Pictures of complete positivity in arbitrary dimension

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Abstract

Two fundamental contributions to categorical quantum mechanics are presented. First, we generalize the CP-construction, that turns any dagger compact category into one with completely positive maps, to arbitrary dimension. Second, we axiomatize when a given category is the result of this construction.

1 Introduction

Since the start of categorical quantum mechanics [2], dagger compactness has played a key role in most constructions, protocol derivations and theorems. To name two:

- Selinger's CPM-construction, which associates to any dagger compact category of pure states and operations a corresponding dagger compact category of mixed states and operations [18];
- Environment structures, an axiomatic substitute for the CPM-construction which proved to be particularly useful in the derivation of quantum protocols [5, 8].

It is well know that assuming compactness imposes finite dimension when exporting these results to the Hilbert space model [11]. This paper introduces variations of each the above two results that rely on dagger structure alone, and in the presence of compactness reduce to the above ones. Hence, these variations accommodate interpretation not just in the dagger compact category of finite dimensional Hilbert spaces and linear maps, but also in the dagger category of Hilbert spaces of arbitrary dimension and continuous linear maps. We show:

- that the generalized CPM-construction indeed corresponds to the usual definitions of infinite-dimensional quantum information theory;
- that the direct correspondence between the CPM-construction and environment structure (up to the so-called doubling axiom) still carries trough.

The next two sections each discuss one of our two variations in turn.

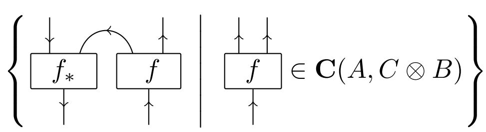
Related work While there are previous results dealing with the transition to a noncompact setting in some way or another, *e.g.* [1, 11, 12], what is particularly appealing about the results in this paper is that they still allow the diagrammatic representations of symmetric monoidal categories [15, 19]. The variation of the CPM-construction relying solely on dagger structure was thus far only publicized as an internal research report [3]; by relating it to the usual setting of infinite-dimensional quantum information theory, this paper justifies the construction.

Classical information can be modeled in categorical quantum mechanics using so-called classical structures [7]. It is not clear whether these survive CP-constructions; see also [13] in this volume. The environment structures of Section 3 could be a useful tool in this investigation. Also, the current work seems well suited to an abstract proof of Naimark's dilation theorem [6].

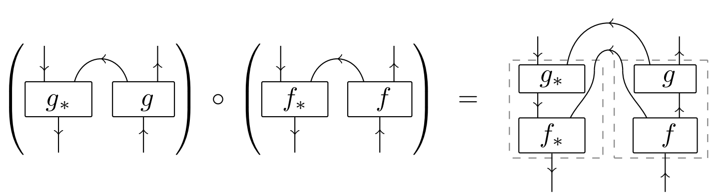
2 Complete positivity

We start by recalling the *CPM-construction* [18], that, given a dagger compact category $\mathbf{C}$, produces a new dagger compact category $\mathrm{CPM}(\mathbf{C})$ as follows. For an introduction to dagger (compact) categories and their graphical calculus, we refer to [2, 16, 19]. When wires of both types A and A^* arise in one diagram, we will decorate them with arrows in opposite directions. When possible we will suppress coherence isomorphisms in formulae. Finally, recall that $(\_)_*$ reverses the order of tensor products, so f_* has type $A^* \to B^* \otimes C^*$ when $f\colon A \to C \otimes B$ [18].

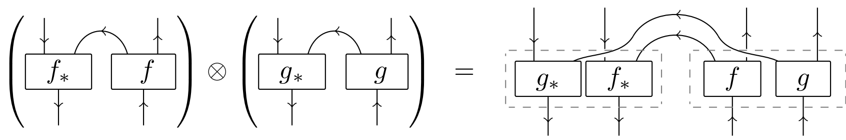
- The objects of $\mathrm{CPM}(\mathbf{C})$ are the same as those of $\mathbf{C}$.
- The morphisms $A \to B$ of $\mathrm{CPM}(\mathbf{C})$ are those morphisms of $\mathbf{C}$ that can be written in the form $(1 \otimes \eta^\dagger \otimes 1)(f_* \otimes f)\colon A^* \otimes A \to B^* \otimes B$ for some morphism $f\colon A \to C \otimes B$ and object C in $\mathbf{C}$.

  $\mathrm{CPM}(\mathbf{C})(A,B) =$ 

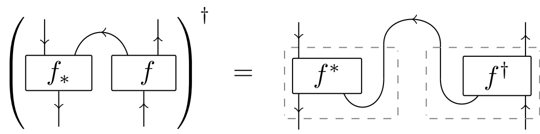
  We call C the *ancillary system* of $(1 \otimes \eta^\dagger \otimes 1)(f_* \otimes f)$, and f its *Kraus morphism.*
- Identities are inherited from $\mathbf{C}$, and composition is defined as follows.



- The tensor unit I and the tensor product of objects are inherited from $\mathbf{C}$, and the tensor product of morphisms is defined as follows.



- The dagger is defined as follows.



- Finally, the cup $\eta_A \colon I \to A^* \otimes A$ in CPM($\mathbf{C}$) is given by $(\eta_A)_* \otimes \eta_A = \eta_A \otimes \eta_A$ in $\mathbf{C}$ (*i.e.* with ancillary system I and Kraus morphism η_A in $\mathbf{C}$).

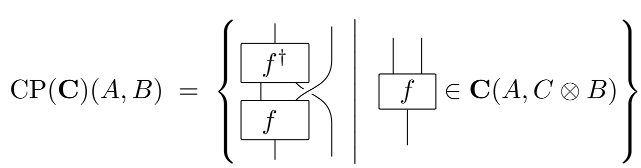
If $\mathbf{C}$ is the dagger compact category **fdHilb** of finite-dimensional Hilbert spaces and linear maps, then CPM(**fdHilb**) is precisely the category of finite-dimensional Hilbert spaces and completely positive maps [18]. To come closer to the traditional setting, we may identify the objects H of CPM(**fdHilb**) with their algebras of operators $B(H)$.

The notion of complete positivity makes perfect sense for normal linear maps between von Neumann algebras $B(H)$ for Hilbert space H of arbitrary dimension. We now present a CP-construction that works on dagger symmetric monoidal categories that are not necessarily compact, and that reduces to the previous construction in the compact case. Subsequently we prove that applying this construction to the category of Hilbert spaces indeed results in the traditional completely positive maps as morphisms.

Definition 1. Let $\mathbf{C}$ be a dagger symmetric monoidal category. We define a new category CP($\mathbf{C}$).

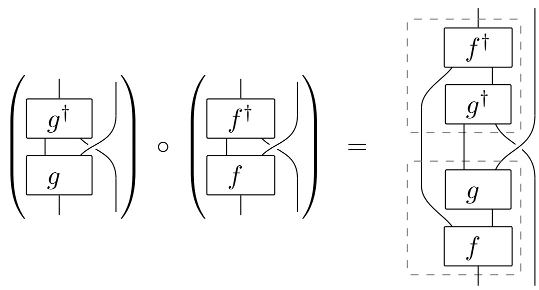
- The objects of CP($\mathbf{C}$) are those of $\mathbf{C}$.

- The morphisms $A \to B$ of CP($\mathbf{C}$) are those morphisms of $\mathbf{C}$ that can be written in the form $(f^\dagger \otimes 1)(1 \otimes \sigma)(f \otimes 1) \colon B \otimes A \to B \otimes A$ for some morphism $f \colon A \to B \otimes C$ and object C in $\mathbf{C}$, where $\sigma \colon B \otimes B \to B \otimes B$ is the swap.

  We call C the *ancillary system* of $(f^\dagger \otimes 1)(1 \otimes \sigma)(f \otimes 1)$, and f its *Kraus morphism*.

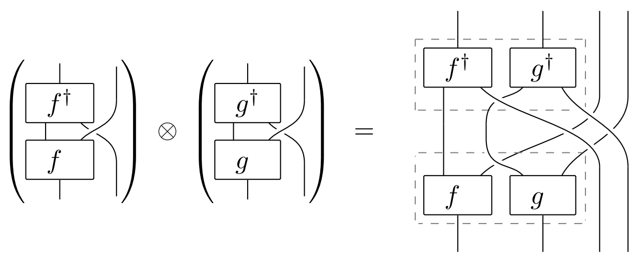
- Composition is defined as follows.



- The identity $A \to A$ in $\mathrm{CP}(\mathbf{C})$ is the swap map $\sigma\colon A \otimes A \to A \otimes A$ in $\mathbf{C}$, with $C = I$.

Diagrammatic manipulations show that $\mathrm{CP}(\mathbf{C})$ indeed is a well-defined category.

Proposition 2. *For any dagger symmetric monoidal category* $\mathbf{C}$*, the category* $\mathrm{CP}(\mathbf{C})$ *is symmetric monoidal. The tensor unit* I *and tensor products of objects are as in* $\mathbf{C}$*, and the tensor product of morphisms is defined as follows.*

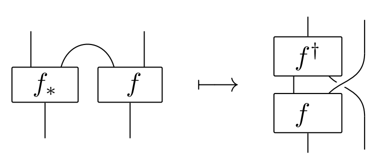


Proof. Laborious but straightforward. □

Notice that $\mathrm{CP}(\mathbf{C})$ does not obviously have a dagger, even though $\mathbf{C}$ does: the CP-construction sends a dagger symmetric monoidal category to a symmetric monoidal category that no longer necessarily has a dagger. Therefore it is not straightforwardly a monad.

Proposition 3. *If* $\mathbf{C}$ *is a dagger compact category, then* $\mathrm{CPM}(\mathbf{C})$ *and* $\mathrm{CP}(\mathbf{C})$ *are isomorphic as dagger symmetric monoidal categories.*

Proof. The isomorphism $\mathrm{CPM}(\mathbf{C}) \to \mathrm{CP}(\mathbf{C})$ is given by $A \mapsto A$ on objects, and acts as follows on morphisms.



This assignment is clearly invertible. To see that it gives a well-defined functor preserving daggers and symmetric monoidal structure takes light work. □

We now embark on showing that the CP-construction is not just meaningless manipulation of diagrams that just happens to coincide with the traditional setting in the case of finite-dimensional Hilbert spaces.

To do so, we first recall the definition of quantum operations in the Heisenberg picture, as it is usually stated in infinite-dimensional quantum information theory [14]. A function $\varphi\colon A \to B$ between von Neumann algebras is *unital* when $\varphi(1) = 1$, and *positive* when for each $a \in A$ there is $b \in B$ such that $\varphi(a^*a) = b^*b$. When it is linear, the function φ is called *normal* when it preserves suprema of increasing nets of projections, or equivalently, when it is ultraweakly continuous. Finally, φ is *completely positive* when $\varphi \otimes 1\colon A \otimes M_n \to B \otimes M_n$ is positive for all $n \in \mathbb{N}$, where M_n is the von Neumann algebra of n-by-n complex matrices. (See e.g. [21, 3.3] or [17, p26].)

Definition 4. A *quantum operation* is a normal completely positive linear map between von Neumann algebras.[1] Hilbert spaces H and quantum operations $B(H) \to B(K)$ form a category that we denote by **QOperations**.

To see that CP(**Hilb**) is isomorphic to **QOperations**, we will rely on two classical theorems: Stinespring's dilation theorem, showing that quantum operations can be written as *-homomorphisms on larger algebras, and Dixmier's structure theorem for normal *-homomorphisms. A **-homomorphism* is a linear map $\pi\colon A \to B$ between C*-algebras that satisfies $\pi(ab) = \pi(a)\pi(b)$ and $\pi(a^*) = \pi(a)^*$.

Let us emphasize that the following results hold for arbitrary Hilbert spaces, not just separable ones.

Theorem 5 (Stinespring)**.** *Let A be a von Neumann algebra. For any completely positive linear map $\varphi\colon A \to B(H)$, there exist a unital *-homomorphism $\pi\colon A \to B(K)$ and a continuous linear $v\colon H \to K$ such that $\varphi(a) = v^\dagger \pi(a) v$.*

Proof. See [20], [21, IV.3.6(ii)] or [17, 4.1]. □

Theorem 6 (Dixmier)**.** *Every normal *-homomorphism $\varphi\colon B(H) \to B(K)$ factors as $\varphi = \varphi_3\varphi_2\varphi_1$ for:*

- *an* ampliation $\varphi_1\colon B(H) \to B(H \otimes H')\colon f \mapsto f \otimes 1_{H'}$ *for some Hilbert space H';*
- *an* induction $\varphi_2\colon B(H \otimes H') \to B(K')\colon f \mapsto pfp$ *for the projection $p \in B(H \otimes H')$ onto some Hilbert subspace $K' \subseteq H \otimes H'$;*
- *a* spatial isomorphism $\varphi_3\colon B(K') \to B(K)\colon f \mapsto u^\dagger f u$ *for a unitary $u\colon K \to K'$.*

Proof. See [10, I.4.3] or [21, IV.5.5]. □

[1] Not every completely positive linear map between von Neumann algebras is normal. In fact, a positive linear map is normal if and only if it is weak-* continuous [9, 46.5].

Corollary 7. *A linear map $\varphi\colon B(H) \to B(K)$ is a quantum operation if and only if it is of the form $\varphi(f) = g^\dagger(f \otimes 1)g$ for some Hilbert space H' and continuous linear map $g\colon K \to H \otimes H'$.*

Proof. Given a quantum channel φ, combine Theorems 5 and 6 to get $\varphi(f) = v^\dagger u^\dagger p(f \otimes 1)puv$; taking $g = puv$ brings φ into the required form.

Conversely, one easily checks directly via Definition 4 that a map φ of the given form is indeed normal, and completely positive. □

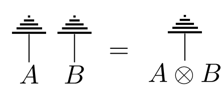
Theorem 8. CP(**Hilb**) *and* **QOperations** *are isomorphic as symmetric monoidal categories.*

Proof. The objects are already equal. The isomorphism sends a morphism of CP(**Hilb**) with Kraus morphism g to the quantum operation $g^\dagger(\_ \otimes 1)g$. □

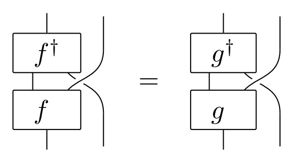
3 Environment structures

This section axiomatizes when a given category is of the form CP(**C**), by generalizing [5, 8].

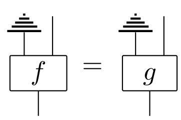
Definition 9. An *environment structure* for a dagger symmetric monoidal category **C** is a dagger symmetric monoidal supercategory $\widehat{\mathbf{C}}$ with the same objects, in which each object A has a morphism $\top_A\colon A \to I$, depicted as , satisfying the following axioms.

(a) We have $\top_I = 1_I$, and for all objects A and B: 

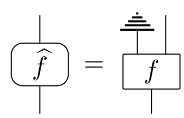
in $\widehat{\mathbf{C}}$;

(b) For all $f, g \in \mathbf{C}(A, C \otimes B)$: 

in **C** $\iff$



in $\widehat{\mathbf{C}}$;

(c) For each $\widehat{f} \in \widehat{\mathbf{C}}(A, B)$ there is $f \in \mathbf{C}(A, C \otimes B)$ such that 

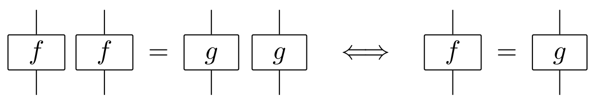
in $\widehat{\mathbf{C}}$.

Morphisms in $\widehat{\mathbf{C}}$ are depicted with rounded corners.

Intuitively, if we think of the category **C** as consisting of pure states, then the supercategory $\widehat{\mathbf{C}}$ consists of mixed states. The maps $\top$ 'ground' a system within the environment. Condition (c) then reads that every mixed state can be seen as a pure state in an extended system; the lack of knowledge carried in the ancillary system represents the variables relative to which we mix.

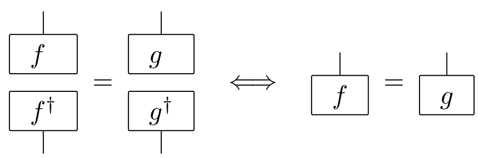
Remark. The inclusion $\mathbf{C} \hookrightarrow \widehat{\mathbf{C}}$ given by an environment structure is different from the canonical functor $\mathbf{C} \to \mathrm{CP}(\mathbf{C})$ given by $f \mapsto \sigma(f \otimes f^\dagger)$. After all, for $\mathbf{C} = \mathbf{Hilb}$, the latter identifies global phases and therefore is not faithful. However, as we will see in Theorem 12 below, the image $\mathbf{D}$ of the canonical functor $\mathbf{C} \to \mathrm{CP}(\mathbf{C})$ does come with an environment structure in which $\widehat{\mathbf{D}} = \mathrm{CP}(\mathbf{C})$. We may think of $\mathbf{D}$ as a *double* of $\mathbf{C}$. In fact, it turns out this characterizes categories of the form $\mathrm{CP}(\mathbf{C})$ if and only if they satisfy the following axiom.

Definition 10. A symmetric monoidal category satisfies the *doubling axiom* when

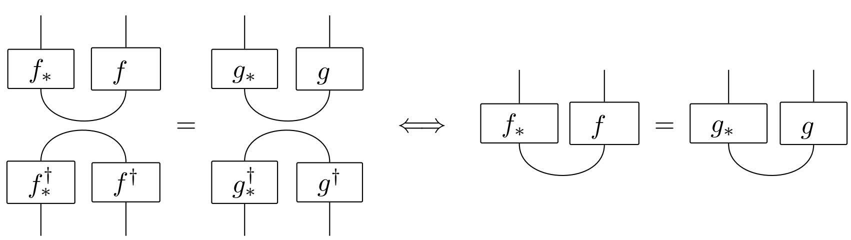


for all parallel morphisms f and g.

When we follow through the isomorphism of Proposition 3, it is clear that Definition 9 reduces to its namesake that is only defined in the compact case [8]. Definition 10 also relates to previously defined properties in the compact case, which we now review. Recall that a dagger compact category $\mathbf{C}$ satisfies the so-called *preparation-state agreement axiom* [4] when $f \circ f^\dagger = g \circ g^\dagger$ implies $f = g$ for all morphisms f and g with domain I:



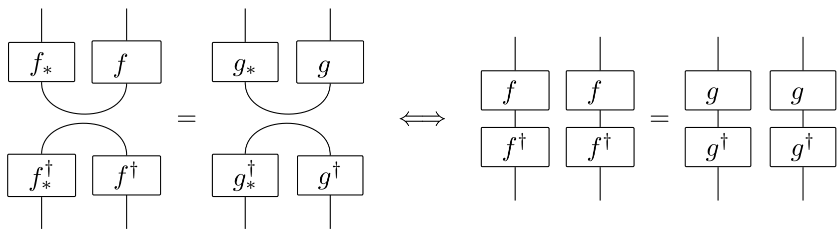
So if $\mathbf{C}$ is a dagger compact category, then $\mathrm{CPM}(\mathbf{C})$ satisfies the preparation-state agreement axiom when



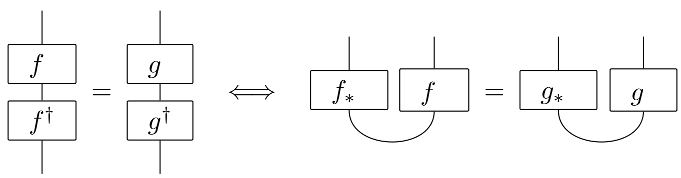
for all morphisms f and g with a common domain. The doubling axiom (for $\mathbf{C}$) does not necessarily imply the preparation-state agreement axiom (for $\mathbf{C}$), as can be seen in *e.g.* $\mathbf{C} = \mathbf{fdHilb}$. In fact, the preparation-state agreement axiom (for $\mathbf{C}$) is equivalent to the doubling axiom (for $\mathbf{C}$) for parallel morphisms with domain I instead of all parallel morphisms. Nevertheless, the doubling axiom is stronger than the preparation-state axiom in the sense of the following proposition.

Proposition 11. *If a dagger compact category* **C** *satisfies the doubling axiom, then* CPM(**C**) *satisfies the preparation-state agreement axiom.*

Proof. Let f and g be morphisms with a common domain. Elementary graphical manipulations yield:



By the doubling axiom for **C**, the latter is equivalent to $f \circ f^\dagger = g \circ g^\dagger$. Finally, by using graphical manipulation again, we obtain:

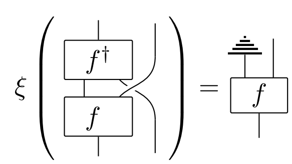


Hence CPM(**C**) satisfies the state-preparation agreement axiom. ∎

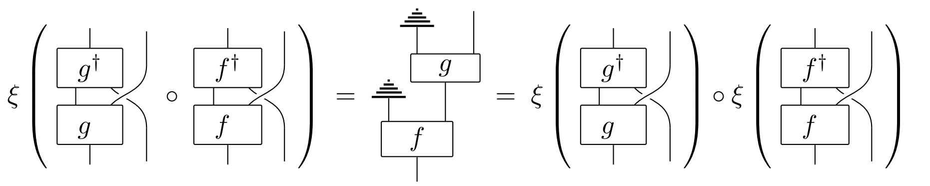
We now show that the same relationship between the CPM-construction and environment structures as in the compact case holds unabated for our noncompact variations.

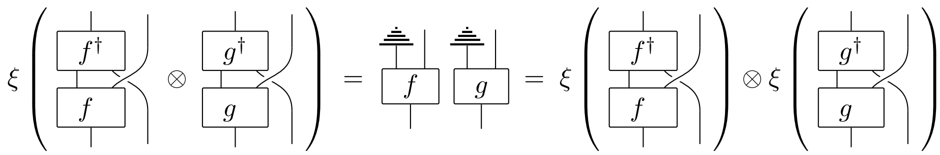
Theorem 12. *If a symmetric monoidal category* **C** *comes with an environment structure, then there exists an isomorphism* $\xi\colon \mathrm{CP}(\mathbf{C}) \to \widehat{\mathbf{C}}$ *of symmetric monoidal categories. Moreover,* CP(**C**) *allows such an axiomatic presentation if and only if it obeys the doubling axiom.*

Proof. Define $\xi(A) = A$ on objects, and



on morphisms. This map is well-defined and injective by Definition 9(b), and is surjective by Definition 9(c). It preserves composition and monoidal structure

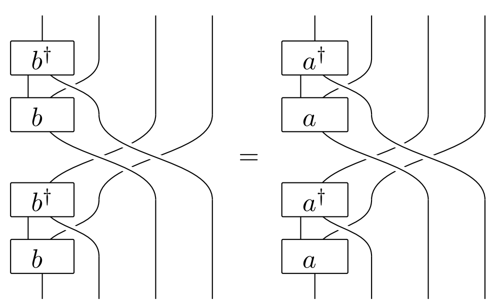




by Definition 9(a). It also clearly preserves identities. Thus ξ is an isomorphism of symmetric monoidal categories.

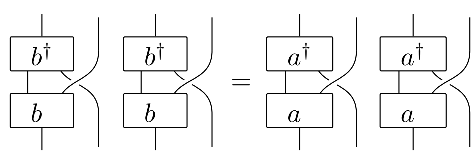
For the second statement, let $\mathbf{D}$ be the image of the canonical functor $\mathbf{C} \to \mathrm{CP}(\mathbf{C})$, set $\widehat{\mathbf{D}} = \mathrm{CP}(\mathbf{C})$, and define $\top_A$ to be the morphism $A \to I$ in $\mathrm{CP}(\mathbf{C})$ with Kraus morphism $1_A \colon A \to A$ in $\mathbf{C}$. Then Definition 9(a) and (c) are automatically satisfied. It now suffices to show that the doubling axiom holds for $\mathrm{CP}(\mathbf{C})$ if and only if Definition 9(b) is satisfied. After all, the latter is the case precisely when $\top_A$ form a well-defined environment structure.

Consider the equation



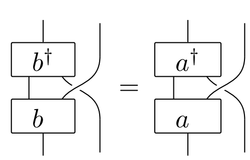
(1)

in $\mathbf{C}$. By elementary graphical manipulations we see that (1) is equivalent to the following equation.



(2)

Now, on the one hand, the doubling axiom for $\mathrm{CP}(\mathbf{C})$ means precisely that (2) is equivalent to the following equation.



(3)

On the other hand, we will now show (3) is equivalent to (1) if and only if Definition 9(b) is satisfied. But (3) unfolds to the right-hand side of Definition 9(b) when we write out definitions (of morphisms in CP($\mathbf{C}$), of composition in CP($\mathbf{C}$), and of $\top$) and use Proposition 2. Similarly, (1) unfolds to the left-hand side of Definition 9(b) by writing out definitions (of morphisms in CP($\mathbf{C}$), and of $\mathbf{D}$).

Thus CP($\mathbf{C}$) obeys the doubling axiom if and only it satisfies Definition 9(b).

□

References

[1] Samson Abramsky, Richard Blute, and Prakhash Panangaden. Nuclear and trace ideals in tensored *-categories. *Journal of Pure and Applied Algebra*, 143:3–47, 1999.

[2] Samson Abramsky and Bob Coecke. A categorical semantics of quantum protocols. In *Logic in Computer Science 19*, pages 415–425. IEEE Computer Society, 2004.

[3] Bob Coecke. Complete positivity without positivity and without compactness. Technical Report PRG-RR-07-05, Oxford University Computing Laboratory, 2007.

[4] Bob Coecke. De-linearizing linearity: Projective quantum axiomatics from strong compact closure. *Electronic Notes in Theoretical Computer Science*, 170:49–72, 2007.

[5] Bob Coecke. Axiomatic description of mixed states from selinger's cpm-construction. *Electronic Notes in Theoretical Computer Science*, 210:3–13, 2008.

[6] Bob Coecke and Éric O. Paquette. Povms and Naimark's theorem without sums. In *QPL IV*, volume 210 of *ENTCS*, pages 15–31, 2008.

[7] Bob Coecke and Duško Pavlović. Quantum measurements without sums. In *Mathematics of Quantum Computing and Technology*, pages 559–596. Taylor and Francis, 2007.

[8] Bob Coecke and Simon Perdrix. Environment and classical channels in categorical quantum mechanics. In *CSL'10/EACSL'10*, pages 230–244. Springer, 2010.

[9] John B. Conway. *A course in operator theory.* American Mathematical Society, 2000.

[10] Jacques Dixmier. *Von Neumann algebras.* North-Holland, 1969.

[11] Chris Heunen. Compactly accessible categories and quantum key distribution. *Logical Methods in Computer Science*, 4(4), 2008.

[12] Chris Heunen. An embedding theorem for Hilbert categories. *Theory and Applications of Categories*, 22(13):321–344, 2009.

[13] Chris Heunen and Sergio Boixo. Completely positive classical structures and sequentializable quantum protocols. In *QPL VIII*, 2011.

[14] Alexander S. Holevo. *Statistical structure of quantum theory*. Springer, 2001.

[15] A. Joyal and R. Street. The geometry of tensor calculus I. *Advances in Mathematics*, 88:55–112, 1991.

[16] G. Max Kelly and Miguel L. Laplaza. Coherence for compact closed categories. *Journal of Pure and Applied Algebra*, 19:193–213, 1980.

[17] Vern Paulsen. *Completely bounded maps and operators algebras*. Cambridge University Press, 2002.

[18] Peter Selinger. Dagger compact closed categories and completely positive maps. In *Quantum Programming Languages*, volume 170 of *ENTCS*, pages 139–163. Elsevier, 2007.

[19] Peter Selinger. A survey of graphical languages for monoidal categories. In *New Structures for Physics*, volume 813 of *Lecture Notes in Physics*, pages 289–355. Springer, 2010.

[20] W. Forrest Stinespring. Positive functions on C*-algebras. *Proceedings of the American Mathematical Society*, 6(2):211–216, 1955.

[21] Masamichi Takesaki. *Theory of Operator Algebra I*. Encyclopaedia of Mathematical Sciences. Springer, 1979.