Completely positive classical structures and sequentializable quantum protocols

Chris Heunen*
University of Oxford
heunen@cs.ox.ac.uk

Sergio Boixo†
USC & Harvard University
sergio@boixo.com

We study classical structures in various categories of completely positive morphisms: on sets and relations, on cobordisms, on a free dagger compact category, and on Hilbert spaces. As an application, we prove that quantum maps with commuting Kraus operators can be sequentialized. Hence such protocols are precisely as robust under general dephasing noise when entangled as when sequential.

1 Introduction

There are two ways to model classical information in categorical quantum mechanics. Originally, biproducts were used to capture classical information external to the category at hand [2]. Later, so-called classical structures emerged as a way to model classical data internal to the category itself [12, 13, 3].

The setting of most interest to quantum information theory is that of completely positive maps, which was abstracted in Selinger's CP-construction [26]. This construction need not preserve biproducts. If desired, biproducts have to be freely added again. In fact, the counterexample given in [26] is the category of finite-dimensional Hilbert spaces and completely positive maps, which is of course the crucial model for quantum mechanics.

On the other hand, the CP-construction does preserve classical structures. However, it might introduce new classical structures, that would therefore not model classical information. The current paper addresses the (non)existence of such noncanonical classical structures in categories of completely positive maps. We prove that there are no noncanonical completely positive classical structures in several categories: sets and relations, cobordisms, and the free dagger compact category on one generator.

For the main event, we then consider the category of finite-dimensional Hilbert spaces and completely positive maps. We cannot close the question there entirely yet. But we make enough progress to enable an application to quantum metrology [15]. Many quantum metrology protocols operate parallelly on a maximally entangled state. For some protocols, such as phase estimation, frame synchronization, and clock synchronization, there exists an equivalent sequential version, in which entanglement is traded for repeated operations on a simpler quantum state. We can extend the class of known sequentializable protocols to those whose Kraus operators commute.

The setup of the paper is simple: Section 2 introduces the necessary ingredients, Section 3 considers completely positive classical structures, and Section 4 outlines their application to sequentializable quantum protocols. Much of this work has been done while both authors were at the Institute for Quantum Information at the California Institute of Technology. We thank Peter Selinger for pointing out [20], and David Pérez García, Robert König, Peter Love and Spiros Michalakis for discussions.

*Supported by the Netherlands Organisation for Scientific Research (NWO).
†Supported by FIS2008-01236 and Defense Advanced Research Projects Agency award N66001-09-1-2101.

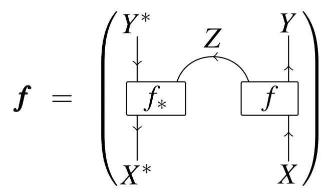
Submitted to:
QPL 2011

2 Preliminaries

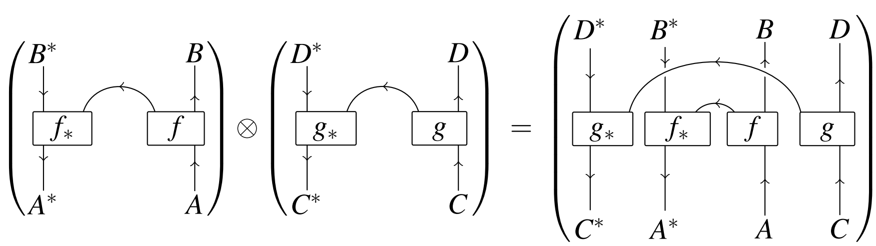
For basics on dagger compact categories we refer to [2, 12, 27]. A *classical structure* in a dagger compact category is a morphism $\delta\colon X \to X \otimes X$ satisfying $(\mathrm{id} \otimes \delta^\dagger) \circ (\delta \otimes \mathrm{id}) = \delta \circ \delta^\dagger = (\delta^\dagger \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta)$, $\delta^\dagger \circ \delta = \mathrm{id}$, and $\sigma \circ \delta = \delta$, where $\sigma\colon X \otimes X \to X \otimes X$ is the swap isomorphism [12, 13, 3]. We depict δ as . In the category **fdHilb** of finite-dimensional Hilbert spaces, classical structures correspond to orthonormal bases [13, 3]: an orthonormal basis $\{|i\rangle\}_i$ induces a classical structure by $|i\rangle \mapsto |ii\rangle$, and the basis is retrieved from a classical structure as the set of nonzero *copyables* $\{x \in X \mid \delta(x) = x \otimes x\} \setminus \{0\}$. This legitimizes using classical structures to model (the copying of) classical data.

The other main ingredient we work with is the *CP-construction*, that we now briefly recall, turning a dagger compact closed category $\mathbf{C}$ into another one by taking the completely positive morphisms [26]. The latter is given by lifting the Stinespring characterization of completely positive maps to a definition:

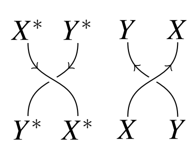
- The objects of $\mathrm{CP}(\mathbf{C})$ are those of $\mathbf{C}$. To distinguish the category they live in, we will denote objects of $\mathbf{C}$ by X, and objects of $\mathrm{CP}(\mathbf{C})$ by $\boldsymbol{X}$.
- The morphisms $\boldsymbol{X} \to \boldsymbol{Y}$ in $\mathrm{CP}(\mathbf{C})$ are the morphisms of $\mathbf{C}$ of the form $(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}) \circ (f_* \otimes f)$ for some $f\colon X \to Z \otimes Y$ in $\mathbf{C}$ called *Kraus morphism*. In graphical [27] terms:



  To distinguish their home category, we denote morphisms in $\mathbf{C}$ as f, and those in $\mathrm{CP}(\mathbf{C})$ as $\boldsymbol{f}$.
- Composition and dagger are as in $\mathbf{C}$.
- The tensor product of objects is as in $\mathbf{C}$.
- The tensor product of morphisms is given by $(\boldsymbol{f} \otimes \boldsymbol{g}) = ((123)(4)) \circ (f_* \otimes f \otimes g_* \otimes g) \circ ((132)(4))$, where the group theoretic notation $((132)(4))$ denotes the canonical permutation built out of identities and swap morphisms using tensor product and composition. Graphically, this looks as follows.



- The swap isomorphism $\boldsymbol{\sigma}\colon \boldsymbol{X} \otimes \boldsymbol{Y} \to \boldsymbol{Y} \otimes \boldsymbol{X}$ in $\mathrm{CP}(\mathbf{C})$ is given by $\sigma \otimes \sigma$.



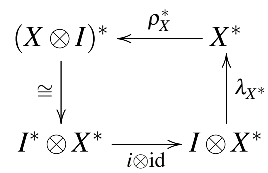
There are two functors of particular interest between $\mathbf{C}$ and $\mathrm{CP}(\mathbf{C})$. They form the components of a natural transformation when varying the base category $\mathbf{C}$ [20].

Lemma 1. *Consider the functors*

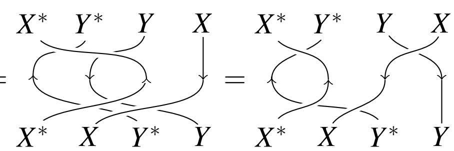
$$F\colon \mathbf{C} \to \mathrm{CP}(\mathbf{C}) \qquad F(X) = \boldsymbol{X} \qquad F(f) = f_* \otimes f,$$
$$G\colon \mathrm{CP}(\mathbf{C}) \to \mathbf{C} \qquad G(\boldsymbol{X}) = X^* \otimes X \qquad G(\boldsymbol{f}) = (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}) \circ (f_* \otimes f).$$

Then F is a symmetric (strict) monoidal functor that preserves dagger, and G is a symmetric (strong) monoidal functor that preserves dagger.

Proof. The first statement is easily verified by taking identities for the required morphisms $\boldsymbol{I} \to F(I)$ and $F(X) \otimes F(Y) \to F(X \otimes Y)$. As to the second statement, G preserves daggers almost by definition. To verify that G is (strong) monoidal, one can take the required morphisms $\varphi\colon I \to G(\boldsymbol{I}) = I^* \otimes I$ and $\varphi_{X,Y}\colon G(\boldsymbol{X}) \otimes G(\boldsymbol{Y}) = X^* \otimes X \otimes Y^* \otimes Y \to Y^* \otimes X^* \otimes X \otimes Y \cong G(\boldsymbol{X} \otimes \boldsymbol{Y})$ to be the canonical isomorphisms of that type. In particular, $\varphi_{X,Y} = (123)(4)$. This makes the required diagrams commute. For example, the 'left unit' condition boils down to $(\rho_X)_* = \lambda_{X^*} \circ (i \otimes \mathrm{id}_{X^*})$, where i is the canonical isomorphism $I^* \to I$. This follows from unitarity of ρ_X, because $(\rho_X)_* = (\rho_X)^{\dagger *} = (\rho_X^{-1})^* = (\rho_X^*)^{-1}$, and the diagram



commutes by the coherence theorem for compact closed categories. Finally,

$\varphi \circ \sigma =$


$= \boldsymbol{\sigma} \circ \varphi,$

so G is in fact symmetric strong monoidal. □

3 Completely positive classical structures

This section considers classical structures in categories $\mathrm{CP}(\mathbf{C})$ for various dagger compact categories $\mathbf{C}$, starting from the following corollary of Lemma 1.

Corollary 2. *Every classical structure on X in $\mathbf{C}$ induces a classical structure on $\boldsymbol{X}$ in* $\mathrm{CP}(\mathbf{C})$. *Every classical structure on $\boldsymbol{X}$ in* $\mathrm{CP}(\mathbf{C})$ *induces a classical structure on $X^* \otimes X$ in $\mathbf{C}$.* □

Classical structures in $\mathrm{CP}(\mathbf{C})$ induced by classical structures in $\mathbf{C}$ are called *canonical*. We will prove that for $\mathbf{C} = \mathbf{Rel}$, $\mathbf{C} = \mathbf{Cob}$, and for $\mathbf{C}$ the free dagger compact category on one generator, any classical structure in $\mathrm{CP}(\mathbf{C})$ is canonical. We conjecture the same is the case for $\mathbf{C} = \mathbf{fdHilb}$ and provide strong evidence for this conjecture.

Sets and relations Let us start with the category **Rel** of sets and relations as a toy example. In general, a relation $R \subseteq (X \times X) \times (Y \times Y)$ is completely positive when

$$(x',x)R(y',y) \iff (x,x')R(y,y'), \tag{1}$$
$$(x',x)R(y',y) \implies (x,x)R(y,y). \tag{2}$$

By [23], a classical structure on an object X in CP(**Rel**) is a direct sum of Abelian groups, the union of whose carrier sets is $X \times X$. Additionally, the group multiplication must be a completely positive relation in the above sense. This means the following, writing the group additively:

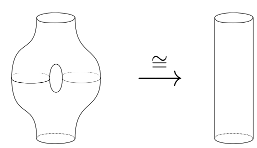
$$(a,b)+(c,d)=(e,f) \Longleftrightarrow (c,d)+(a,b)=(f,e), \tag{3}$$

$$(a,b)+(c,d)=(e,f) \Longrightarrow (c,d)+(c,d)=(f,f). \tag{4}$$

In particular, if we take $(a,b)=0$, then (3) implies that $(c,d)=(c,d)+(a,b)=(d,c)$. Hence it is already forced that $c=d$. That is, the classical structure on X in CP(**Rel**) must be canonical. This proves the following theorem.

Theorem 3. *Any classical structure in* CP(**Rel**) *is canonical.* □

Cobordisms Now consider the category **Cob** of 1d compact closed manifolds and 2d cobordisms between them is a dagger compact category. It is the free dagger compact category on a classical structure [19], and is very interesting from the point of view of quantum field theory [4]. However, it has the property that every isometry is automatically unitary: if we are to have a homeomorphism



the left-hand cobordism cannot actually have nonzero genus. Therefore **Cob** has no nontrivial classical structures at all, as does CP(**Cob**). This makes the following proposition trivial, but nevertheless true.

Proposition 4. *Any classical structure in* CP(**Cob**) *is canonical.* □

Free dagger compact category Let us now have a look at the free dagger compact category **C** on one generator. It is described explicitly in [1], whose notation we adopt here. Concretely, the objects of **C** are $(m,n) \in \mathbb{N}^2$. Morphisms $(m,n) \to (p,q)$ exist only when $m+q=n+p$, and are pairs (s,π) of a natural number s and a permutation $\pi \in S(m+q)$. The identity is $(0,\mathrm{id})$, and composition is given by the so-called execution formula as $(t,\sigma)\circ(s,\pi)=(s+t,\mathrm{ex}(\sigma,\pi))$, see [1]. The monoidal structure is $(m,n)\otimes(p,q)=(m+p,n+q)$ on objects and $(s,\pi)\otimes(t,\sigma)=(s+t,\pi+\sigma)$ on morphisms. Finally, the dagger is given by $(s,\pi)^\dagger=(s,\pi^{-1})$. It turns out that the situation is similar to that with cobordisms.

Proposition 5. *Let* **C** *be the free dagger compact category on one generator. Any classical structure in* CP(**C**) *is canonical.*

Proof. We will show that in both categories **C** and CP(**C**), the only morphisms $\delta\colon X \to X\otimes X$ satisfying $\delta^\dagger\circ\delta=\mathrm{id}$ are the trivial ones with $X=I$. First of all, if $\delta=(s,\pi)\colon (m,n)\to(m,n)\otimes(m,n)=(2m,2n)$ is a morphism in **C**, then $m=n$, and hence $\pi\in S(3n)$. Now $\delta^\dagger\circ\delta=(s,\pi^{-1})\circ(s,\pi)=(2s,\mathrm{ex}(\pi^{-1},\pi))$. Writing both π and $\mathrm{ex}(\pi^{-1},\pi)$ as matrices, as in [1], we find $\mathrm{ex}(\pi^{-1},\pi)_{12}=\pi_{11}\circ\pi_{11}^{-1}=1$. But $\mathrm{id}_{12}=0$. So $\delta^\dagger\circ\delta=\mathrm{id}$ only when $m=n=0$. That is, **C** only has trivial classical structures.

Similarly, working out the upper-right entry of the matrix of $\boldsymbol{\delta}^\dagger\circ\boldsymbol{\delta}$ for a morphism $\boldsymbol{\delta}=(\boldsymbol{s},\boldsymbol{\pi})$ in CP(**C**), we find that it is the union of $\pi_{11}\circ\pi_{11}^{-1}$ with some other terms. In particular, it contains id. Hence $\boldsymbol{\delta}^\dagger\circ\boldsymbol{\delta}=\mathbf{id}$ only if $\mathrm{dom}(\boldsymbol{\delta})=\boldsymbol{I}$, so CP(**C**) only has trivial classical structures, too. □

Finite-dimensional Hilbert spaces The rest of this section concentrates on the category CP(**fdHilb**). Here, we can identify the object $\boldsymbol{X}$ with $\mathbb{C}^n$, and $X^* \otimes X$ with the Hilbert space $M_n(\mathbb{C})$ of n-by-n matrices under the Hilbert-Schmidt inner product $\langle \rho \,|\, \sigma \rangle = \mathrm{Tr}(\rho^\dagger \sigma)$. One strategy to investigate classical structures in this category, inspired by the strategy that worked so well for **Rel**, could be as follows:

1. Start with a classical structure $\boldsymbol{\delta}$ on $\mathbb{C}^n$ in CP(**fdHilb**).
2. Apply Corollary 2 to obtain a classical structure $\delta = G(\boldsymbol{\delta})$ on $M_n(\mathbb{C})$.
3. Apply [13] to obtain an orthonormal basis $\{\alpha\}$ for $M_n(\mathbb{C})$.
4. Investigate when $\delta \colon |\alpha\rangle \mapsto |\alpha\alpha\rangle$ is a completely positive map.

Unfortunately, the last step proves to be quite difficult. A first thought might be to use Choi's theorem, resulting in the following characterization.

Theorem 6. *Classical structures $\boldsymbol{\delta}$ on $\mathbb{C}^n$ in* CP(**fdHilb**) *correspond to orthonormal bases α of $M_n(\mathbb{C})$ for which $\Delta_{j''k''j'k'jk} = \sum_\alpha \overline{\alpha_{jk}} \alpha_{j'k'} \alpha_{j''k''}$ are matrix entries of a positive semidefinite $\Delta \colon (\mathbb{C}^n)^{\otimes 3} \to (\mathbb{C}^n)^{\otimes 3}$.*

Proof. Setting $|\psi\rangle = \sum_i |ii\rangle$ shows that

$$\delta(\rho) = \sum_\alpha \langle \psi| (\alpha^\dagger \otimes \mathrm{id}) \circ (\rho \otimes \mathrm{id}) |\psi\rangle \, (\alpha \otimes \alpha) = \sum_\alpha \mathrm{Tr}(\alpha^\dagger \rho) \cdot (\alpha \otimes \alpha),$$

so that in particular $\delta(|i\rangle\langle j|) = \sum_\alpha \overline{\alpha_{ij}} \cdot (\alpha \otimes \alpha)$. Recall that Choi's theorem states that a morphism $\delta \colon X^* \otimes X \to Y^* \otimes Y$ in **fdHilb** is completely positive if and only if the map $\Delta \colon X \otimes Y \to X \otimes Y$ given by $\langle aj|\Delta|bk\rangle = \langle a|\,\delta(|j\rangle\langle k|)\,|b\rangle$ for $a,b \in Y$ and $j,k \in X$ is positive semidefinite [22, 3.14]. Applying Choi's theorem to the map δ defined by $\alpha \mapsto \alpha \otimes \alpha$ and taking $a = |j''j'\rangle$ and $b = |k''k'\rangle$ yields

$$\Delta_{j''k''j'k'jk} = \left\langle j''j' \right| \delta(|j\rangle\langle k|) \left| k''k' \right\rangle = \sum_\alpha \overline{\alpha_{jk}} \left\langle j''j' \right| \alpha \otimes \alpha \left| k''k' \right\rangle = \sum_\alpha \overline{\alpha_{jk}} \alpha_{j'k'} \alpha_{j''k''}. \qquad \square$$

The condition of the previous theorem is not vacuous. For example, the normalized Pauli matrices

$$\alpha_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \alpha_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \alpha_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \alpha_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

form an orthonormal basis for $M_2(\mathbb{C})$. But their Choi matrix Δ is not even positive:

$$\sqrt{2}\alpha_1 + \alpha_2 + \alpha_4 = \begin{pmatrix} 1 + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 - \frac{1}{\sqrt{2}} \end{pmatrix} \geq 0,$$

as its eigenvalues are $\{2, 0\}$, but its image under δ is

$$\sqrt{2}\alpha_1 \otimes \alpha_1 + \alpha_2 \otimes \alpha_2 + \alpha_4 \otimes \alpha_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} - \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} - \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} + \frac{1}{2} \end{pmatrix},$$

which has an eigenvalue $\frac{1}{2}(\sqrt{2} - 2) < 0$ and therefore is not positive semidefinite.

Numerical evidence 7. The condition of Theorem 6 is easy to verify with computer algebra software. We can also generate random orthonormal bases when $X = \mathbb{C}^n$, *i.e.* when $X^* \otimes X$ is the C*-algebra $M_n(\mathbb{C})$ of n-by-n-matrices: generate n^2 random matrices; with large probability they are linearly independent; use the Gram-Schmidt procedure to turn them into an orthonormal basis of M_n. Running this test for 1000 randomly chosen orthonormal bases did not provide a counterexample to our conjecture for $n = 2,3$.

The previous Theorem and Numerical evidence neither provide much intuition nor have direct physical significance. We end this section by providing some more natural sufficient conditions, and by listing a large number of necessary and sufficient conditions equivalent to our conjecture. This partial progress is enough to enable novel physical applications outlined in the next section.

Proposition 8. *Let $\{\alpha_1,\ldots,\alpha_N\}$ be a orthonormal basis of $X^* \otimes X$, and let δ be the induced classical structure in* **fdHilb**. *If δ is completely positive, then:*

(a) the set of copyables is closed under adjoints: $\{\alpha_1,\ldots,\alpha_N\} = \{\alpha_1^\dagger,\ldots,\alpha_N^\dagger\}$;

(b) copyables are closed under absolute values: $\{|\alpha_1|,\ldots,|\alpha_N|\} \subseteq \{0,\alpha_1,\ldots,\alpha_N\}$;

(c) the map δ satisfies $\delta(\mathrm{id})\delta(ab) = \delta(a)\delta(b)$ for all $a,b\colon X \to X$;

(d) positive semidefinite copyables commute pairwise.

Proof. To see (a), notice that if δ is positive, then it preserves (names of) adjoints [6, Lemma 2.3.1], so that in particular $\delta(\alpha_j^\dagger) = \delta(\alpha_j)^\dagger = \alpha_j^\dagger \otimes \alpha_j^\dagger$. Therefore $\alpha_j^\dagger$ is (also) one of the basis vectors.

For (b) and (c) we first prove the auxiliary result that $ab = 0$ if and only if $\langle a\,|\,b\rangle = 0$ for positive definite matrices a,b. Say $a = x^\dagger x$ and $b = y^\dagger y$. If $ab = 0$, then $0 = \mathrm{Tr}(x^\dagger x y^\dagger y) = \langle a\,|\,b\rangle$. Conversely, if $0 = \langle a\,|\,b\rangle = \mathrm{Tr}(x^\dagger x y^\dagger y) = \mathrm{Tr}(y x^\dagger x y^\dagger) = \|x y^\dagger\|^2$ then $xy^\dagger$=0 and hence $ab = 0$.

By isometry of δ we can therefore conclude that $\delta(a)\delta(b) = 0$ whenever $ab = 0$ for positive a,b. Hence (c) follows from [14, Theorem 2], as well as the fact that δ preserves absolute values. Hence if α is copyable, then $\delta(|\alpha|) = |\delta(\alpha)| = |\alpha \otimes \alpha| = |\alpha| \otimes |\alpha|$, and so $|\alpha|$ is copyable, too, establishing (b).

Finally, (d) follows from a generalized version of the no-cloning theorem [5, Corollary 3]. □

Lemma 9. *If δ is completely positive, then $\delta^\dagger(\mathrm{id})$:*

(a) is positive semidefinite;

(b) is invertible, and hence positive definite;

(c) satisfies $\delta^\dagger(\mathrm{id}) \geq \mathrm{id}$, and hence its eigenvalues are at least 1.

Proof. To see (a), notice that id is a positive definite matrix and that $\delta^\dagger$ is a positive map. For (b), apply [10, Lemma 2.2] to get a completely positive map d such that $d^\dagger(\mathrm{id}) = \mathrm{id}$ and $\delta^\dagger = (\sqrt{f} \otimes \sqrt{f}) \circ d$, where $f = \delta^\dagger(\mathrm{id})$, and observe that

$$\dim(X)^2 = \mathrm{rank}(\delta^\dagger) = \mathrm{rank}((\sqrt{f} \otimes \sqrt{f}) \circ d^\dagger) \leq \mathrm{rank}(\sqrt{f} \otimes \sqrt{f}) \leq \dim(X)^2,$$

so that $\mathrm{rank}(\sqrt{f}) = \dim(X)$, whence $\sqrt{f}$ is invertible. Therefore also $\delta^\dagger(\mathrm{id})$ is invertible.

Notice that $\delta(\mathrm{id})^\dagger = \delta(\mathrm{id}^\dagger) = \delta(\mathrm{id})$ since δ is positive by Proposition 8(a). The map $\frac{1}{\|\delta(\mathrm{id})\|_{\mathrm{op}}}\delta(\mathrm{id})$ is a contraction, and so [6, Proposition 1.3.1] the matrix

$$\frac{1}{\|\delta(\mathrm{id})\|_{\mathrm{op}}} \begin{pmatrix} \mathrm{id} & \delta(\mathrm{id}) \\ \delta(\mathrm{id}) & \mathrm{id} \end{pmatrix}$$

is positive semidefinite. Because $\delta^\dagger$ is completely positive and $\delta^\dagger \circ \delta(\mathrm{id}) = \mathrm{id}$, therefore also the matrix

$$\frac{1}{\|\delta(\mathrm{id})\|_{\mathrm{op}}} \begin{pmatrix} \delta^\dagger(\mathrm{id}) & \mathrm{id} \\ \mathrm{id} & \delta^\dagger(\mathrm{id}) \end{pmatrix}$$

is positive semidefinite. It follows from (b) and [6, Theorem 1.3.3] that

$$\frac{1}{\|\delta(\mathrm{id})\|_{\mathrm{op}}}\delta^\dagger(\mathrm{id}) \geq \frac{1}{\|\delta(\mathrm{id})\|_{\mathrm{op}}} \left(\frac{1}{\|\delta(\mathrm{id})\|_{\mathrm{op}}}\delta^\dagger(\mathrm{id}) \right)^{-1} \frac{1}{\|\delta(\mathrm{id})\|_{\mathrm{op}}},$$

and therefore $\delta^\dagger(\mathrm{id}) \geq (\delta^\dagger(\mathrm{id}))^{-1}$. Finally, since $\delta^\dagger(\mathrm{id})$ is diagonalizable by (a), this yields $\delta^\dagger(\mathrm{id}) \geq \mathrm{id}$, establishing (c). □

Theorem 10. *The following are equivalent:*

- *(a) A classical structure $\boldsymbol{\delta}$ on $\boldsymbol{X}$ in* $\mathrm{CP}(\mathbf{fdHilb})$ *is canonical;*
- *(b) δ preserves pure states;*
- *(c) δ is a trace-preserving map;*
- *(d) $\delta^\dagger$ is unital: $\delta^\dagger(\mathrm{id}_{X\otimes X}) = \mathrm{id}_X$;*
- *(e) $\delta^\dagger(\mathrm{id}_{X\otimes X}) \leq \mathrm{id}_X$;*
- *(f) δ is a trace-nonincreasing map;*
- *(g) $\delta^\dagger(\mathrm{id}_{X\otimes X})$ has operator norm at most 1;*[1]
- *(h) $\mathrm{id}_X = \sum A$ for some subset A of the copyables;*
- *(i) $\mathrm{Tr}(\alpha) = 1$ for positive semidefinite copyables α;*
- *(j) $\mathrm{rank}(\alpha) = 1$ for nonzero copyables α;*
- *(k) if α is copyable, then so is $\alpha^\dagger\alpha$;*
- *(l) the set of copyables is closed under composition: $\alpha_i\alpha_j \in \{0, \alpha_1, \ldots, \alpha_N\}$;*
- *(m) $\delta(\mathrm{id}_X)$ is idempotent;*
- *(n) the ancilla in a minimal Kraus decomposition of δ is one-dimensional.*

Proof. **(b⇒a)** If $\boldsymbol{\delta}$ sends pure states $|\psi\rangle\langle\psi|$ to pure states $|\varphi\rangle\langle\varphi|$, it defines a linear map g sending $|\psi\rangle$ to the unique eigenvector $|\varphi\rangle$ of $\boldsymbol{\delta}(|\psi\rangle\langle\psi|)$, so that $\boldsymbol{\delta} = g_* \otimes g$. One then readily verifies that the axioms for classical structures on $\boldsymbol{\delta}$ imply that g is a classical structure in **fdHilb**.

(c⇒b) Since completely positive trace-preserving maps send quantum states to quantum states (with eigenvalues bounded above by 1), isometry of δ yields

$$\mathrm{Tr}\left(\boldsymbol{\delta}(|\psi\rangle\langle\psi|)^2\right) = \mathrm{Tr}((|\psi\rangle\langle\psi|)^2) = 1,$$

and therefore $\boldsymbol{\delta}$ preserves purity.

[1] One might think that isometry of δ means that it has operator norm 1, and hence so does $\delta^\dagger$, and hence (by applying the Russo-Dye theorem [6, Corollary 2.3.8]) so does $\delta^\dagger(\mathrm{id})$. However, the first operator norm is taken with respect to the Hilbert-Schmidt norm on $M_n(\mathbb{C})$, and the last operator norm is taken with respect to the operator norm on $M_n(\mathbb{C})$. Hence all one can conclude from this reasoning is that $\dim(X)^{-1/2} \leq \|\delta^\dagger(\mathrm{id})\|_{\mathrm{op}} \leq \dim(X)^{1/2}$.

(d⇔c) This equivalence is well-known, see *e.g.* [18]; it is also easily seen using the graphical calculus.

(e⇔d) One direction is trivial; the other follows from Lemma 9(c).

(g⇒d) Combining Lemma 9(c) with (g) gives that all eigenvalues of $\delta^\dagger(\mathrm{id})$ are 1, and hence that $\delta^\dagger(\mathrm{id})$ is (unitarily similar to) the identity matrix.

(f⇔e) Clearly (c), and hence (e), implies (f), so it suffices to show (f)⇒(e). If δ is trace-nonincreasing, then $\mathrm{Tr}(a - a\delta^\dagger(\mathrm{id})) = \langle \mathrm{id} - \delta^\dagger(\mathrm{id}) \,|\, a\rangle = \langle id \,|\, a\rangle - \langle \mathrm{id} \,|\, \delta(a)\rangle = \mathrm{Tr}(a) - \mathrm{Tr}(\delta(a)) \geq 0$ for all a. If $x_i > 0$ is the ith eigenvalue of $\delta^\dagger(\mathrm{id})$, then by taking $a = \mathrm{diag}(0,\ldots,1,\ldots,0)$ with a single 1 in the ith place, we obtain $x_i - 1 \leq 0$. Hence $\delta^\dagger(\mathrm{id}) \leq \mathrm{id}$.

(h⇒g) $\delta^\dagger(\mathrm{id}) = \delta^\dagger((\sum_{\alpha\in A}\alpha)\otimes(\sum_{\beta\in A}\beta)) = \sum_{\alpha,\beta\in A}\delta^\dagger(\alpha\otimes\beta) = \sum_{\alpha\in A}\alpha$, so $\|\delta^\dagger(\mathrm{id})\|_{\mathrm{op}} = \|\mathrm{id}\|_{\mathrm{op}} = 1$.

(i⇒h) Use Plancherel's theorem to get $\mathrm{id} = \sum_\alpha \langle \mathrm{id} \,|\, \alpha\rangle \alpha = \sum_\alpha \mathrm{Tr}(\alpha)\alpha$. Now if α is copyable but not positive semidefinite, then by Proposition 8(b) we have $\mathrm{Tr}(\alpha) = \mathrm{Tr}(|\alpha|) = 0$ iff $|\alpha| = 0$ iff $\alpha = 0$. Hence we can take $A = \{\alpha \mid \mathrm{Tr}(\alpha) = 1\}$.

(j⇒i) If α is positive semidefinite, then by assumption (j) we have $\mathrm{rank}(\alpha) = 1$, so that $\alpha = U\,|\phi\rangle\,\langle\phi|\,U^\dagger$ for some unit vector ϕ and unitary U. Therefore $\mathrm{Tr}(\alpha) = \mathrm{Tr}(|\phi\rangle\,\langle\phi|) = \langle\phi\,|\,\phi\rangle = 1$.

(k⇒j) if $\alpha \neq 0$ then also $\alpha^\dagger\alpha \neq 0$. Hence $\mathrm{rank}(\alpha) = \mathrm{rank}(\alpha^\dagger\alpha) = 1$.

(l⇒k) follows from Proposition 8(a).

(m⇒l) If $\delta(\mathrm{id})$ is idempotent, then it is a projection by Proposition 8(a). By Proposition 8(c) it commutes with the image of δ, and hence is contained in the support projection of that image. Hence $\delta(\mathrm{id})\delta(a) = \delta(a)$ for all matrices a, so that δ preserves multiplication by Proposition 8(c), from which (l) follows.

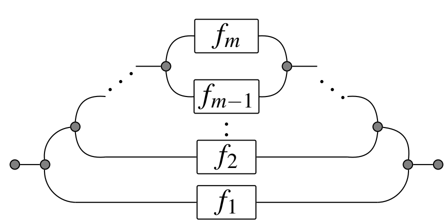
(a⇒m) If δ is canonical, then α are of the form $|i\rangle\,\langle j|$, so that $\mathrm{id} = \sum_i |i\rangle\,\langle i|$, and we obtain $\delta(\mathrm{id})^2 = (\sum_i \delta(|i\rangle\,\langle i|))(\sum_j \delta(|j\rangle\,\langle j|)) = \sum_i |ii\rangle\,\langle ii| = \delta(\mathrm{id})$.

(a⇔n) is just a reformulation in terms of the Kraus decomposition; the ancilla space is the one called Z in the definition of $\mathrm{CP}(\mathbf{C})$ in Section 2.

□

4 Sequentializable quantum protocols

Up to normalization, classical structures in **fdHilb** produce a state that plays an important role in many protocols studied in quantum information and computation, namely the *maximally entangled state*. This state is produced by postcomposing the unit of the classical structure with as many comultiplications as are needed to get an m-party state. Typical examples, frequently encountered in quantum metrology, are optimal *phase estimation* protocols. They can be modeled graphically as the following diagram, read left to right:



(5)

In one of most common cases, the diagram is interpreted in the category **fdHilb** and the morphisms f_j are all identical phase gates $e^{-i\phi\sigma_z/2}$, with unknown phase ϕ and Pauli matrix σ_z. This entangled protocol produces an estimator $\hat{\phi}$ of the unknown phase with an uncertainty scaling $\delta\hat{\phi} \propto 1/m$, which is called the Heisenberg limit [15, 8]. It follows from the *generalized spider theorem* [11] that if the linear maps f_j are "generalized phases" that are compatible with the classical structure, then the previous diagram is equivalent to the following one

(6)

for any permutation $\pi \in S(m)$. This implies that the entangled parallel protocol (5) is equivalent to a sequential protocol (6) with the same uncertainty scaling. This equivalence has been studied in frame synchronization [24] and clock synchronization [7, 16] between two parties. In quantum computation, the transformation between sequential and entangled protocols has also been used to study the computational power of quantum circuits with restricted length [21, 17].

Categorically, the wires and boxes in the diagrams above can also be interpreted as finite-dimensional Hilbert spaces and quantum maps, *i.e.* trace-preserving completely positive maps. These are the maps quantum information theory concerns [18]. For quantum computation, it makes sense to relax to trace-nonincreasing completely positive maps [25]. Neither form a dagger compact category in themselves, but we may regard them as living within CP(**fdHilb**). The equivalence between (5) and (6) holds unabated for quantum maps $\boldsymbol{f}$ in CP(**fdHilb**), as long as they are compatible with the classical structure.

This interpretation has the advantage that general quantum maps are able to model *noisy* estimation protocols. Hence we can now, fully generally, address the question of whether the parallel protocol with maximally entangled states is more fragile in its response to noise than the corresponding sequential protocol. On the one hand, physical intuition tells us that this might be the case, given that entanglement is considered to be a very delicate resource in general. On the other hand, the equivalence between both diagrams derived from the categorical machinery means that for certain kinds of noise the entangled protocol is not more fragile than the sequential one. This generalizes the equivalences of clock synchronization protocols under noise that have been studied for two-dimensional Hilbert space in [7]. By Theorem 10, classical structures are quantum maps if and only if they are canonical, and hence correspond to an orthonormal basis. The only question left is which quantum maps are compatible with that basis. We now give without proof the explicit form of such maps, referring to [9] for details.

Being a completely positive operator, a quantum map can be written as $\rho \mapsto \sum_s b_s \rho b_s^\dagger$, where b_s are the Kraus operators. Denote the nth roots of unity by $\omega_j = e^{-i2\pi j/n}$, where n is the dimension of the underlying Hilbert space. The quantum maps that are compatible with a classical structure are precisely those which can be expressed with Kraus operators of the form

$$b_s = \sqrt{r_s} \sum_j e^{-i(\phi_j + 2\pi js/n)} \left|j\right\rangle \left\langle j\right|,$$

where $\{|j\rangle\}$ is the basis defined by the classical structure, ϕ_j define arbitrary phase rotations, and r_j are positive constants parametrizing general dephasing noise, satisfying $\sum_j r_j = 1$. Maps without dephasing (*i.e.* pure rotations) are obtained by choosing $r_s = \delta_{s,0}$.

Thus, when the above quantum metrology protocols are modeled with such maps, usage of maximally entangled states is equally robust as, and has equal uncertainty scaling to, the corresponding sequential version.

References

[1] Samson Abramsky (2005): *Abstract Scalars, Loops, and Free Traced and Strongly Compact Closed Categories*. In: *Algebra and Coalgebra in Computer Science*, CALCO'05, Springer, pp. 1–30, doi:10.1007/11548133_1.

[2] Samson Abramsky & Bob Coecke (2004): *A categorical semantics of quantum protocols*. In: *Logic in Computer Science 19*, IEEE Computer Society, pp. 415–425, doi:10.1109/LICS.2004.1319636.

[3] Samson Abramsky & Chris Heunen (2011): *H*-algebras and nonunital Frobenius algebras: first steps in infinite-dimensional categorical quantum mechanics*. *Clifford Lectures, AMS Proceedings of Symposia in Applied Mathematics* .

[4] John C. Baez (2006): *Structural foundations of quantum gravity*, chapter Quantum quandaries: a category-theoretic perspective, pp. 240–265. Oxford University Press.

[5] Howard Barnum, Jonathan Barrett, Matthew Leifer & Alexander Wilce (2007): *A generalized no-broadcasting theorem*. *Physical Review Letters* 99, p. 240501, doi:10.1103/PhysRevLett.99.240501.

[6] Rejandra Bhatia (2007): *Positive definite matrices*. Princeton University Press.

[7] Sergio Boixo, Carlton M. Caves, Animesh Datta & Anil Shaji (2006): *On decoherence in quantum clock synchronization*. *Laser Physics* 16(11), pp. 1525–1532. Available at `arXiv:quant-ph/0605013`.

[8] Sergio Boixo, Steven T. Flammia, Carlton M. Caves & John M. Geremia (2007): *Generalized Limits for Single-Parameter Quantum Estimation*. *Physical Review Letters* 98(9), pp. 090401–4, doi:10.1103/PhysRevLett.98.090401.

[9] Sergio Boixo & Chris Heunen (2011): *Entangled and sequential quantum protocols with dephasing*. *Submitted to Physical Review Letters* Available at `arXiv:1108.3569`.

[10] Man-Duen Choi & Edward G. Effros (1977): *Injectivity and Operator spaces*. *Journal of Functional Analysis* 24, pp. 156–209, doi:10.1016/0022-1236(77)90052-0.

[11] Bob Coecke & Ross Duncan (2011): *Interacting quantum observables: categorical algebra and diagrammatics*. *New Journal of Physics* 13(4), p. 043016, doi:10.1088/1367-2630/13/4/043016.

[12] Bob Coecke & Éric O. Paquette (2010): *Categories for the practising physicist*. In: *New structures for Physics*, *Lecture Notes in Physics* 813, Springer, pp. 173–286.

[13] Bob Coecke, Duško Pavlović & Jamie Vicary (2009): *A new description of orthogonal bases*. *Mathematical Structures in Computer Science* Available at `arXiv:0810.0812`.

[14] L. Terrell Gardner (1979): *Linear maps of C^*-algebras preserving the absolute value*. *Proceedings of the American Mathematical Society* 76(2), pp. 271–278, doi:10.1090/S0002-9939-1979-0537087-0.

[15] Vittorio Giovannetti, Seth Lloyd & Lorenzo Maccone (2006): *Quantum Metrology*. *Physical Review Letters* 96(1), pp. 010401–4, doi:10.1103/PhysRevLett.96.010401.

[16] Brendon L. Higgins, Dominic W. Berry, Stephen D. Bartlett, Howard M. Wiseman & Geoff J. Pryde (2007): *Entanglement-free Heisenberg-limited phase estimation*. *Nature* 450(7168), pp. 393–396, doi:10.1038/nature06257.

[17] Peter Høyer & Robert Špalek (2005): *Quantum fan-out is powerful*. *Theory of computing* 1, pp. 81–103, doi:10.4086/toc.2005.v001a005.

[18] Michael Keyl & Reinhard F. Werner (2007): *Channels and maps*. In Dagmar Bruß & Gerd Leuchs, editors: *Lectures on Quantum Information*, Wiley-VCH, pp. 73–86.

[19] Joachim Kock (2003): *Frobenius algebras and 2-D Topological Quantum Field Theories*. *London Mathematical Society Student Texts* 59, Cambridge University Press.

[20] Micah Blake McCurdy & Peter Selinger (2006): *Basic dagger compact closed categories*. Unpublished manuscript, Dalhousie University.

[21] Cristopher Moore & Martin Nilsson (2002): *Parallel quantum computation and quantum codes*. *SIAM Journal on Computing* 31(3), pp. 799–815, doi:10.1137/S0097539799355053.

[22] Vern Paulsen (2002): *Completely bounded maps and operators algebras*. Cambridge University Press.

[23] Duško Pavlović (2009): *Quantum and classical structures in nondeterministic computation*. In Peter Bruza et al., editor: *Third International symposium on Quantum Interaction*, *Lecture Notes in Artificial Intelligence* 5494, Springer, pp. 143–157, doi:10.1007/978-3-642-00834-4_13.

[24] Terry Rudolph & Lov Grover (2003): *Quantum Communication Complexity of Establishing a Shared Reference Frame*. *Physical Review Letters* 91(21), p. 217905, doi:10.1103/PhysRevLett.91.217905.

[25] Peter Selinger (2004): *Towards a semantics for higher-order quantum computation*. In: *Quantum Programming Languages. TUCS General Publication*, 33, pp. 127–143.

[26] Peter Selinger (2007): *Dagger compact closed categories and completely positive maps*. In: *Quantum Programming Languages*, *Electronic Notes in Theoretical Computer Science* 170, Elsevier, pp. 139–163, doi:10.1016/j.entcs.2006.12.018.

[27] Peter Selinger (2010): *A survey of graphical languages for monoidal categories*. In: *New Structures for Physics*, *Lecture Notes in Physics* 813, Springer, pp. 289–355.