category equivalent to **fHilb**, while taking S to be the Boolean semiring $\{0, 1\}$, with trivial involution, gives **fRel**.

- An infinitary generalization of $\mathbf{Mat}(\mathbb{C})$ is given by $\mathbf{Mat}_{\ell^2}(\mathbb{C})$. This category has arbitrary sets as objects, and as morphisms matrices $M \colon X \times Y \to \mathbb{C}$ such that for each $x \in X$, the family $\{M(x, y)\}_{y \in Y}$ is ℓ^2-summable; and for each $y \in Y$, the family $\{M(x, y)\}_{x \in X}$ is ℓ^2-summable. The category **Hilb** is equivalent to a (nonfull) subcategory of $\mathbf{Mat}_{\ell^2}(\mathbb{C})$ [12, Theorem 3.1.7].

Graphical Calculus We also briefly recall the graphical calculus for symmetric monoidal dagger categories [35]. This can be seen as a two-dimensional version of *Dirac notation*, which allows equational reasoning to be performed graphically in a sound and complete fashion. A morphism $f\colon X \to Y$ is represented as $\underset{X}{\overset{Y}{\boxed{f}}}$, the identity on X simply becomes $\underset{X}{\overset{X}{|}}$, and composition and tensor products appear as follows.



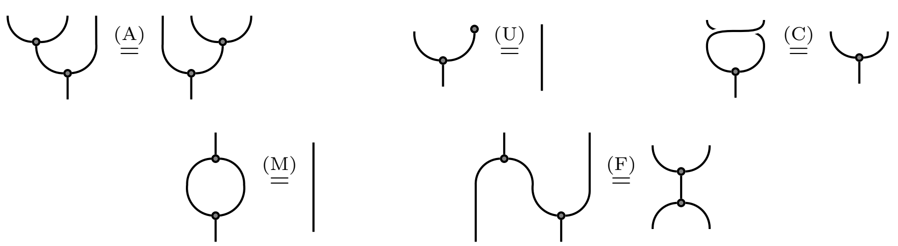
The symmetry isomorphism σ is drawn as ⤫. The dagger is represented graphically by a horizontal reflection.

2.1 Dagger Frobenius algebras

Frobenius algebras are a classic notion in mathematics [31]. A particular form of such algebras was introduced in the general setting of monoidal dagger categories by Coecke and Pavlović in [18]. In their version, a *dagger Frobenius structure* on an object A in a dagger monoidal category is a commutative comonoid $(I \xleftarrow{\varepsilon} A \xrightarrow{\delta} A \otimes A)$ satisfying certain additional equations:

$$
\begin{aligned}
(\mathrm{id}_A \otimes \delta) \circ \delta &= (\delta \otimes \mathrm{id}_A) \circ \delta, && \text{(A)}\\
(\mathrm{id}_A \otimes \varepsilon) \circ \delta &= \mathrm{id}_A, && \text{(U)}\\
\sigma \circ \delta &= \delta, && \text{(C)}\\
\delta^\dagger \circ \delta &= \mathrm{id}_A, && \text{(M)}\\
\delta \circ \delta^\dagger &= (\delta^\dagger \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \delta). && \text{(F)}
\end{aligned}
$$

These equations become more perspicuous when represented diagrammatically, as below. Here, we draw the comultiplication δ as ⑂, and the counit ε as ⫯.



A 'right-handed version' of the Frobenius law (F) follows from (C); in the non-commutative case we should add this symmetric version to axiom (F).

2.2 Dagger Frobenius algebras in quantum mechanics

Frobenius algebras provide a high-level algebraic way of talking about *orthonormal bases*, and hence can be seen as modeling quantum mechanical *observables*.

To put this in context, we recall the *no-cloning theorem* [38], which says that there is no quantum evolution (*i.e.* unitary operator) $f\colon H \to H \otimes H$ such that, for any $|\phi\rangle \in H$,

$$f|\phi\rangle = |\phi\rangle \otimes |\phi\rangle.$$

A general form of no-cloning holds for structural reasons in categorical quantum mechanics [3]. In particular, there is no *natural*, *i.e.* uniform or basis-independent, family of diagonal morphisms in a compact closed category, unless the category collapses, so that endomorphisms are scalar multiples of the identity.

However, if we drop naturality, we *can* define such maps in **Hilb** in a basis-dependent fashion. Moreover, it turns out that such maps can be used to *uniquely determine* bases. Firstly, consider *copying maps*, which can be defined in arbitrary dimension: for a given basis $\{|i\rangle\}_{i\in I}$ of H, define $\delta\colon H \to H \otimes H$ by (continuous linear extension of) $|i\rangle \mapsto |ii\rangle$.

For example, consider the map $\delta_{\mathrm{std}}\colon \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ defined by

$$|0\rangle \mapsto |00\rangle, \qquad |1\rangle \mapsto |11\rangle.$$

By construction, this copies the elements of the computational basis — and *only* these, as in general

$$\delta_{\mathrm{std}}(\alpha|0\rangle + \beta|1\rangle) = \alpha|00\rangle + \beta|11\rangle \neq (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle).$$

Next, consider *deleting maps* $\varepsilon\colon H \to \mathbb{C}$ defined by (linear extension of) $|e_i\rangle \mapsto 1$. In contrast to copying, these can be defined in *finite dimension only*. It is straightforward to verify that these maps define a dagger Frobenius structure on H. Moreover, the following result provides a striking converse.

Theorem 1 *[19] Orthonormal bases of a finite-dimensional Hilbert space H are in one-to-one correspondence with dagger Frobenius structures on H.* □

This result in fact follows easily from previous results in the literature on Frobenius algebras [1]; we will give a short proof from the established literature in Section 4.4.

Another result provides a counterpart—at first sight displaying very different looking behaviour—in the category **Rel**.

Theorem 2 *[32] Dagger Frobenius structures in the category* **Rel** *correspond to disjoint unions of abelian groups.* □

We shall provide a different proof of this result in Section 5.1, which makes no use of units, and hence generalizes to a wide range of other situations, such as locally bifinite and quantale-valued relations, and positive ℓ_2-matrices.

2.3 The problem

The notion of Frobenius structure as defined above, which requires a unit, limits us to the *finite-dimensional case* in **Hilb**, as the following lemma shows.

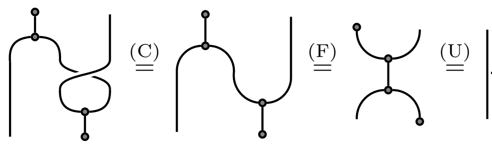
Lemma 3 *A Frobenius algebra in* **Hilb** *is unital if and only if it is finite-dimensional.*

Proof Sufficiency is shown in [29, 3.6.9], while necessity follows from [27, Corollary to Theorem 4]. □

In fact, a Frobenius structure on an object A gives rise to a *compact* (or *rigid*) structure on A, with A as its own dual (see [6]). Indeed, define $\eta = \delta \circ \varepsilon^\dagger \colon I \to A \otimes A$. In the category **fHilb**, for example, $\eta \colon \mathbb{C} \to \mathbb{C}^2 \otimes \mathbb{C}^2$ is an *entangled state preparation*:

$$\eta_{\mathrm{std}} = \delta_{\mathrm{std}} \circ \varepsilon^\dagger_{\mathrm{std}} = (1 \mapsto \delta_{\mathrm{std}}(|0\rangle + |1\rangle)) = (1 \mapsto |00\rangle + |11\rangle).$$

In general it is easy to see that η indeed provides a dagger compact structure on A, with $A^* = A$:



As is well-known, a compact structure exists only for finite-dimensional spaces in **Hilb**. Thus to obtain a notion capable of being extended beyond the finite-dimensional case, we need to drop the assumption of a unit.

3 H*-algebras

We begin our investigation of suitable axioms for a notion of algebra which can characterize orthonormal bases in arbitrary dimension by recalling the axioms for Frobenius structures.

$$(\mathrm{id}_A \otimes \delta) \circ \delta = (\delta \otimes \mathrm{id}_A) \circ \delta \tag{A}$$

$$(\mathrm{id}_A \otimes \varepsilon) \circ \delta = \mathrm{id}_A \tag{U}$$

$$\sigma \circ \delta = \delta \tag{C}$$

$$\delta^\dagger \circ \delta = \mathrm{id}_A \tag{M}$$

$$\delta \circ \delta^\dagger = (\delta^\dagger \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \delta) \tag{F}$$

We note in passing that there is some redundancy in the definition of Frobenius structure.

Lemma 4 *In any dagger monoidal category, (M) and (F) imply (A).*

Proof First, four applications of (F) yield

$$\begin{aligned}(\delta \otimes \mathrm{id}) \circ \delta \circ \delta^\dagger &= (\delta \otimes \mathrm{id}) \circ (\delta^\dagger \circ \mathrm{id}) \circ (\mathrm{id} \otimes \delta) \\ &= (\mathrm{id} \otimes \delta^\dagger \otimes \mathrm{id}) \circ (\delta \otimes \delta) \\ &= (\mathrm{id} \otimes \delta) \circ (\mathrm{id} \otimes \delta^\dagger) \circ (\delta \otimes \mathrm{id}) \\ &= (\mathrm{id} \otimes \delta) \circ \delta \circ \delta^\dagger.\end{aligned}$$

So by (M), we have $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta$. $\square$

The other axioms of a Frobenius algebra—(F), (C) and (M)—are independent:

- Group algebras of noncommutative groups [8, Example 4] satisfy everything except for (C).
- Any nontrivial commutative Hopf algebra satisfies everything except for (F).
- The trivial algebra, given by $\delta(a) = 0$, satisfies everything except for (M).[1]

It is worth noting that under additional assumptions, such as unitality and enrichment in abelian groups, (A) and (M) are known to imply (F) [30, Section 6].

We shall now *redefine* a Frobenius algebra[2] in a dagger monoidal category to be an object A equipped with a comultiplication $\delta : A \to A \otimes A$ satisfying (A), (C), (M) and (F). A Frobenius algebra which additionally has an arrow $\varepsilon \colon A \to I$ satisfying (U) will explicitly be called *unital*.

3.1 Regular representation as pointwise abstraction

As we have seen, unital Frobenius algebras allow us to define compact, and hence closed, structure. How much of this can we keep in key examples such as **Hilb**?

The category **Hilb** has well-behaved duals, since $H \cong H^{**}$, and indeed there is a conjugate-linear isomorphism $H \cong H^*$. However, it is *not* the case that the tensor unit $\mathbb{C}$ is exponentiable in **Hilb**, since if it was, we would have a bounded linear evaluation map

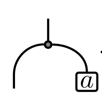
$$H \otimes H^* \to \mathbb{C},$$

and hence its adjoint $\mathbb{C} \to H \otimes H^*$, and a compact structure.

We shall now present an axiom which captures what seems to be the best we can do in general in the way of a 'transfer of variables'. It is, indeed, a general form, meaningful in any monoidal dagger category, of a salient structure in functional analysis.

Suppose we have a comultiplication $\delta \colon A \to A \otimes A$, and hence a multiplication $\mu = \delta^\dagger \colon A \otimes A \to A$. We can *curry* the multiplication (this process is also called λ-abstraction [10]) for *points*—this is just the regular representation![3] Thus we have a function $R \colon \mathbf{D}(I, A) \to \mathbf{D}(A, A)$ defined by

$$R(a) = \mu \circ (\mathrm{id} \otimes a) =$$



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[1] This example falsifies the second entry in the table on page 11 of [19], which should be amended with the requirement that δ is monic. Together with [8], this also shows that $\ker(\delta) = 0$, *i.e.* δ being monic, is the weakest requirement for which one can prove that entry.

[2] In the literature the unital version is more specifically termed a special commutative dagger Frobenius algebra (sometimes also called a separable algebra, or a Q-system). As we will only be concerned with these kinds of Frobenius algebras, we prefer to keep terminology simple and dispense with the adjectives.

[3] As we are in a commutative context, there is no need to distinguish between left and right regular representations.

If μ is associative, this is a semigroup homomorphism.

3.2 Axiom (H)

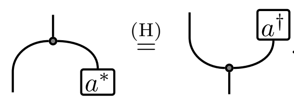
An endomorphism homset $\mathbf{D}(A, A)$ in a dagger category $\mathbf{D}$ is not just a monoid, but a *monoid with involution*, because of the dagger. We say that (A, μ) *satisfies axiom (H)* if there is an operation $a \mapsto a^*$ on $\mathbf{D}(I, A)$ such that R becomes a homomorphism of involutive semigroups, *i.e.*

$$R(a^*) = R(a)^\dagger$$

for every $a \colon I \to A$. This unfolds to

$$\mu \circ (a^* \otimes \mathrm{id}) = (a^\dagger \otimes \mathrm{id}) \circ \mu^\dagger; \tag{H}$$

or diagrammatically:



Thus $a \mapsto a^*$ is indeed a 'transfer of variables'.

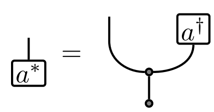
3.3 Relationships between axioms (F) and (H)

The rest of this section compares axioms (F) and (H) at the abstract level of monoidal dagger categories.

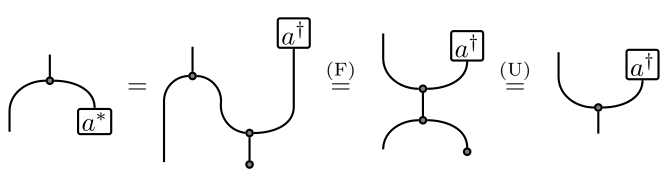
The following observation by Coecke, Pavlović and Vicary is the central idea in their proof of Theorem 1.

Lemma 5 *In any dagger monoidal category, (F) and (U) imply (H).*

Proof Define $a^* = (a^\dagger \otimes \mathrm{id}) \circ \delta \circ \varepsilon^\dagger$.



This indeed satisfies (H).



□

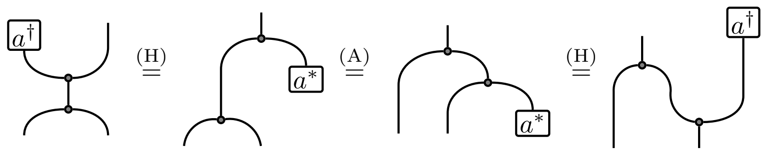
Recall that a category is *well-pointed* if the following holds:

$$f = g \colon A \to B \iff f \circ x = g \circ x \text{ for all } x \colon I \to A. \tag{WP}$$

All the categories listed in our Examples are well-pointed in this sense.

Lemma 6 *In a well-pointed dagger monoidal category, (H) and (A) imply (F).*

Proof For any $a\colon I \to A$ we have the following.

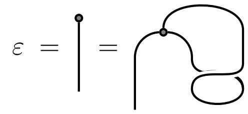


Then (F) follows from well-pointedness. □

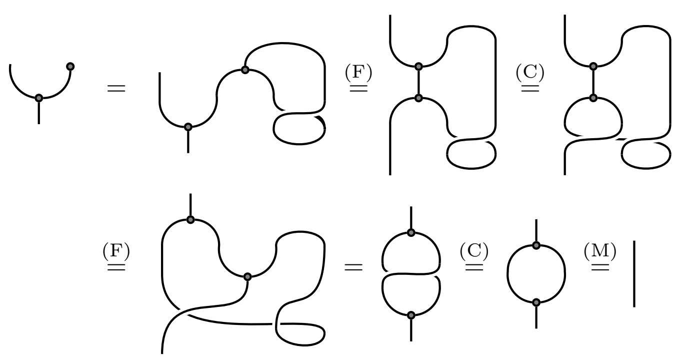
Lemma 5 is strengthened by the following proposition, which proves that compactness implies unitality.

Proposition 7 *Any Frobenius algebra in a dagger compact category is unital.*

Proof [14, Remark (1) on page 503] Suppose that $\delta\colon A \to A \otimes A$ is a nonunital Frobenius algebra in a compact category. Define $\varepsilon\colon A \to I$ as follows.



Then the following holds, where we draw the unit and counit of compactness by caps and cups (without dots).



That is, (U) holds. □

Thus in the unital, well-pointed case, (F) and (H) are essentially equivalent. Our interest is, of course, in the nonunital case. To explain the provenance of the (H) axiom, and its implications for obtaining a correspondence with orthonormal bases in **Hilb** in arbitrary dimension, we shall now study the situation in the concrete setting of Hilbert spaces.

4 H*-algebras in Hilb

We begin by revisiting Theorem 1. How should the correspondence between Frobenius algebras and orthonormal bases be expressed mathematically? In fact, the content of this result is really a *structure theorem* of a classic genre in algebra [7]. The following theorem, the *Wedderburn structure theorem*, is the prime example; it was subsequently generalized by Artin, and there have been many subsequent developments.

Theorem 8 (Wedderburn, 1908) *Every finite-dimensional semisimple algebra is isomorphic to a product of full matrix algebras. In the commutative case over the complex numbers, this has the form: the algebra is isomorphic to a product of one-dimensional complex algebras.* □

To see the connection between the Wedderburn structure theorem and Theorem 1, consider the coalgebra A determined by an orthonormal basis $\{|i\rangle\}$ on a Hilbert space:

$$\delta\colon |i\rangle \mapsto |ii\rangle.$$

This is isomorphic as a coalgebra to a direct sum of one-dimensional coalgebras

$$\delta_{\mathbb{C}}\colon \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}, \qquad 1 \mapsto 1 \otimes 1.$$

To say that a Frobenius algebra corresponds to an orthonormal basis is exactly to say that it is isomorphic as a coalgebra to a Hilbert space direct sum of one-dimensional coalgebras:

$$A \;\cong\; \bigoplus_I (\mathbb{C}, \delta_{\mathbb{C}}),$$

where the cardinality of I is the dimension of H. Applying dagger, this is equivalent to A being isomorphic as an *algebra* to the direct sum of one-dimensional *algebras*

$$A \;\cong\; \bigoplus_I (\mathbb{C}, \mu_{\mathbb{C}}), \qquad \mu_{\mathbb{C}}\colon 1 \otimes 1 \mapsto 1.$$

In this case, we say that the Frobenius algebra *admits the structure theorem*, making the view of bases as (co)algebras precise.

4.1 H*-algebras

There is a remarkable generalization of the Wedderburn structure theorem to an infinite-dimensional setting, in a classic paper from 1945 by Warren Ambrose on 'H*-algebras' [8]. He defines an H*-algebra[4] as a (not necessarily unital) Banach algebra based on a Hilbert space H, such that for each $x \in H$ there is an $x^* \in H$ with

$$\langle xy \,|\, z\rangle = \langle y \,|\, x^* z\rangle$$

[4]This is unrelated to the notion termed 2-H*-algebra in [9].

for all $y, z \in H$, and similarly for right multiplication. Note that

$$\langle xy \,|\, z\rangle = (\mu \circ (x \otimes y))^{\dagger} \circ z = (x^{\dagger} \otimes y^{\dagger}) \circ \mu^{\dagger} \circ z,$$

where we identify points $x \in H$ with morphisms $x\colon \mathbb{C} \to H$, and similarly

$$\langle y \,|\, x^* z\rangle = y^{\dagger} \circ \mu \circ (x^* \otimes z).$$

Using the well-pointedness of **Hilb**, it is easy to see that this is equivalent to the (H) condition! The following two lemmas show that the assumptions (A), (C), (M) and (H) indeed result in an H*-algebra.

Lemma 9 *A monoid in* **Hilb** *satisfying (M) is a Banach algebra.*

Proof The condition (M) implies that $P = \mu^{\dagger} \circ \mu$ is a projector:

$$P^2 = \mu^{\dagger} \circ \mu \circ \mu^{\dagger} \circ \mu = \mu^{\dagger} \circ \mu = P$$

and clearly $P = P^{\dagger}$. Hence a monoid in **Hilb** satisfying (M) is a Banach algebra, since:

$$\begin{aligned} \|xy\|^2 &= \langle xy \,|\, xy\rangle \\ &= (x^{\dagger} \otimes y^{\dagger}) \circ \mu^{\dagger} \circ \mu \circ (x \otimes y) \\ &= \langle x \otimes y \,|\, P(x \otimes y)\rangle \\ &\leq \langle x \otimes y \,|\, x \otimes y\rangle \\ &= \langle x \,|\, x\rangle \langle y \,|\, y\rangle \\ &= \|x\|^2 \|y\|^2. \end{aligned}$$

□

Remark In fact, it can be shown that the multiplication of a semigroup in **Hilb** satisfying (H) is automatically continuous, so that after adjusting by a constant, the semigroup is a Banach algebra [26, Corollary 2.2].[5]

The following lemma establishes *properness*, which corresponds to x^* being the unique vector with the property defining H*-algebras. It follows that $R(x^*)$ is the adjoint of $R(x)$.

Lemma 10 *Suppose $\delta\colon A \to A \otimes A$ in* **Hilb** *satisfies (A) and (H). Then (M) implies properness, i.e. $aA = 0 \Rightarrow a = 0$. Hence (M) holds if and only if the regular representation is monic.*

Proof By [8, Theorem 2.2], A is the direct sum of its trivial ideal A' and a proper H*-algebra A''. Here, the trivial ideal is $A' = \{a \in A \mid aA = 0\}$. Since the direct sum of Hilbert spaces is a dagger biproduct, we have $\delta = \delta' \oplus \delta''\colon A \to A \otimes A$, where $\delta'\colon A' \to A' \otimes A'$ and $\delta''\colon A'' \to A'' \otimes A''$. The latter two morphisms are again dagger monic as a consequence of (M). So the multiplication $\delta'^{\dagger}$ of A' is epic, which forces $A' = 0$. □

[5]Hence Proposition 11 and Theorem 12 can be altered to show that a monoid in **Hilb** satisfying properness, (A), (C), and (H) (but not necessarily (M)!), corresponds to an *orthogonal* basis. This may have consequences for attempts to classify multipartite entanglement according to various Frobenius structures [17].

The following proposition summarizes the preceding discussion.

Proposition 11 *Any structure (A, μ) in* **Hilb** *satisfying (A), (H) and (M) is an H*-algebra (and also satisfies (F)); and conversely, an H*-algebra satisfies (A), (H) and (M), and hence also (F).* □

Ambrose proved a complete structure theorem for H*-algebras, of which we now state the commutative case.

Theorem 12 (Ambrose, 1945) *A proper commutative H*-algebra (of arbitrary dimension) is isomorphic to a Hilbert space direct sum of one-dimensional algebras.* □

This is equivalent to asserting isomorphism qua coalgebras. So it is exactly the result we are after! Rather than relying on Ambrose's results, we now give a direct, conceptual proof, using a few notions from Gelfand duality for commutative Banach algebras.

4.2 Copyables and semisimplicity

A *copyable element* of a semigroup $\delta \colon A \to A \otimes A$ in a monoidal category is a semigroup homomorphism to it from the canonical semigroup on the monoidal unit. More precisely, a copyable element is a morphism $a \colon I \to A$ such that $(a \otimes a) \circ \delta = \delta \circ a$. In a well-pointed category such as **Hilb**, we can speak of a copyable element of δ as a point $a \in A$ with $\delta(a) = a \otimes a$.[6]

Proposition 13 *Assuming only (A), nonzero copyable elements are linearly independent.*

Proof [24, Theorem 10.18(ii)] Suppose $\{a_0, \dots, a_n\}$ is a minimal nonempty linearly dependent set of nonzero copyables. Then $a_0 = \sum_{i=1}^{n} \alpha_i a_i$ for suitable coefficients $\alpha_i \in \mathbb{C}$. So

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i (a_i \otimes a_i) &= \sum_{i=1}^{n} \alpha_i \delta(a_i) \\ &= \delta(a_0) \\ &= (\sum_{i=1}^{n} \alpha_i a_i) \otimes (\sum_{j=1}^{n} \alpha_j a_j) \\ &= \sum_{i,j=1}^{n} \alpha_i \alpha_j (a_i \otimes a_j).\end{aligned}$$

By minimality, $\{a_1, \dots, a_n\}$ is linearly independent. Hence $\alpha_i^2 = \alpha_i$ for all i, and $\alpha_i \alpha_j = 0$ for $i \neq j$. So $\alpha_i = 0$ or $\alpha_i = 1$ for all i. If $\alpha_j = 1$, then $\alpha_i = 0$ for

[6]Copyable elements are also called *primitive* in the context of C*-bigebras [24], and *grouplike* in the study of Hopf algebras [37, 28].

all $i \neq j$, so $a_0 = a_j$. By minimality, then $j = 1$ and $\{a_0, a_j\} = \{a_0\}$, which is impossible. So we must have $\alpha_i = 0$ for all i. But then $a_0 = 0$, which is likewise a contradiction. □

Proposition 14 *Assuming only (M), nonzero copyable elements have unit norm.*

Proof If $\delta(a) = a \otimes a$, then $\|a\| = \|\delta(a)\| = \|a \otimes a\| = \|a\|^2$. □

Proposition 15 *Assuming only (F), copyable elements are pairwise orthogonal.*

Proof [19, Corollary 4.7] Let a, b be copyables. Then:

$$\begin{aligned}
\langle a \,|\, a\rangle \cdot \langle a \,|\, a\rangle \cdot \langle b \,|\, a\rangle &= \langle a \otimes a \otimes b \,|\, a \otimes a \otimes a\rangle \\
&= \langle(\delta \otimes \mathrm{id})(a \otimes b) \,|\, (\mathrm{id} \otimes \delta)(a \otimes a)\rangle \\
&= \langle a \otimes b \,|\, (\delta^\dagger \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta)(a \otimes a)\rangle \\
&= \langle a \otimes b \,|\, (\mathrm{id} \otimes \delta^\dagger) \circ (\delta \otimes \mathrm{id})(a \otimes a)\rangle \\
&= \langle(\mathrm{id} \otimes \delta)(a \otimes b) \,|\, (\delta \otimes \mathrm{id})(a \otimes a)\rangle \\
&= \langle a \otimes b \otimes b \,|\, a \otimes a \otimes a\rangle \\
&= \langle a \,|\, a\rangle \cdot \langle b \,|\, a\rangle \cdot \langle b \,|\, a\rangle.
\end{aligned}$$

Analogously $\langle b \,|\, b\rangle\langle b \,|\, b\rangle\langle a \,|\, b\rangle = \langle b \,|\, b\rangle\langle a \,|\, b\rangle\langle a \,|\, b\rangle$. Hence, if $\langle a \,|\, a\rangle \neq 0$ and $\langle b \,|\, a\rangle \neq 0$, then $\langle a \,|\, a\rangle = \langle b \,|\, a\rangle$ and $\langle b \,|\, b\rangle = \langle a \,|\, b\rangle$. So $\langle a \,|\, a\rangle, \langle b \,|\, b\rangle \in \mathbb{R}$ and $\langle a \,|\, a\rangle = \langle a \,|\, b\rangle = \langle b \,|\, a\rangle = \langle b \,|\, b\rangle$. Now suppose $\langle a \,|\, b\rangle \neq 0$. Then $\langle a - b \,|\, a - b\rangle = \langle a \,|\, a\rangle - \langle a \,|\, b\rangle - \langle b \,|\, a\rangle + \langle b \,|\, b\rangle = 0$. So $a - b = 0$, *i.e.* $a = b$. Hence the copyables are pairwise orthogonal. □

Applying dagger, a copyable element of A corresponds exactly to a comonoid homomorphism $(\mathbb{C}, \delta_{\mathbb{C}}) \to (A, \delta)$:

$$\begin{array}{ccc}
1 & \longmapsto & 1 \otimes 1 \\
\downarrow & & \downarrow \\
a & \longmapsto & a \otimes a.
\end{array}$$

We have already seen that copyable elements correspond exactly to algebra homomorphisms

$$(A, \mu) \to (\mathbb{C}, \mu_{\mathbb{C}}),$$

i.e. to *characters* of the algebra—the elements of the Gelfand spectrum of A [33]. This leads to our first characterization of when a (nonunital) Frobenius algebra in **Hilb** corresponds to an orthonormal basis.

Theorem 16 *A Frobenius algebra in* **Hilb** *admits the structure theorem and hence corresponds to an orthonormal basis if and only if it is semisimple.*

PROOF We first consider sufficiency. Form a direct sum of one-dimensional coalgebras indexed by the copyables of (A, δ). This will have an isometric embedding as a coalgebra into (A, δ):

$$e\colon \bigoplus_{\{a \mid \delta(a) = a \otimes a\}} (\mathbb{C}, \delta_{\mathbb{C}}) \to (A, \delta).$$

The image S of e is a closed subspace of A, and has an orthonormal basis given by the images of the characters of A qua copyables. The structure theorem holds if the image of e spans A.

Given $a \in A$ and a character c, the evaluation $c(a)$ gives the Fourier coefficient of a at the basis element of S corresponding to c. Now S will be the whole of A if and only if distinct vectors have distinct projections on S, *i.e.* if and only if distinct vectors have distinct Gelfand transforms $\hat{a}\colon c \mapsto c(a)$. Hence the Ambrose structure theorem holds when the Gelfand representation is injective, which holds if and only if the algebra is semisimple.

Necessity is easy to see from the form of a direct sum of one-dimensional algebras, as the lattice of ideals is a complete atomic boolean algebra, where the atoms are the generators of the algebras. □

We shall restate the previous theorem in terms of axiom (H), so that we have a characterization that lends itself to categories other than **Hilb**.

Proposition 17 *A Frobenius algebra in* **Hilb** *satisfies (H) if and only if it is semisimple, and hence admits the structure theorem.*

PROOF That proper H*-algebras are semisimple follows from the results in [8]. Conversely, $\bigoplus_I (\mathbb{C}, \mu_{\mathbb{C}})$ is easily seen to satisfy (H); we can define x^* by taking conjugate coefficients in the given basis. □

4.3 Categorical formulation

We can recast these results into a categorical form. Recall that there is a functor $\ell^2\colon \mathbf{PInj} \to \mathbf{Hilb}$ on the category of sets and partial injections [11, 21]. It sends a set X to the Hilbert space $\ell^2(X) = \{\varphi\colon X \to \mathbb{C} \mid \sum_{x \in X} |\varphi(x)|^2 < \infty\}$, which is the free Hilbert space on X that is equipped with an orthonormal basis, *i.e.* an H*-algebra, in a sense we will now make precise. Firstly, we make Frobenius and H*-algebras into a category. While other choices of morphisms can fruitfully be made [23], the following one suits our current purposes.

Definition 18 Let $\mathbf{D}$ be a symmetric monoidal dagger category. We denote by $\mathbf{HStar}(\mathbf{D})$ the category whose objects are H*-algebras in $\mathbf{D}$, and by $\mathbf{Frob}(\mathbf{D})$ the category whose objects are Frobenius algebras in $\mathbf{D}$. A morphism $(A, \delta) \to (A', \delta')$ in both categories is a morphism $f\colon A \to A'$ in $\mathbf{D}$ satisfying $(f \otimes f) \circ \delta = \delta' \circ f$ and $f^\dagger \circ f = \mathrm{id}$.

Proposition 19 *Every object in* **PInj** *carries a unique H*-algebra structure, namely* $\delta(a) = (a, a)$. *In other words, the categories* $\mathbf{HStar}(\mathbf{PInj})$, $\mathbf{Frob}(\mathbf{PInj})$, *and* **PInj** *are isomorphic.*

Proof Let $\delta = (A \leftarrowtail\!\!\!\!\!\leftarrow \delta_1 \!-\!\!\!\prec D \rightarrowtail \delta_2 \twoheadrightarrow A \times A)$ be an object of **HStar**(**PInj**). Because of (M), we may assume that $\delta_1 = \mathrm{id}$. By (C), we find that δ_2 is a tuple of some $d \colon A \to A$ with itself. It follows from (A) that $d = d \circ d$. Finally, since **PInj** is well-pointed, δ satisfies (F) by Lemma 6. Writing out what (F) means gives

$$\{((d(b), b), (b, d(b))) \mid b \in A\} = \{((c, d(c)), (d(c), c)) \mid c \in A\}.$$

Hence for all $b \in A$, there is $c \in A$ with $b = d(c)$ and $d(b) = c$. Taking $b = d(a)$ we find that $c = a$, so that for all $a \in A$ we have $d \circ d(a) = a$. Therefore $d = d \circ d = \mathrm{id}$. We conclude that δ is the diagonal function $a \mapsto (a, a)$.

To establish the isomorphism of categories, we exhibit an inverse F to the forgetful functor $U \colon \mathbf{Frob}(\mathbf{PInj}) \to \mathbf{PInj}$ given by $(A, \delta) \mapsto A$ on objects and $f \mapsto f$ on morphisms. Define $F(A) = (A, \Delta)$ on objects, and $F(f) = f$ on morphisms. One easily verifies that F is a well-defined functor, *i.e.* that every morphism of **PInj** commutes with diagonals. Unicity of H*-algebra structure guarantees that $F \circ U = \mathrm{Id}$ and $U \circ F = \mathrm{Id}$. □

As a corollary one finds that an object in **PInj** with its unique H*-algebra structure is unital if and only if it is a singleton set.

Since the Hilbert space $\ell^2(X)$ comes with a chosen basis induced by X, the ℓ^2 construction is in fact a functor $\ell^2 \colon \mathbf{HStar}(\mathbf{PInj}) \to \mathbf{HStar}(\mathbf{Hilb})$. Conversely, there is a functor U in the other direction taking an H*-algebra to the set of its copyables; this is functorial by [8, Example 3]. These two functors are adjoints:

$$\mathbf{HStar}(\mathbf{PInj}) \underset{U}{\overset{\ell^2}{\underset{\bot}{\rightleftarrows}}} \mathbf{HStar}(\mathbf{Hilb}).$$

The Ambrose structure theorem, Theorem 12, can now be restated as saying that this adjunction is in fact an equivalence.

Similarly, there is an adjunction between **Frob**(**PInj**) and **Frob**(**Hilb**), but it is not yet clear if this is an equivalence, too, *i.e.* if **Frob**(**Hilb**) and **HStar**(**Hilb**) are equivalent categories. In fact, this question is the central issue of the rest of this paper, and will lead to the main open question in Section 4.5 to follow. In the meantime, we shall use the categorical formulation to give different characterizations of when Frobenius algebras in **Hilb** admit the structure theorem.

4.4 Further conditions

There are in fact a number of conditions on Frobenius algebras in **Hilb** which are equivalent to admitting the structure theorem. This section gives two more.

Theorem 20 *A Frobenius algebra in* **Hilb** *is an H*-algebra, and hence corresponds to an orthonormal basis, if and only if it is a directed colimit (in* **Frob**(**Hilb**)*) of unital Frobenius algebras.*

PROOF Given an orthonormal basis $\{|i\rangle\}_{i\in I}$ for A, define $\delta\colon A \to A\otimes A$ by (continuous linear extension of) $\delta|i\rangle = |ii\rangle$. For finite subsets F of I, define $\delta_F\colon \ell^2(F) \to \ell^2(F)\otimes\ell^2(F)$ by $\delta_F|i\rangle = |ii\rangle$. These are well-defined objects of **Frob**(**Hilb**) by Theorem 1. Since F is finite, every δ_F is a unital Frobenius algebra in **Hilb**. Together they form a (directed) diagram in **Frob**(**Hilb**) by inclusions $i_{F\subseteq F'}\colon \ell^2(F) \hookrightarrow \ell^2(F')$ if $F \subseteq F'$; the latter are well-defined morphisms since $\delta_{F'}\circ i_{F\subseteq F'}|i\rangle = |ii\rangle = (i_{F\subseteq F'}\otimes i_{F\subseteq F'})\circ\delta_F|i\rangle$. Finally, we verify that δ is the colimit of this diagram. The colimiting cocone is given by the inclusions $i_F\colon \ell^2(F) \hookrightarrow A$; these are morphisms $i_F\colon \delta_F \to \delta$ in **Frob**(**Hilb**) since $\delta\circ i_F = (i_F\otimes i_F)\circ\delta_F$, that are easily seen to form a cocone. Now, if $f_F\colon \delta_F \to (A',\delta')$ form another cocone, define $m\colon X\to X'$ by (continuous linear extension of) $m|i\rangle = f_{\ell^2(\{|i\rangle\})}|i\rangle$. Then $m\circ i_F|i\rangle = f_{\ell^2(\{|i\rangle\})}|i\rangle = f_F|i\rangle$ for $i\in F$, so that indeed $m\circ i_F = f_F$. Moreover, m is the unique such morphism. Thus δ is indeed a colimit of the δ_F.

Conversely, suppose (A,δ) is a colimit of some diagram $d\colon \mathbf{I}\to\mathbf{Frob}(\mathbf{Hilb})$. We will show that the nonzero copyables form an orthonormal basis for A. By Lemmas 13 and 15, it suffices to prove that they span a dense subspace of A. Let $a\in A$ be given. Since the colimiting cocone morphisms $c_i\colon A_i\to A$ are jointly epic, the union of their images is dense in A, and therefore a can be written as a limit of $c_i(a_i)$ with $a_i\in A_i$ for some of the $i\in\mathbf{I}$. These a_i, in turn, can be written as linear combinations of elements of copyables of A_i by Theorem 1. Now, c_i maps copyables into copyables, and so we have written a as a limit of linear combinations of copyables of A. Hence the copyables of A spans a dense subspace of A, and therefore form an orthonormal basis.

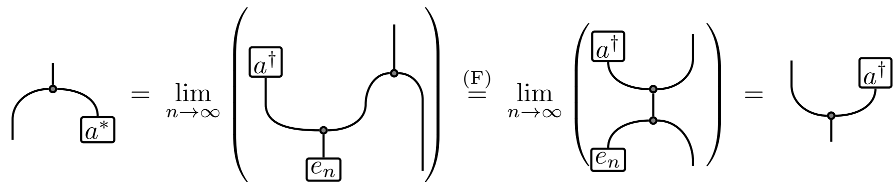
Finally, we verify that these two constructions are mutually inverse. Starting with a δ, one obtains $E = \{e \mid \delta(e) = e\otimes e\}$, and then $\delta'\colon A\to A\otimes A$ by (continuous linear extension of) $\delta'(e) = e\otimes e$ for $e\in E$. The definition of E then gives $\delta' = \delta$.

Conversely, starting with an orthonormal basis $\{|i\rangle\}_{i\in I}$, one obtains $\delta\colon A\to A\otimes A$ by (continuous linear extension of) $\delta|i\rangle = |ii\rangle$, and then $E = \{a\in A \mid \delta(a) = a\otimes a\}$. It is trivial that $\{|i\rangle \mid i\in I\}\subseteq E$. Moreover, we know that E is linearly independent by 13. Since it contains a basis, it must therefore be a basis itself. Hence indeed $E = \{|i\rangle \mid i\in I\}$. □

For *separable* Hilbert spaces, there is also a characterization in terms of approximate units as follows.

Theorem 21 *A Frobenius algebra on a separable Hilbert space in* **Hilb** *is an H*-algebra, and hence corresponds to an orthonormal basis, if and only if there is a sequence e_n such that $e_n a$ converges to a for all a, and $(\mathrm{id}\otimes a^\dagger)\circ\delta(e_n)$ converges.*

PROOF Writing $a_n^* = (\mathrm{id}\otimes a^\dagger)\circ\delta(e_n)$, by assumption $a^* = \lim_{n\to\infty} a_n^*$ is well-defined. Since morphisms in **Hilb** are continuous functions and composition preserves continuity, (H) holds by the following argument.



Hence approximate units imply (H). Conversely, using the Ambrose structure theorem, Theorem 12, we can always define e_n to be the sum of the first n copyables. □

Summarizing, we have the following result.

Theorem 22 *For a Frobenius algebra in* **Hilb***, the following are equivalent:*

(a) it is induced by an orthonormal basis;

(b) it admits the structure theorem;

(c) it is semisimple;

(d) it satisfies axiom (H);

(e) it is a directed colimit (with respect to isometric homomorphisms) of finite-dimensional unital Frobenius algebras.

Moreover, if the Hilbert space is separable, these are equivalent to:

(f) it has a suitable form of approximate identity. □

We see that the finite-dimensional result follows immediately from our general result and Lemma 5, which shows that the algebra is C* and hence semisimple. In fact, the influential thesis [1] (see also [2, 29]) already observes explicitly (and in much wider generality) that:

- If (M) holds, a unital Frobenius algebra is semisimple [1, Theorem 2.3.3].
- A commutative semisimple unital Frobenius algebra is a direct sum of fields [1, Theorem 2.2.5].

Thus the only additional ingredient required to obtain Theorem 1 is the elementary Lemma 15.

4.5 The main question

The main remaining question in our quest for a suitable notion of algebra to characterize orthonormal bases in arbitrary dimension is the following.

In the presence of (A), (C), and (M), does (F) imply (H)?

We can ask this question for the central case of **Hilb**, and for monoidal dagger categories in general.

If the answer is positive, then nonunital Frobenius algebras give us the right notion of observable to use in categorical quantum mechanics. If it is negative, we may consider adopting (H) as the right axiom instead of (F).

At present, these questions remain open, both for **Hilb** and for the general case. However, positive results have been achieved for a large family of categories; these will be described in the following section. We shall conclude this section by further narrowing down the question in the category **Hilb**.

Recall that the *Jacobson radical* of a commutative ring is the intersection of all its maximal regular ideals, and that a ring is called *radical* when it equals its Jacobson radical.

Proposition 23 *Frobenius algebras A in* **Hilb** *decompose as a direct sum*

$$A \cong S \oplus R$$

of (co)algebras, where S is an H-algebra and R is a radical algebra.*

Proof Let a be a copyable element of a Frobenius algebra A in **Hilb**. Consider the embedding into A of a as a one-dimensional algebra. This embedding is a kernel, since it is isometric and its domain is finite-dimensional. Observe that this embedding is an algebra homomorphism as well as a coalgebra homomorphism, because copyables are idempotents by (M). Now it follows from [23, Lemma 19] that also the orthogonal complement of the embedding is both an algebra homomorphism and a coalgebra homomorphism. Finally, Frobenius algebra structure restricts along such embeddings by [23, Proposition 9].

We can apply this to the embedding of the closed span of all copyables of A, and conclude that A decomposes (as a (co)algebra) into a direct sum of its copyables and the orthogonal subspace. By definition, the former summand is semisimple, and is hence a H*-algebra by Proposition 17. The latter summand by construction has no copyables and hence no characters, and is therefore radical. □

This shows how the Jacobson radical of a Frobenius algebra sits inside it in a very simple way. Indeed, we are left with not just a nonsemisimple algebra, but a radical one, which is the opposite of a semisimple algebra—an algebra is semisimple precisely when its Jacobson radical is zero. Hence, in the category **Hilb**, our main remaining question above reduces to finding out whether R must be zero, as follows.

Does there exist a nontrivial radical Frobenius algebra?

Although there is an extensive literature about commutative radical Banach algebras, including a complete classification that in fact ties in with approximate units [20], this question seems to be rather difficult.

5 H*-algebras in categories of relations and positive matrices

We have been able to give a complete analysis of nonunital Frobenius algebras in several (related) cases, including:

- categories of relations, and locally bifinite relations, valued in cancellative quantales;
- nonnegative matrices with ℓ^2-summable rows and columns.

The common feature of these cases can be characterized as *the absence of destructive interference.*

The main result we obtain is as follows.

Theorem 24 *Nonunital Frobenius algebras in all these categories decompose as direct sums of abelian groups, and satisfy (H).*

The remainder of this section is devoted to the proof of this theorem. Our plan is as follows. Firstly, we shall prove the result for **Rel**, the category of sets and relations. In this case, our main question is already answered directly by Proposition 7 and Theorem 22. Moreover, the result in this case has appeared in [32]. However, our proof is quite different, and in particular makes no use of units. This means that it can be carried over to the other situations mentioned above.

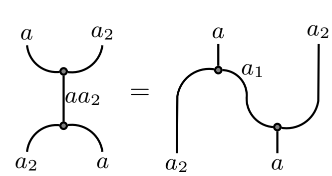
5.1 Frobenius algebras in Rel and lbfRel

We assume given a set A, and a Frobenius algebra structure on it given by a relation $\Delta \subseteq A \times (A \times A)$. We shall write ∇ for $\Delta^\dagger$.

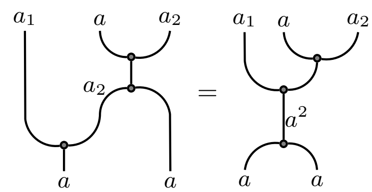
Definition 25 Define $x \sim y$ if and only if $(x,y)\nabla z$ for some z. By (M), the relation ∇ is single-valued and surjective. Therefore, we may also use multiplicative notation xy (suppressing the ∇), and write $x \sim y$ to mean that xy is defined.

Lemma 26 *The relation $\sim$ is reflexive.*

PROOF Let $a \in A$. By (M), we have $a = a_1 a_2$ for some $a_1, a_2 \in A$. Then $(a_2, a)(\mathrm{id} \otimes \Delta)(a_2, a_1, a_2)$ and $(a_2, a_1, a_2)(\nabla \otimes \mathrm{id})(a, a_2)$ by (C), so by (F) we have $(a_2, a)\Delta \circ \nabla(a, a_2)$, so that aa_2 is defined. Diagrammatically, we annotate the lines with elements to show they are related by that morphism.



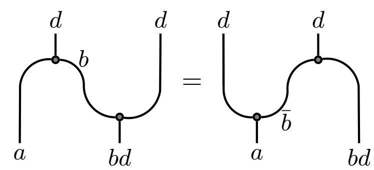
Also $(a,a)(\Delta \otimes \mathrm{id})(a_1, a_2, a)$ and $(a_1, a_2, a)(\mathrm{id} \otimes (\Delta \circ \nabla))(a_1, a, a_2)$, so by (F) we have $(a,a)(\mathrm{id} \otimes \Delta) \circ \Delta \circ \nabla(a_1, a, a_2)$, so that a^2 is defined.



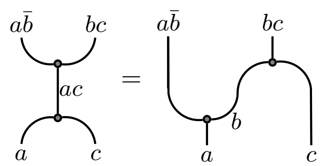
That is, $a \sim a$. □

Lemma 27 *The relation $\sim$ is transitive.*

Proof Suppose that $a \sim b$ and $b \sim c$. Then $d = ab$ is defined. By Lemma 26, then $(ab)d = d^2$ is defined. Hence by (A), also $a(bd)$ is defined. Applying (F) now yields $\bar{b}$ such that $a = d\bar{b}$.



It now follows from (C) that $a = d\bar{b} = a\bar{b}b$; in particular $a\bar{b}$ is defined. But then also $ac = (a\bar{b}b)c = (a\bar{b})(bc)$ is seen to be defined by the assumption $b \sim c$ and another application of (F).



Hence $a \sim c$. □

Proposition 28 *The Frobenius algebra A is a disjoint union of totally defined commutative semigroups, each satisfying (F).*

Proof By the previous two lemmas and (C), the relation $\sim$ is a equivalence relation. Hence A is a disjoint union of the equivalence classes under $\sim$. By definition of $\sim$, the multiplication ∇ is totally defined on these equivalence classes. Moreover, they inherit the properties (M), (C), (A) and (F) from A. □

Lemma 29 *[25] A semigroup S is a group if and only if $\forall a \in S[aS = S = Sa]$.*

Proof The condition $aS = S$ means $\forall b \exists c[b = ac]$. If S is a group, this is obviously fulfilled by $c = a^{-1}b$. For the converse, fix $a \in S$. Applying the condition with $b = a$ yields c such that $a = ac$. Define $e = c$, and let $x \in S$. Then applying the condition with $b = x$ gives c with $x = ac$. Hence $ex = eac = ac = x$. Thus S is a monoid with (global) unit e. Applying the condition once more, with $a = x$ and $b = e$ yields x^{-1} with $xx^{-1} = e$. □

Theorem 30 *A is a disjoint union of commutative groups.*

Proof Let A' be one of the equivalence classes of A, and consider $a, b \in A'$. This means that $a \sim b$. As in the proof of Lemma 27, there is a $\bar{b}$ such that $a = \bar{b}ba$. Putting $c = \bar{b}a$ thus gives $\forall a, b \in A' \exists c \in A'[a = cb]$. In other words, $aA' = A'$ and similarly $A' = A'a$ for all $a \in A$. Hence A' is a (commutative) group by Lemma 29. □

The following theorem already follows from Proposition 7 and Theorem 22, but now we have a direct proof that also carries over to the theorem after it, which does not follow from the earlier results.

Theorem 31 *In* **Rel***, Frobenius algebras satisfy (H), and the conditions (F) and (H) are equivalent in the presence of the other axioms for Frobenius algebras.*

Proof This follows directly from Lemma 30, since we can define $a^* = a^{-1}$, where a^{-1} is the inverse in the disjoint summand containing a. More precisely, a point of A in **Rel** will be a subset of A, and we apply the definition $a^* = a^{-1}$ pointwise to this subset. This assignment is easily seen to satisfy (H). □

Theorem 32 *In* **lbfRel***, Frobenius algebras are disjoint unions of abelian groups and hence satisfy (H), and the conditions (F) and (H) are equivalent in the presence of the other axioms for Frobenius algebras.*

Proof The proof above made no use of units, and is equally valid in **lbfRel**.□

5.2 Quantale-valued relations

We shall now consider categories of the form $\mathbf{Rel}(Q)$, where Q is a commutative, cancellative quantale. Recall that a commutative quantale [34] is a structure $(Q, \cdot, 1, \leq)$, where $(Q, \cdot, 1)$ is a commutative monoid, and $(Q, \leq)$ is a partial order which is a complete lattice, *i.e.* it has suprema of arbitrary subsets. In particular, the supremum of the empty set is the least element of the poset, written 0. The multiplication is required to distribute over arbitrary joins, *i.e.*

$$x \cdot (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} x \cdot y_i, \qquad (\bigvee_{i \in I} x_i) \cdot y = \bigvee_{i \in I} x_i \cdot y.$$

The quantale is called *cancellative* if

$$x \cdot y = x \cdot z \;\Rightarrow\; x = 0 \vee y = z.$$

An example is given by the extended nonnegative reals $[0, \infty]$ with the usual ordering, and multiplication as the monoid operation. Note that the only nontrivial example when the monoid operation is idempotent, *i.e.* when the quantale is a locale, is the two-element boolean algebra $\mathbf{2} = \{0, 1\}$, since in the idempotent case $x \cdot 1 = x \cdot x$ for all x. We write **canQuant** for the category of cancellative quantales which are nontrivial, *i.e.* in which $0 \neq 1$.

Proposition 33 *The two element boolean algebra is terminal in* **canQuant***.*

Proof The unique homomorphism $h\colon Q \to \mathbf{2}$ sends 0 to itself, and everything else to 1. Preservation of sups holds trivially, and cancellativity implies that multiplication is preserved. □

The category $\mathbf{Rel}(Q)$ has sets as objects; morphisms $R\colon X \relbar\mapstochar\rightarrow Y$ are Q-valued matrices, *i.e.* functions $X \times Y \to Q$. Composition is relational composition evaluated in Q, *i.e.* if $R\colon X \relbar\mapstochar\rightarrow Y$ and $S\colon Y \relbar\mapstochar\rightarrow Z$, then

$$S \circ R(x, z) = \bigvee_{y \in Y} R(x, y) \cdot S(y, z).$$

It is easily verified that this yields a category, with identities given by diagonal matrices; that it has a monoidal structure induced by cartesian product; and that is has a dagger given by matrix transpose, *i.e.* relational converse. Thus $\mathbf{Rel}(Q)$ is a symmetric monoidal dagger category, and the notion of Frobenius algebra makes sense in it. Note that $\mathbf{Rel}(\mathbf{2})$ is just $\mathbf{Rel}$.

A homomorphism of quantales $h\colon Q \to R$ induces a (strong) monoidal dagger functor $h^*\colon \mathbf{Rel}(Q) \to \mathbf{Rel}(R)$, which transports Frobenius algebras in $\mathbf{Rel}(Q)$ to Frobenius algebras in $\mathbf{Rel}(R)$. In particular, by Proposition 33, a Frobenius algebra $\Delta\colon A \relbar\mapstochar\rightarrow A \times A$ in $\mathbf{Rel}(Q)$ has a reduct $h^*\Delta\colon A \relbar\mapstochar\rightarrow A \times A$ in $\mathbf{Rel}$. Hence Theorems 30 and 31 apply to this reduct. The remaining degree of freedom in the Frobenius algebra in $\mathbf{Rel}(Q)$ is which elements of Q can be assigned to the elements of the matrix.

Suppose that we have a Frobenius algebra $\Delta\colon A \relbar\mapstochar\rightarrow A \times A$ in $\mathbf{Rel}(Q)$. We write $M\colon (A \times A) \times A \to Q$ for the matrix function corresponding to the converse of Δ, and we write $M(a, b, c)$ rather than $M((a, b), c)$.

Because the unique homomorphism $Q \to \mathbf{2}$ reflects 0, an entry $M(a, b, c)$ is nonzero if and only if the corresponding relation $(a, b)(h^*\nabla)c$ holds. Applying Theorem 30, this immediately implies that for each $a, b \in A$, there is exactly one $c \in A$ such that $M(a, b, c) \neq 0$.

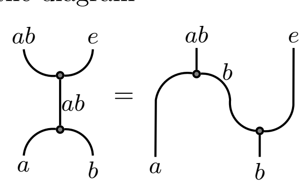
We can use this observation to apply similar diagrams to those used in our proofs for $\mathbf{Rel}$ in order to obtain constraints on the values taken by the matrix in Q.

Proposition 34 *With notation as above:*

(a) If e is an identity element in one of the disjoint summands, then $M(a, e, a) = M(b, e, b)$ for all a, b in that disjoint summand. We write q_e for this common value.

(b) For all $a, b \in A$, we have $M(a, b, ab)^2 = 1$.

Proof For (a), consider the diagram

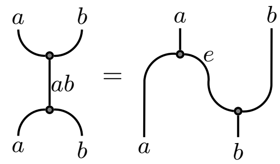


This implies the equation

$$M(a,b,ab)\cdot M(ab,e,ab) = M(a,b,ab)\cdot M(b,e,b)$$

and hence, by cancellativity, $M(ab,e,ab) = M(b,e,b)$. Hence for any c, taking $b = a^{-1}c$, $M(c,e,c) = M(b,e,b)$.

For (b), consider the diagram



This implies the equation $M(a,b,ab)^2 = q_e^2$. Now applying (M), for each c we obtain that

$$\bigvee\{M(a,b,ab)^2 \mid ab = c\} = 1.$$

As all the terms in this supremum are the same, $M(a,b,ab)^2 = 1$. □

Thus if $q^2 = 1$ implies $q = 1$ in Q, the matrix M is in fact valued in $\mathbf{2}$. Otherwise, we can choose square roots of unity for the entries.

Theorem 35 *Let Q be a cancellative quantale. Suppose that $q^2 = 1$ implies $q = 1$ in Q. Then every Frobenius algebra in $\mathbf{Rel}(Q)$ satisfies (H).* □

5.3 Positive ℓ^2 matrices

We now consider the case of matrices in $\mathbf{Mat}_{\ell^2}(\mathbb{C})$ valued in the non-negative reals. These form a monoidal dagger subcategory of $\mathbf{Mat}_{\ell^2}(\mathbb{C})$, which we denote by $\mathbf{Mat}_{\ell^2}(\mathbb{R}^+)$. Note that the semiring $(\mathbb{R}^+, +, 0, \times, 1)$ has a unique 0-reflecting semiring homomorphism to $\mathbf{2}$. Hence a Frobenius algebra in $\mathbf{Mat}_{\ell^2}(\mathbb{R}^+)$ has a reduct to one in $\mathbf{Rel}$ via this homomorphism. Just as before, we can apply Theorem 30 to this reduct.

We have the following analogue to Proposition 34, where $M\colon (A\times A)\times A \to \mathbb{R}^+$ is the matrix realizing the Frobenius algebra structure.

Proposition 36 *The function M is constant on each disjoint summand of A.*

PROOF We can use the same reasoning as in Proposition 34(a) to show that, if e is an identity element in one of the disjoint summands, then for all a, b in that disjoint summand, $M(a,e,a) = M(b,e,b)$. We write r_e for this common value.

Using the same reasoning as in Proposition 34(b) one finds $M(a,b,ab)^2 = r_e^2$. Since we are in $\mathbb{R}^+$, this implies $M(a,b,ab) = r_e$, so that M is constant on each disjoint summand. □

Proposition 37 *Each disjoint summand is finite, and the common value of M on that summand is $1/\sqrt{d}$, where d is the cardinality of the summand.*

PROOF Applying (M), for each c in the summand we obtain that

$$\sum_{ab=c} M(a,b,c)^2 = 1.$$

Since the summand is a group, for each c and a there is a unique b such that $ab = c$. Moreover, by Proposition 36, all the terms in this sum are equal. Thus the sum must be finite, with the number of terms d the cardinality of the summand. We can therefore rewrite the equation as $dr_e^2 = 1$, and hence $r_e = 1/\sqrt{d}$. □

Theorem 38 *Every Frobenius algebra in* $\mathbf{Mat}_{\ell^2}(\mathbb{R}^+)$ *satisfies (H).*

PROOF We make the same pointwise assignment $x^* = x^{-1}$ on the elements of A as in the proof of Theorem 31, with the weight $1/\sqrt{d}$ determined by the summand. □

In the case when the matrix represents a bounded linear map in **Hilb**, we can apply Theorem 22, and obtain:

Proposition 39 *If a Frobenius algebra in* **Hilb** *can be represented by a non-negative real matrix, then it corresponds to a direct sum of one-dimensional algebras, and hence to an orthonormal basis.*

Conversely, if a Frobenius algebra in **Hilb** satisfies (H), it is induced by an orthonormal basis, and hence has a matrix representation with nonnegative entries. Therefore we have found another equivalent characterization of when (F) implies (H) in **Hilb** to add to our list in Theorem 22:

Proposition 40 *A Frobenius algebra A in* **Hilb** *satisfies (H) if and only if there is a basis of A such that the matrix of the comultiplication has nonnegative entries when represented on that basis.*

5.4 Discussion

How different is the situation with Frobenius algebras in these matrix categories from **Hilb**? In fact, it is not as different as it might at first appear.

- The category **Hilb** is equivalent to a full subcategory of the dagger monoidal category of complex matrices with ℓ^2-summable rows and columns. The 'only' assumption needed but not satisfied is positivity.
- The result 'looks' different, but beware. Consider *group rings* (or algebras) over the complex numbers, for finite abelian groups. They can easily be set up to fulfil all our axioms, including (U), so that Theorem 1 applies, and they decompose as direct sums of one-dimensional algebras. But the isomorphism which gives this decomposition may be quite non-obvious.[7]

[7] It would be interesting, for example, to know the computational complexity of determining this isomorphism, given a presentation of the group. As far as we know, this question has not been studied.

Note that copyable elements are idempotents, so the only group element which is copyable is the identity.

- This decomposition result indeed shows that group rings over the complex numbers are very weak invariants of the groups. The group rings of two finite abelian groups will be isomorphic if the groups have the same order [36]!
- However, this is highly sensitive to which field we are over. Group algebras over the *rationals* are isomorphism invariants of groups [26].

6 Outlook

We are still investigating our main question, of whether (F) implies (H), in **Hilb** and elsewhere.

Beyond this, we see the following main lines for continuing a development of categorical quantum mechanics applicable to infinite-dimensional situations.

- We are now able to consider observables with infinite discrete spectra. Beyond this lie continuous observables and projection-valued measures; it remains to be seen how these can be analyzed in the setting of categorical quantum mechanics.
- Complementary observables should be studied in this setting. The bialgebra approach studied in [16] is based on axiomatizing *mutually unbiased bases*, and does not extend directly to the infinite-dimensional case. However, complementary observables are studied from a much more general perspective in works such as [13], and this should provide a good basis for suitable categorical axiomatizations.
- This leads on to another point. There may be other means, within the setting of categorical quantum mechanics, of representing observables, measurements and complementarity, which may be more flexible than the Frobenius algebra approach, and in a sense more natural, since tensor product structure is not inherent in the basic notion of measurement. Methodologically, one should beware of concluding over-hastily that a particular approach is canonical, simply on the grounds that it captures the standard notion in finite-dimensional Hilbert spaces. There may be several ways of doing this, and some more definitive characterization would be desirable.
- A related investigation to the present one is the work on nuclear and traced ideals in [4]. It seems likely that some combination of the ideas developed there, and those we have studied in this paper, will prove fruitful.

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