# No-go theorems for functorial localic spectra of noncommutative rings

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January 31, 2011

**Abstract**

Any functor from the category of C*-algebras to the category of locales which assigns to each commutative C*-algebra its Gelfand spectrum must be trivial on algebras of $n$-by-$n$ matrices for $n \geq 3$. The same obstruction applies to the Zariski, Stone, and Pierce spectra. The possibility of spectra in categories other than that of locales is briefly discussed.

A recent article [7] by Reyes shows that any functor $\mathbf{Cstar}^{\mathrm{op}} \to \mathbf{Top}$ which assigns to each commutative C*-algebra its Gelfand spectrum must be trivial on the matrix algebras $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$. The introduction of that paper suggests that this might be because the spectrum ought to be a "space without points". This suggests that a functorial extension which is nontrivial on the matrix algebras $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$ might still be possible in locale theory, where there is a sensible notion of pointfree space. Unfortunately, this is not the case: we show that any functor $\mathbf{Cstar}^{\mathrm{op}} \to \mathbf{Loc}$ which assigns to each commutative C*-algebra its Gelfand spectrum must be trivial on $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$. The same obstruction applies to the Zariski, Stone, and Pierce spectra.

## 1 Locale-theoretic preliminaries

Locales and topological spaces are closely related apart from a few subtle differences. One of the most important is that in these categories limits are, in general, computed differently. Initially one might hope that for this reason Reyes' result does not apply to locales, but it turns out that it does. The key observation is that, although limits in spaces and locales differ in general, they coincide for those spaces (locales) that arise as spectra.

**Proposition 1.** *Both compact regular and compact completely regular locales are closed under limits in* **Loc**.

*Proof.* The product of compact (completely) regular locales is again (completely) regular and compact [4, III.1.6, III.1.7, IV.1.5]. The equalizer of $f, g\colon A \to B$ is a closed sublocale of $A$ whenever $B$ is (completely) regular [4, III.1.3] and a closed sublocale of a compact (completely) regular locale is again (completely) regular and compact [4, III.1.2, IV.1.5]. □

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**Proposition 2.** *The limit in* **Loc** *of a diagram of coherent locales and coherent morphisms is coherent. In addition, the mediating morphisms are coherent.*

*Proof.* A (morphism of) locale(s) is coherent if it lies in the image of the functor Idl: **DLat** $\to$ **Frm**, which is faithful and left adjoint to the forgetful functor [4, II.2.11]. ☐

**Corollary 3.** *Stone locales are closed under limits in* **Loc**.

*Proof.* A locale is Stone when it is both compact regular and coherent. Any continuous morphism between Stone spaces is coherent. ☐

**Corollary 4.** *The functors* Spec: $\mathbf{cCstar}^{\mathrm{op}} \to$ **Loc** *and* Idl: $\mathbf{Bool}^{\mathrm{op}} \to$ **Loc** *which send commutative C*-algebras to their Gelfand spectra and boolean algebras to their Stone spectra preserve all limits.*

*Proof.* The functor Spec is part of a duality between commutative C*-algebras and compact completely regular locales, and therefore certainly preserves all limits as a functor to the category of such locales. But as these locales are closed under limits, it also preserves all limits when considered as a functor to the category of all locales. The Stone case is completely analogous. ☐

## 2 Main results

A locale is trivial when it is an initial object in **Loc**, *i.e.* when it satisfies $0 = 1$. Let us call an object $R$ *Kochen–Specker* when there is a morphism $\mathbb{M}_n(\mathbb{C}) \to R$ for some $n \geq 3$. This class includes at least the matrix rings $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$ themselves. For our purposes it can often be widened; for example, by [2] Theorem 7 below still holds when we include in the class of Kochen–Specker algebras the von Neumann algebras without an $\mathbb{M}_2(\mathbb{C})$ factor.

**Lemma 5.** *If a functor* $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Loc}$ *is trivial on* $\mathbb{M}_n(\mathbb{C})$ *for* $n \geq 3$*, then it is trivial on all Kochen–Specker rings.*

*Proof.* If $f\colon \mathbb{M}_n(\mathbb{C}) \to R$ is a ring morphism, then $Ff\colon FR \to F\mathbb{M}_n(\mathbb{C})$ is a locale morphism to the trivial locale, and so $FR$ must be trivial. ☐

**Theorem 6.** *Any functor* $\mathbf{Cstar}^{\mathrm{op}} \to \mathbf{Loc}$ *which assigns to each commutative C*-algebra its Gelfand spectrum must be trivial on all Kochen–Specker C*-algebras.*

*Proof.* For any C*-algebra $A$, let $\mathcal{C}(A)$ be the diagram of commutative subalgebras under inclusion. Define $G(X)$ to be the limit in **Loc** of Spec$(A)$ with $A \in \mathcal{C}(X)$.

1. $G$ is a functor $\mathbf{Cstar}^{\mathrm{op}} \to \mathbf{Loc}$ which assigns to each C*-algebra its Stone spectrum.

2. It is the terminal such functor.

3. Since Spec preserves limits, $G(A)$ can equally well be computed by first taking the colimit of $\mathcal{C}(A)$ in **cCstar** and then its Gelfand spectrum.

4. But for $A = \mathbb{M}_n(\mathbb{C})$ with $n \geq 3$, the colimit of $\mathcal{C}(A)$ in **cCstar** yields the 0-dimensional C*-algebra; this is the Kochen–Specker theorem [6]. Hence on these C*-algebras $G$ yields the trivial locale.

5. If $F\colon \mathbf{Cstar}^{\mathrm{op}} \to \mathbf{Loc}$ is any other functor which assigns to each C*-algebra its Gelfand spectrum, then finality of $G$ guarantees maps $FA \to GA$ for all C*-algebras $A$. Hence $FA$ is trivial if $A = \mathbb{M}_n(\mathbb{C})$ for $n \geq 3$.

Combining the above observations and the previous lemma yields the statement of the theorem. □

In a similar vein one proves the following three variations: for Gelfand spectra in the category **Neumann** of von Neumann algebras and normal *-homomorphisms; for Stone spectra in the category **PBoolean** of partial boolean algebras and partial homomorphisms (see [6, 1]); and for Stone spectra in the category **OML** of orthomodular lattices and their homomorphisms. Denote the functor $\mathbf{Cstar} \to \mathbf{PBoolean}$ taking projections by Proj.

**Theorem 7.** *Any functor* $\mathbf{Neumann}^{\mathrm{op}} \to \mathbf{Loc}$ *which assigns to each von Neumann algebra its Gelfand spectrum must be trivial on all Kochen–Specker von Neumann algebras.* □

**Theorem 8.** *Any functor* $F\colon \mathbf{PBoolean}^{\mathrm{op}} \to \mathbf{Loc}$ *which assigns to each boolean algebra its Stone spectrum must be trivial on* $\mathrm{Proj}(\mathbb{M}_n(\mathbb{C}))$ *for* $n \geq 3$. □

**Theorem 9.** *Any functor* $\mathbf{OML}^{\mathrm{op}} \to \mathbf{Loc}$ *which assigns to each boolean algebra its Stone spectrum must be trivial on* $\mathrm{Proj}(\mathbb{M}_n(\mathbb{C}))$ *for* $n \geq 3$. □

The Pierce spectrum of a commutative ring, *i.e.* the Stone spectrum of its boolean algebra of idempotents, requires the following slightly adapted proof.

**Theorem 10.** *Any functor* $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{Loc}$ *which assigns to each commutative ring its Pierce spectrum must be trivial on all Kochen–Specker rings.*

*Proof.* Define the functor $G$ as before: it first considers the diagram $\mathcal{C}(R)$ of all commutative subrings of a ring $R$, and then takes the limit of their Pierce spectra. Alternatively, one may first take idempotents of $R$ and regard these as a partial boolean algebra $X$, and then take the limit in **Loc** of $\mathrm{Idl}(A)$ with $A \in \mathcal{C}(X)$. But this results in the trivial locale for $\mathrm{Proj}(\mathbb{M}_n(\mathbb{C}))$ for $n \geq 3$, and so by functoriality of $F$ in Theorem 8 also for the larger partial boolean algebra $X$. Hence $G$ is trivial on $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$ and the argument proceeds as before. □

For the Zariski spectrum we argue slightly differently (and nonconstructively) by reducing the result to the one by Reyes.

**Theorem 11.** *Any functor* $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{Loc}$ *which assigns to each commutative ring its Zariski spectrum must be trivial on all Kochen–Specker rings.*

*Proof.* Define the functor $G$ as before, and note that we take a limit of a diagram of coherent locales and coherent morphisms. As such a limit is coherent and coherent locales are spatial (by the prime ideal theorem), its triviality on matrix algebras $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$ follows from the work of Reyes. □

## 3 Discussion

Our main results prove an obstruction to direct functorial extensions of various spectra, taking values in locales. There is also no hope for values in the following categories instead of **Loc**.

- The functor $\mathrm{Sh}\colon \mathbf{Loc} \to \mathbf{Topos}$ that takes sheaves preserves limits [5, C.1.4.8], so the obstruction for **Loc** also holds for **Topos**.
- The forgetful functor from ringed toposes to **Topos** reflects initial objects, so replacing **Loc** by that category does not help either.
- The forgetful functor from the category of schemes (possibly over noncommutative rings) to the category **Top** of topological spaces reflects initial objects, so there is no use in replacing locales by schemes.

On the other hand, one could read our main result positively. There *is* hope for a functorial spectrum taking values in the following categories.

- The inclusion functor $\mathbf{Loc} \to \mathbf{Quantale}^{\mathrm{op}}$ does not preserve limits. It has a left adjoint [9, 12.2], but that does not reflect initial objects. Hence it is viable to search for quantale-valued spectra.
- The Bohrification construction, which inspired most of this (see [3, 1]), is not a direct extension of the Gelfand spectrum and hence escapes the hypothesis of our main theorem.
- One could consider different morphisms between rings/algebras, and hence take a different view of these objects, to obtain a functorial spectrum resembling a space (see the discussion in [7]).
- For example, there *is* an interesting functor $F$ from **Cstar** to the category of *quantum frames*, that for commutative C*-algebras comes down to the Gelfand spectrum [8]. This does not contradict the above results, because there is no forgetful functor from quantum frames to either quantales or locales: indeed, $F(\mathbb{M}_3(\mathbb{C}))$ consists of closed right ideals of $\mathbb{M}_3(\mathbb{C})$, and therefore is not trivial.

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