Completely positive projections and biproducts

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The recently introduced CP*–construction unites quantum channels and classical systems, subsuming the earlier CPM–construction in categorical quantum mechanics. We compare this construction to two earlier attempts at solving this problem: freely adding biproducts to CPM, and freely splitting idempotents in CPM. The CP*–construction embeds the former, and embeds into the latter, but neither embedding is an equivalence in general.

1 Introduction

Two of the authors recently introduced the so-called *CP*–construction*, turning a category $\mathbf{V}$ of abstract state spaces into a category $\mathrm{CP}^*[\mathbf{V}]$ of abstract C*-algebras and completely positive maps [4]. It accommodates both quantum channels and classical systems in a single category. Moreover, it allows nonstandard models connecting to the well-studied theory of groupoids. In particular, it subsumes the earlier CPM–construction, which gives the subcategory $\mathrm{CPM}[\mathbf{V}]$ of the CP*–construction of abstract matrix algebras [8], and adds classical information to it.

There have been earlier attempts at uniting quantum channels and classical systems [8, 9]. This paper compares the CP*–construction to two of them: freely adding biproducts to $\mathrm{CPM}[\mathbf{V}]$, and freely splitting the dagger idempotents of $\mathrm{CPM}[\mathbf{V}]$. These new categories are referred to as $\mathrm{CPM}[\mathbf{V}]^{\oplus}$ and $\mathrm{Split}[\mathrm{CPM}[\mathbf{V}]]$, respectively. We will prove that the CP*–construction lies in between these two: there are full and faithful functors

$$\mathrm{CPM}[\mathbf{V}]^{\oplus} \to \mathrm{CP}^*[\mathbf{V}] \to \mathrm{Split}[\mathrm{CPM}[\mathbf{V}]].$$

When $\mathbf{V}$ is the category of finite-dimensional Hilbert spaces, both outer categories provide “enough space” to reason about classical and quantum data, because any finite-dimensional C*-algebra is a direct sum of matrix algebras (as in $\mathrm{CPM}[\mathbf{FHilb}]^{\oplus}$), and a certain orthogonal subspace of a larger matrix algebra (as in $\mathrm{Split}[\mathrm{CPM}[\mathbf{FHilb}]]$). However, a priori it is unclear whether the second construction captures too much: it may contain many more objects than simply mixtures of classical and quantum state spaces, although none have been discovered so far [9, Remark 4.9]. On the other hand, for $\mathbf{V} \neq \mathbf{FHilb}$, the first construction may not capture enough: there may be interesting objects that are not just sums of quantum systems. For this reason, it is interesting to study $\mathrm{CP}^*[\mathbf{V}]$, because the nonstandard models suggest it captures precisely the right amount of interesting objects.

To be a bit more precise, we will prove that if $\mathbf{V}$ has biproducts, then $\mathrm{CP}^*[\mathbf{V}]$ inherits them. The universal property of the free biproduct completion then guarantees the first embedding above. We will show that this embedding is not an equivalence in general.

For the second embedding, we construct the associated dagger idempotent of an object in $\mathrm{CP}^*[\mathbf{V}]$, and prove that the notions of complete positivity in $\mathrm{CP}^*[\mathbf{V}]$ and $\mathrm{Split}[\mathrm{CPM}[\mathbf{V}]]$ coincide, giving rise to a full and faithful functor. Finally, we will show that this embedding is not an equivalence in general either.

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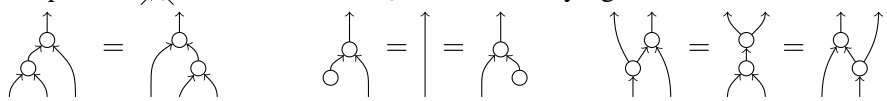
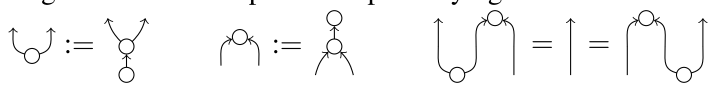
The CP*–construction

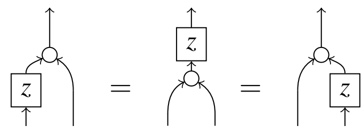
To end this introduction, we very briefly recall the CP*–construction. For more information, we refer to [4]. Let $\mathbf{V}$ be a dagger compact category (see [1, 8]). A *dagger Frobenius algebra* is an object A together with morphisms  : $A \otimes A \to A$ and : $I \to A$ satisfying:

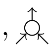


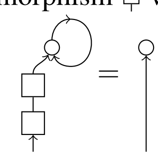
Any dagger Frobenius algebra defines a cap and a cup satisfying the snake identities.



A map $z\colon A \to A$ is *central* for  when:



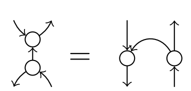
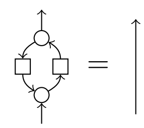
A map $g\colon A \to A$ is *positive* if $g = h^\dagger \circ h$ for some h. A dagger Frobenius algebra $(A,$ , $)$ is *normalisable* if it comes with a central, positive isomorphism  with:

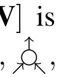
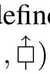
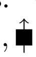
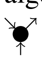
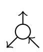


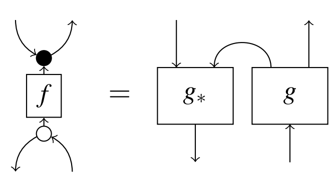
A normalisable dagger Frobenius algebra is *normal* when  $= 1_A$.

Remark 1.1. It will often be more convenient to use the *action* and *coaction* maps associated with a Frobenius algebra, defined as follows:

Using these maps, we can prove alternative forms of the Frobenius and normalisability equations (see Lemmas 2.9 and 2.10 of [4]). These are:

 and  (1)

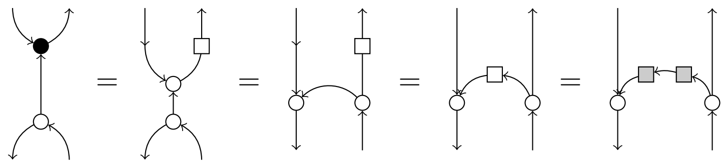
Finally, the category $\mathrm{CP}^*[\mathbf{V}]$ is defined as follows. Objects are normalisable dagger Frobenius algebras in $\mathbf{V}$. Morphisms $(A,$ , , $) \to (B,$ , , $)$ are morphisms $f\colon A \to B$ in $\mathbf{V}$ such that  $\circ f \circ$  is *completely positive*, i.e., such that

 (2)

for some object X and morphism $g\colon A \to X \otimes B$ in $\mathbf{V}$. This category inherits the dagger compact structure from $\mathbf{V}$ [4, Theorem 3.4]. We write $\mathrm{CP}^*_{\mathrm{n}}[\mathbf{V}]$ for the full subcategory whose objects are normal dagger Frobenius algebras in $\mathbf{V}$. Recall that $\mathrm{CPM}[\mathbf{V}]$ has the same objects as $\mathbf{V}$, and morphisms $A \to B$ are completely positive maps $A^* \otimes A \to B^* \otimes B$ [8, Definition 4.18].

Lemma 1.2. *Any normalisable dagger Frobenius algebra in* $\mathbf{V}$ *is isomorphic in* $\mathrm{CP}^*[\mathbf{V}]$ *to a normal one.*

Proof. For an object $(A, \ , \ , \ )$ of $\mathrm{CP}^*[\mathbf{V}]$, define $\ = \ \circ \ $ and $\ = \ ^{-1} \circ \ $. It follows from centrality and self-adjointness of that $(A, \ , \ )$ is a dagger Frobenius algebra in $\mathbf{V}$. Moreover $\ = \ $, and so $(A, \ , \ )$ is normal. Finally, 1_A is a well-defined morphism from $(A, \ , \ , \ )$ to $(A, \ , \ , 1_A)$ in $\mathrm{CP}^*[\mathbf{V}]$: if $\ = \ ^\dagger \circ \ $, then



where the second equality follows from (1). The morphism 1_A is a unitary isomorphism. □

Remark 1.3. The previous lemma shows that $\mathrm{CP}^*[\mathbf{V}]$ and $\mathrm{CP}^*_{\mathrm{n}}[\mathbf{V}]$ are dagger equivalent. Therefore, all properties that we can prove in one automatically transfer to the other, as long as the properties in question are invariant under dagger equivalence. All results in this paper are of this kind, and hence we lose no generality by assuming that all normalisable dagger Frobenius algebras have been normalised.

2 Splitting idempotents

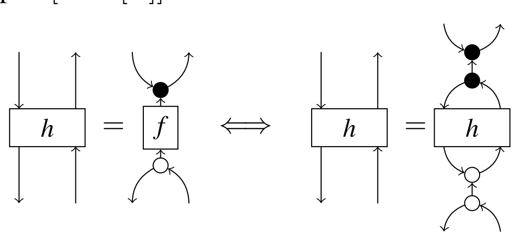
This section exhibits a canonical full and faithful functor from $\mathrm{CP}^*[\mathbf{V}]$ into $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{V}]]$. This functor is not an equivalence for $\mathbf{V} = \mathbf{Rel}$. It is not known whether it is an equivalence for $\mathbf{V} = \mathbf{FHilb}$. However, we characterise its image, showing that the image is equivalent to the full subcategory of $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{FHilb}]]$ consisting of unital dagger idempotents.

Definition 2.1. Let $\mathscr{I}$ be a class of pairs (X, p), where X is an object of $\mathbf{V}$, and $p\colon X \to X$ is a morphism in $\mathbf{V}$ satisfying $p^\dagger = p = p \circ p$, called a *dagger idempotent* or *projection*. The category $\mathrm{Split}_{\mathscr{I}}[\mathbf{V}]$ has $\mathscr{I}$ as objects. Morphisms $(X, p) \to (Y, q)$ in $\mathrm{Split}_{\mathscr{I}}[\mathbf{V}]$ are morphisms $f\colon X \to Y$ in $\mathbf{V}$ satisfying $f = q \circ f \circ p$.

If $\mathscr{I}$ is closed under tensor, then $\mathrm{Split}_{\mathscr{I}}[\mathbf{V}]$ is dagger compact [9, Proposition 3.16]. When $\mathscr{I}$ is the class of all dagger idempotents in $\mathbf{V}$, we also write $\mathrm{Split}^\dagger[\mathbf{V}]$ instead of $\mathrm{Split}_{\mathscr{I}}[\mathbf{V}]$.

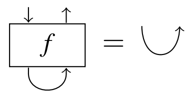
Lemma 2.2. *Let* $\mathbf{V}$ *be any dagger compact category. Then there is a canonical functor* $F\colon \mathrm{CP}^*[\mathbf{V}] \to \mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{V}]]$ *acting as* $F(A, \ , \ , \ ) = (A, \ \circ \ \circ \ \circ \ )$ *on objects, and as* $F(f) = \ \circ \ \circ f \circ \ \circ \ $ *on morphisms. It is full, faithful, and strongly dagger symmetric monoidal.*

Proof. First, note that $p = F(A, \ , \ , \ )$ is a well-defined object of $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{V}]]$: clearly $p = \ \circ \ \circ \ \circ \ = p^\dagger$; also, it follows from (1) that $p \circ p = p$ and that p is completely positive. Also, $F(f)$ is a well-defined morphism in $\mathrm{CPM}[\mathbf{V}]$ by (2). By (1), $F(f)$ is in fact a well-defined morphism in $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{V}]]$. Next, F is faithful because $f = \ \circ \ \circ F(f) \circ \ \circ \ $. To show that F is full, note that an arbitrary morphism $h\colon A^* \otimes A \to B^* \otimes B$ in $\mathbf{V}$ is a well-defined morphism in $\mathrm{CP}^*[\mathbf{V}]$ if and only if it is a well-defined morphism in $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{V}]]$:

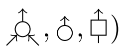


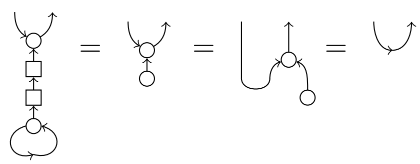
Both $\mathrm{CP}^*[\mathbf{V}]$ and $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{V}]]$ inherit composition, identities, and daggers from $\mathbf{V}$, so F is a full and faithful functor preserving daggers. Similarly, the symmetric monoidal structure in both $\mathrm{CP}^*[\mathbf{V}]$ and $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{V}]]$ is defined in terms of that of $\mathbf{V}$, making F strongly symmetric monoidal. □

It stands to reason that F might become an equivalence by restricting the class $\mathscr{I}$. For example, the following lemma shows that we should at least restrict to splitting unital projections. A completely positive map $f\colon A^* \otimes A \to B^* \otimes B$ is *unital* when:



Lemma 2.3. *The functor F from Lemma 2.2 lands in* $\mathrm{Split}_{\mathscr{I}}[\mathrm{CPM}[\mathbf{V}]]$*, where $\mathscr{I}$ consists of the unital dagger idempotents.*

Proof. Let $(A,$ $)$ be an object of $\mathrm{CP}^*[\mathbf{V}]$. Then $F(A)$ is unital because:



□

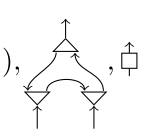
From now on, we will fix $\mathscr{I}$ to be the class of unital dagger idempotents.

Hilbert spaces

When $\mathbf{V} = \mathbf{FHilb}$, the objects of $\mathrm{CP}^*[\mathbf{FHilb}]$ are precisely (concrete) C*-algebras, and the morphisms are completely positive maps in the usual sense of C*-algebras; see [4]. The unital completely positive maps $p \in \mathscr{I}$ are precisely the physically realisable projections. We will prove that F is then an equivalence, by employing a classic theorem by Choi and Effros. It is well-known that the image $f(A)$ of a *-homomorphism $f\colon A \to B$ is a C*-subalgebra of B. The Choi-Effros theorem shows that the image $p(A)$ of a completely positive unital projection $p\colon A \to A$ is a C*-algebra in its own right. In general, it need no longer be a C*-subalgebra; it can have a different multiplication. The following proposition and its proof make precise what we need. Write $\mathbb{M}_n(A)$ for the C*-algebra of n-by-n matrices with entries in A, and simply $\mathbb{M}_n$ for $\mathbb{M}_n(\mathbb{C})$. The assignment $A \mapsto \mathbb{M}_n(A)$ is functorial on the category of C*-algebras and *-homomorphisms. The category $\mathrm{CPM}[\mathbf{FHilb}]$ can be identified with the full subcategory of $\mathrm{CP}^*[\mathbf{FHilb}]$ consisting of the matrix algebras $\mathbb{M}_n$.

Proposition 2.4. *There is a functor* $G\colon \mathrm{Split}_{\mathscr{I}}[\mathrm{CPM}[\mathbf{FHilb}]] \to \mathrm{CP}^*[\mathbf{FHilb}]$ *that sends an object* $(\mathbb{M}_m, p)$ *to its range* $p(\mathbb{M}_m)$*, and a morphism* $f\colon (\mathbb{M}_m, p) \to (\mathbb{M}_n, q)$ *to its underlying function* $f\colon \mathbb{M}_m \to \mathbb{M}_n$.

Proof. Because p is unital, it is certainly contractive, because it has operator norm $\|p\| = \|p(1)\| = \|1\| = 1$. Therefore, a classic theorem by Choi and Effros applies, showing that $p(\mathbb{M}_m)$ is a well-defined C*-algebra under the product $(a,b) \mapsto p(ab)$ [3, Theorem 3.1] (see also [10, Section 2.2]). Hence G is well-defined on objects. Because dagger idempotents dagger split in **FHilb**, this can be denoted graphically as

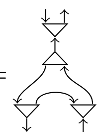
$$G(\mathbb{M}_m \xrightarrow{p} \mathbb{M}_m) = (p(\mathbb{M}_m),$$  $$)$$,

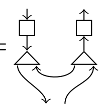
where : $(\mathbb{C}^m)^* \otimes \mathbb{C}^m \to p(\mathbb{M}_m)$ is (a dagger splitting of) p with inclusion : $p(\mathbb{M}_m) \to (\mathbb{C}^m)^* \otimes \mathbb{C}^m$, and is the unique normaliser. That is, we have $1_{p(\mathbb{M}_m)} = p =$ ∘ : $p(\mathbb{M}_m) \to p(\mathbb{M}_m)$.

Choi and Effros [3, Theorem 3.1] also study how positivity in $p(\mathbb{M}_m)$ and $\mathbb{M}_m$ is related. Specifically, they prove that $\mathbb{M}_k(p(\mathbb{M}_m))^+ = \mathbb{M}_k(\mathbb{M}_m)^+ \cap \mathbb{M}_k(p(\mathbb{M}_m))$, where A^+ denotes the positive cone of a C*-algebra A. To see that G is well-defined on morphisms, let $f\colon (\mathbb{M}_m, p) \to (\mathbb{M}_n, q)$ be a morphism in $\mathrm{Split}_{\mathscr{I}}[\mathrm{CPM}[\mathbf{FHilb}]]$. Then $G(f)$ is a well-defined completely positive map $p(\mathbb{M}_m) \to q(\mathbb{M}_n)$ precisely when $x \in \mathbb{M}_k(p(\mathbb{M}_m))^+$ implies $\mathbb{M}_k f(x) \in \mathbb{M}_k(q(\mathbb{M}_n))^+$ for all k. But this is indeed true because $f\colon \mathbb{M}_m \to \mathbb{M}_n$ is a completely positive map satisfying $f = q \circ f \circ p$. Finally, G is clearly functorial. □

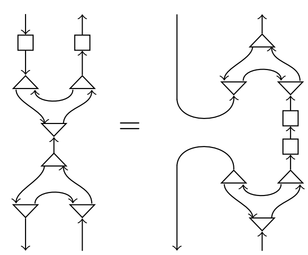
Theorem 2.5. *The functors F and G implement an equivalence between the categories* $\mathrm{CP}^*[\mathbf{FHilb}]$ *and* $\mathrm{Split}_{\mathscr{I}}[\mathrm{CPM}[\mathbf{FHilb}]]$.

Proof. Let $p\colon \mathbb{M}_m \to \mathbb{M}_m$ be a completely positive unital projection. We will show that $F(G(p)) \cong p$. This will establish that F is essentially surjective on objects. Since it is also full and faithful, it follows that F is an equivalence. Using the graphical notation from the proof of Proposition 2.4, define

$g =$  $:\ p(\mathbb{M}_m)^* \otimes p(\mathbb{M}_m) \to \mathbb{M}_m,$

$f =$  $:\ \mathbb{M}_m \to p(\mathbb{M}_m)^* \otimes p(\mathbb{M}_m).$

Then f is in Kraus form, and hence completely positive, by construction. Similarly, g is the composition of $p =$ ∘ , which is completely positive by assumption, and another map that is completely positive by construction. Hence f and g are well-defined morphisms in $\mathrm{CPM}[\mathbf{FHilb}]$. Moreover, by (1), $g \circ f =$ ∘ $= p\colon \mathbb{M}_m \to \mathbb{M}_m$. Also,

$f \circ g =$  $= F(G(p)).$

Therefore $f \circ g = F(G(p))\colon\ p(\mathbb{M}_m)^* \otimes p(\mathbb{M}_m) \to p(\mathbb{M}_m)^* \otimes p(\mathbb{M}_m)$. It follows that $f = F(G(p)) \circ f \circ p$ and $g = p \circ g \circ F(G(p))$, making f and g into well-defined morphisms of $\mathrm{Split}_{\mathscr{I}}[\mathrm{CPM}[\mathbf{FHilb}]]$. In fact, this shows that f and g implement an isomorphism in that category, establishing $F(G(p)) \cong p$.

It now follows that if $A \in \mathrm{CP}^*[\mathbf{FHilb}]$, then $F(G(F(A))) \cong F(A)$, and because F is full and faithful, hence $G(F(A)) \cong A$. It is easy to see that this isomorphism, as well as $F(G(p)) \cong p$, is natural. Thus F and G form an equivalence. □

Remark 2.6. To motivate the need to restrict to the class of unital projections $\mathscr{I}$, let us show that not every object in $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{FHilb}]]$ is unital. We give a counterexample of a completely positive projection that is not even contractive.[1] Take $A = \mathbb{M}_n$, and let $a \in A$ satisfy $a \geq 0$, $\|a\| > 1$, and $\mathrm{Tr}(a) =$

[1] We thank Erling Størmer for discussions on this subject.

$\mathrm{Tr}(a^2)$. For example, we could pick $n = 2$ and

$$a = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

We will define p as the orthogonal projection onto the one-dimensional subspace spanned by a suitable density matrix ρ. Precisely, define $\rho \in A$, $f\colon A \to \mathbb{C}$, and $p\colon A \to A$ by

$$\rho = \frac{a}{\mathrm{Tr}(a)}, \qquad f(x) = \mathrm{Tr}(\rho x), \qquad p(x) = f(x)a.$$

Then $\rho \geq 0$ and $\mathrm{Tr}(\rho) = 1$, so ρ is a density matrix. The adjoint of p with respect to the trace inner product $\langle x \,|\, y \rangle = \mathrm{Tr}(x^\dagger y)$ is $p^\dagger(x) = \rho\,\mathrm{Tr}(ax)$:

$$\mathrm{Tr}(p(x)y) = \mathrm{Tr}(\rho x)\,\mathrm{Tr}(ay) = \mathrm{Tr}(xp^\dagger(y)).$$

Hence p is self-adjoint:

$$p^\dagger(x) = \rho\,\mathrm{Tr}(ax) = \frac{\mathrm{Tr}(ax)a}{\mathrm{Tr}(a)} = \mathrm{Tr}(\rho x)a = p(x).$$

It is also idempotent, because $f(a) = \mathrm{Tr}(\rho a) = \frac{\mathrm{Tr}(a^2)}{\mathrm{Tr}(a)} = 1$:

$$p^2(x) = p(\mathrm{Tr}(\rho x)a) = \mathrm{Tr}(\rho x)p(a) = \mathrm{Tr}(\rho x)\,\mathrm{Tr}(\rho a)a = \mathrm{Tr}(\rho x)a = p(x).$$

Thus p is a well-defined object of $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{FHilb}]]$. But by the Russo-Dye theorem [10, Theorem 1.3.3], the operator norm of p is

$$\|p\| = \|p(1)\| = \|\,\mathrm{Tr}(\rho)a\| = \|a\| > 1.$$

Hence p is not contractive, and in particular, not unital. We leave open the question whether every object of $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{FHilb}]]$ is isomorphic to a unital one.

Sets and relations

Now consider $\mathbf{V} = \mathbf{Rel}$, the category of sets and relations. We will show that in this case, the canonical functor F is not an equivalence, even when restricting to the class of unital projections $\mathscr{I}$, and in fact that there can be no dagger equivalence at all.

Recall from [4] that the category $\mathrm{CP}^*[\mathbf{Rel}]$ has small groupoids $\mathbf{G}$ as objects; morphisms $\mathbf{G} \to \mathbf{H}$ are relations $R\colon\ \mathrm{Mor}(\mathbf{G}) \to \mathrm{Mor}(\mathbf{H})$ satisfying

$$\text{if } gRh, \text{ then } g^{-1}Rh^{-1} \text{ and } 1_{\mathrm{dom}(g)}R1_{\mathrm{dom}(h)}. \tag{3}$$

Notice that $\mathrm{CP}^*[\mathbf{Rel}] = \mathrm{CP}^*_{\mathrm{n}}[\mathbf{Rel}]$ because the only positive isomorphisms in $\mathbf{Rel}$ are identities.

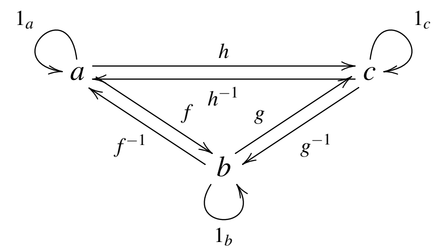
We say that two dagger categories $\mathbf{C}$ and $\mathbf{D}$ are *dagger equivalent* when there exist dagger functors $F\colon\ \mathbf{C} \to \mathbf{D}$ and $G\colon\ \mathbf{D} \to \mathbf{C}$ and natural unitary isomorphisms $G \circ F \cong 1_{\mathbf{C}}$ and $F \circ G \cong 1_{\mathbf{D}}$.

Lemma 2.7. *If all dagger idempotents dagger split in a dagger category* $\mathbf{C}$*, then they do so in any dagger equivalent category* $\mathbf{D}$.

Proof. Let $p\colon X \to X$ be a dagger idempotent in $\mathbf{D}$. Then $G(p)$ is a dagger idempotent in $\mathbf{C}$, and hence dagger splits; say $f\colon G(X) \to Y$ satisfies $G(p) = f^\dagger \circ f$ and $f \circ f^\dagger = 1_Y$. Let u be the unitary isomorphism $X \to F(G(X))$, and set $g = F(f) \circ u\colon X \to F(Y)$. Then $p = g^\dagger \circ g$ and $g \circ g^\dagger = F(1_Y) = 1_{F(Y)}$. □

Theorem 2.8. *The categories* $\mathrm{CP}^*[\mathbf{Rel}]$ *and* $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{Rel}]]$ *cannot be dagger equivalent.*

Proof. By the previous lemma it suffices to exhibit a dagger idempotent in $\mathrm{CP}^*[\mathbf{Rel}]$ that does not dagger split. Let $\mathbf{G}$ be the connected groupoid with 3 objects and 9 morphisms:

$\mathbf{G} =$ 

Write G for the set of morphisms of $\mathbf{G}$, and define $R = \{(x,x) \mid x \in G \setminus \{h,h^{-1}\}\} \subseteq G \times G$. Then R satisfies (3), and hence is a well-defined morphism in $\mathrm{CP}^*[\mathbf{Rel}]$. Moreover, it is a dagger idempotent. Suppose that R dagger splits via some $S \subseteq G \times H$; concretely, this means H is the morphism set of some groupoid $\mathbf{H}$, and S satisfies equation (3), $R = S^\dagger \circ S$, and $S \circ S^\dagger = 1_H$. It follows from $R = S^\dagger \circ S$ that x is related by S to some element of H if and only if x is neither h nor h^{-1}. It also follows from $S^\dagger \circ S = R \subseteq 1_G$ that xSy and $x'Sy$ imply $x = x'$. Hence $S^\dagger$ is single-valued. Furthermore, it follows from $S \circ S^\dagger = 1_H$ that any $y \in H$ relates to some $x \in G$ by $S^\dagger$, and that xSy and xSy' imply $y = y'$. Thus S is (the graph of) a bijection $\{1_a, 1_b, 1_c, f, f^{-1}, g, g^{-1}\} \to H$, and $S^\dagger$ is (the graph of) its inverse. Hence $\mathbf{H}$ must have 7 morphisms.

If $S(f)$ were an endomorphism, then $S(1_a) = 1_{\mathrm{dom}(f)} = S(1_b)$ by (3), contradicting injectivity of S. Similarly, $S(g)$ cannot be an endomorphism. So we may assume that $\mathrm{dom}(S(1_a)) \stackrel{S(f)}{\to} \mathrm{dom}(S(1_b)) \stackrel{S(g)}{\to} \mathrm{dom}(S(1_c))$ with $S(1_a) \neq S(1_b)$. But because $\mathbf{H}$ is a groupoid, there must exist a morphism $\mathrm{dom}(S(1_a)) \to \mathrm{dom}(S(1_b))$, which contradicts the fact that $\mathbf{H}$ can only have 7 morphisms. □

Corollary 2.9. *The functor* $F\colon \mathrm{CP}^*[\mathbf{Rel}] \to \mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{Rel}]]$ *is not an equivalence.*

Proof. Suppose $G\colon \mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{Rel}]] \to \mathrm{CP}^*[\mathbf{Rel}]$ and F form an equivalence with natural isomorphism $\eta_X\colon F(G(X)) \to X$. Let $g\colon X \to Y \in \mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{Rel}]]$. Because every isomorphism in $\mathbf{Rel}$ is unitary, and F preserves daggers,

$$F(G(g^\dagger)) = \eta_X^\dagger \circ g^\dagger \circ \eta_Y = (\eta_Y^\dagger \circ g \circ \eta_X)^\dagger = F(G(g))^\dagger = F(G(g)^\dagger).$$

Since F is faithful, G must also preserve daggers. So F and G in fact form a dagger equivalence. But that contradicts the previous theorem. □

To show that F is not an equivalence even when we restrict to splitting just unital projections $\mathscr{I}$, we need to analyse the isomorphisms in $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{Rel}]]$ further. This is what the rest of this section does.

Lemma 2.10. *Dagger idempotents in* $\mathbf{Rel}$ *are precisely partial equivalence relations.*

Proof. Clearly $R^\dagger = R$ if and only if R is symmetric. Also, $R^2 \subseteq R$ if and only if R is transitive. We will prove that if R is symmetric, then also $R \subseteq R^2$. Suppose xRz. Then also zRx, and so xRx. Hence $xRyRz$ for $y = x$. □

It follows that the category $\mathrm{Split}^\dagger[\mathbf{Rel}]$ has pairs $(X,\sim)$ as objects, where X is a set, and $\sim$ is a partial equivalence relation on X; morphisms $(X,\sim) \to (Y,\approx)$ are relations $R\colon X \to Y$ satisfying $R = {\approx} \circ R \circ {\sim}$. For a partial equivalence relation $\sim$ on X, we write $\mathrm{D}(\sim) = \{x \in X \mid x \sim x\}$.

Lemma 2.11. *Dagger idempotents in* **Rel** *dagger split.*

Proof. Let $\sim$ be a partial equivalence relation on X. Define a splitting relation $R\colon \mathrm{D}(\sim)/{\sim} \to X$ by $R = \{([x]_\sim, x) \mid x \in \mathrm{D}(\sim)\}$. Then

$$\begin{aligned} R^\dagger \circ R &= \{([x]_\sim,[z]_\sim) \mid x,z \in \mathrm{D}(\sim), \exists y \in X\colon x \sim y \sim z\} \\ &= \{([x]_\sim,[z]_\sim) \mid x,z \in \mathrm{D}(\sim), x \sim z\} \\ &= \{([x]_\sim,[x]_\sim) \mid x \in \mathrm{D}(\sim)\} \\ &= 1_{\mathrm{D}(\sim)/\sim}, \end{aligned}$$

and $R \circ R^\dagger = \{(x,z) \mid x,z \in X, \exists y \in \mathrm{D}(\sim)/{\sim}\colon x \sim y \sim z\} = {\sim}$. □

Recall that the category $\mathrm{CPM}[\mathbf{Rel}]$ has sets X as objects; morphisms $X \to Y$ are relations $R\colon X \times X \to Y \times Y$ satisfying

$$(x,x')R(y,y') \implies (x',x)R(y',y) \wedge (x,x)R(y,y). \tag{4}$$

Hence the category $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{Rel}]]$ has pairs $(X,\sim)$ as objects, where X is a set, and $\sim$ is a partial equivalence relation on $X \times X$ satisfying

$$(x,x') \sim (y,y') \implies (x',x) \sim (y',y) \wedge (x,x) \sim (y,y); \tag{5}$$

morphisms $(X,\sim) \to (Y,\approx)$ are relations $R\colon X \times X \to Y \times Y$ satisfying (4) and $R = {\approx} \circ R \circ {\sim}$.

In this description, $F(\mathbf{G}) = (\mathrm{Mor}(\mathbf{G}),\sim)$, where $(a,b) \sim (c,d)$ if and only if $a^{-1}b = c^{-1}d$ (and both compositions are well-defined).

When speaking about a partial equivalence relation $\sim$ on $X \times X$, we will abbreviate $[(x,x')]_\sim$ to $[x,x']_\sim$.

Lemma 2.12. *Objects $(X,\sim)$ and $(Y,\approx)$ are isomorphic in $\mathrm{Split}^\dagger[\mathbf{Rel}]$ if and only if $\mathrm{D}(\sim)/{\sim}$ and $\mathrm{D}(\approx)/{\approx}$ are isomorphic in* **Rel** *(and hence in* **Set***). Furthermore, partial equivalence relations $\sim\colon A \times A \to A \times A$ and $\approx\colon B \times B \to B \times B$ are isomorphic objects of $\mathrm{Split}^\dagger[\mathrm{CPM}[\mathbf{Rel}]]$ if and only if there is a bijection $\alpha\colon \mathrm{D}(\sim)/{\sim} \to \mathrm{D}(\approx)/{\approx}$ satisfying*

$$\alpha([a,a']_\sim) = [b,b']_\approx \implies \alpha([a',a]_\sim) = [b',b]_\sim \wedge \alpha([a,a]_\sim) = [b,b]_\approx. \tag{6}$$

Proof. Suppose $(X,\sim)$ and $(Y,\approx)$ are isomorphic in $\mathrm{Split}^\dagger[\mathbf{Rel}]$. Say relations $R\colon X \to Y$ and $S\colon Y \to X$ satisfy $S \circ R = {\sim}$ and $R \circ S = {\approx}$. Define relations $U\colon \mathrm{D}(\sim)/{\sim} \to \mathrm{D}(\approx)/{\approx}$ and $V\colon \mathrm{D}(\approx)/{\approx} \to \mathrm{D}(\sim)/{\sim}$ by $U = \{([x]_\sim,[y]_\approx) \mid (x,y) \in R\}$ and $V = \{([y]_\approx,[x]_\sim) \mid (y,x) \in S\}$. Then

$$V \circ U = \{([x]_\sim,[x]_\sim) \mid (x,x') \in S \circ R\} = \{([x]_\sim,[x]_\sim) \mid x \in \mathrm{D}(\sim)\} = 1_{\mathrm{D}(\sim)/\sim},$$

and similarly $U \circ V = 1_{\mathrm{D}(\approx)/\approx}$. Hence $\mathrm{D}(\sim)/{\sim}$ and $\mathrm{D}(\approx)/{\approx}$ are isomorphic in **Rel**.

Conversely, assume that $\mathrm{D}(\sim)/{\sim}$ and $\mathrm{D}(\approx)/{\approx}$ are isomorphic in **Rel**. Say $U\colon \mathrm{D}(\sim)/{\sim} \to \mathrm{D}(\approx)/{\approx}$ and $V\colon \mathrm{D}(\approx)/{\approx} \to \mathrm{D}(\sim)/{\sim}$ satisfy $U \circ V = 1_{\mathrm{D}(\approx)/\approx}$ and $V \circ U = 1_{\mathrm{D}(\sim)/\sim}$. Define relations $R\colon X \to Y$ and $S\colon Y \to X$ by $R = \{(x,y) \mid [x]_\sim U [y]_\approx\}$ and $S = \{(y,x) \mid [y]_\approx V [x]_\sim\}$. Then

$${\approx} \circ R \circ {\sim} = \{(x,y) \mid \exists x',y'\colon x \sim x', y \approx y', [x']_\sim U [y']_\approx\} = R,$$

and similarly S is a well-defined morphism of $\mathrm{Split}^{\dagger}[\mathbf{Rel}]$. Also

$$S \circ R = \{(x,x') \mid \exists y\colon [x]_{\sim} U [y]_{\approx} V [x']_{\sim}\} = \{(x,x') \mid ([x]_{\sim},[x']_{\sim}) \in 1_{\mathrm{D}(\sim)/\sim}\} = \sim,$$

and similarly $R \circ S = \approx$. So $(X,\sim)$ and $(Y,\approx)$ are isomorphic in $\mathrm{Split}^{\dagger}[\mathbf{Rel}]$.

In case $X = A \times A$ and $Y = B \times B$, notice that R and S satisfy (4) if and only if the bijection $\alpha = U$ and its inverse $\alpha^{-1} = V$ satisfy (6). Finally, α^{-1} satisfies (6) precisely when α does. □

Lemma 2.13. *If* $\mathbf{G}$ *is a small groupoid and* $F(\mathbf{G}) = (\mathrm{Mor}(\mathbf{G}),\sim)$*, then* $\mathrm{D}(\sim)/\sim$ *is in bijection with* $\mathrm{Mor}(\mathbf{G})$*. Furthermore, an object* $(X,\sim)$ *of* $\mathrm{Split}^{\dagger}[\mathrm{CPM}[\mathbf{Rel}]]$ *is isomorphic to* $F(\mathbf{G})$ *for a small groupoid* $\mathbf{G}$ *if and only if there is a bijection* $\beta\colon \mathrm{Mor}(\mathbf{G}) \to \mathrm{D}(\sim)/\sim$ *satisfying*

$$\beta(g) = [x,x']_{\sim} \implies \beta(g^{-1}) = [x',x]_{\sim} \wedge \beta(1_{\mathrm{dom}(g)}) = [x,x]_{\sim} \tag{7}$$

for all $g \in \mathrm{Mor}(\mathbf{G})$.

Proof. Define functions $\gamma\colon \mathrm{D}(\sim)/\sim \to \mathrm{Mor}(\mathbf{G})$ and $\beta\colon \mathrm{Mor}(\mathbf{G}) \to \mathrm{D}(\sim)/\sim$ by $\gamma([g,f]_{\sim}) = g^{-1}f$ and $\beta(h) = [1_{\mathrm{cod}(h)},h]_{\sim}$. Then

$$\beta \circ \gamma([g,f]_{\sim}) = \gamma(g^{-1}f) = [1_{\mathrm{dom}(g)}, g^{-1}f]_{\sim} = [g,f]_{\sim}$$

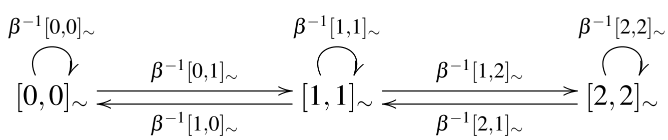
and $\gamma \circ \beta(h) = h$. The second statement now follows from Lemma 2.12. □

Theorem 2.14. *The functor* $F\colon \mathrm{CP}^*[\mathbf{Rel}] \to \mathrm{Split}_{\mathscr{I}}[\mathrm{CPM}[\mathbf{Rel}]]$ *is not an equivalence.*

Proof. In the setting of the second statement of Lemma 2.13, the identities of $\mathbf{G}$ must be the morphisms $\beta^{-1}([x,x]_{\sim})$ for $x \in X$. Therefore we may restrict to groupoids with $\mathrm{Ob}(\mathbf{G}) = \{[x,x]_{\sim} \mid x \in X\}$. Furthermore, it then follows from (7) that $\beta^{-1}[x,x']_{\sim}$ is a morphism $[x,x]_{\sim} \to [x',x']_{\sim}$. The same counterexample as in the proof of Theorem 2.8 now shows that F is not an equivalence. Take $X = \{0,1,2\}$, and let $\sim$ be specified by

$$\begin{array}{lll} (0,0) \sim (0,0), & (1,1) \sim (1,1), & (2,2) \sim (2,2), \\ (0,1) \sim (0,1), & (1,0) \sim (1,0), & \\ (1,2) \sim (1,2), & (2,1) \sim (2,1); & \end{array}$$

no other pairs satisfy $(x,x') \sim (y,y')$. In particular, $(0,2) \not\sim (0,2)$. Then $\sim$ is a partial equivalence relation that satisfies (5), and so $(X,\sim)$ is a well-defined object in $\mathrm{Split}^{\dagger}[\mathrm{CPM}[\mathbf{Rel}]]$. Now suppose that $(X,\sim)$ is isomorphic to $F(\mathbf{G})$. As discussed above, we may assume that $\mathbf{G}$ has three objects $0,1,2$ and seven morphisms, with types as follows.



But this can never be made into a groupoid: there are arrows $[0,0]_{\sim} \to [1,1]_{\sim}$ and $[1,1]_{\sim} \to [2,2]_{\sim}$, but no morphisms $[0,0]_{\sim} \to [2,2]_{\sim}$, so no composition can be defined. We conclude that the essential image of F is not all of $\mathrm{Split}^{\dagger}[\mathrm{CPM}[\mathbf{Rel}]]$.

In fact, $(X,\sim)$ is an object of $\mathrm{Split}_{\mathscr{I}}[\mathrm{CPM}[\mathbf{Rel}]]$, i.e. it is unital (and therefore trace-preserving) precisely when $(x,x) \in D(\sim)$ for all $x \in X$. Since the above counterexample satisfies this, the restriction $F\colon \mathrm{CP}^*[\mathbf{Rel}] \to \mathrm{Split}_{\mathscr{I}}[\mathrm{CPM}[\mathbf{Rel}]]$ is not an equivalence. □

3 Biproducts

This section shows that if $\mathbf{V}$ has biproducts, then so does $\mathrm{CP}^*[\mathbf{V}]$, and there is a full and faithful functor $\mathrm{CPM}[\mathbf{V}]^{\oplus} \to \mathrm{CP}^*[\mathbf{V}]$. Furthermore, this functor is an equivalence for $\mathbf{V} = \mathbf{Hilb}$, but not for $\mathbf{V} = \mathbf{Rel}$.

Early in the development of categorical quantum mechanics, classical information was modelled by biproducts. Since categories of completely positive maps need not inherit biproducts from their base category, biproducts had to be explicitly added to $\mathrm{CPM}[V]$. Later on, Frobenius algebras were proposed as an alternative to biproducts. We now come full circle by proving a satisfying relationship between Frobenius algebras, completely positive maps, and biproducts. This requires quite some detailed (matrix) calculations. We first summarise the basic interaction of biproducts and dual objects.

Recall that a *zero object* is a terminal initial object. A zero object induces unique *zero maps* from any object to any other object that factor through the zero object. A *biproduct* of objects A and B consists of an object $A \oplus B$ together with morphisms $A \underset{p_A}{\overset{i_A}{\rightleftarrows}} A \oplus B \underset{p_B}{\overset{i_B}{\leftrightarrows}} B$, such that $A \oplus B$ is simultaneously a product of A and B with projections p_A and p_B and a coproduct of A and B with injections i_A and i_B, satisfying $p_A \circ i_A = 1_A$, $p_B \circ i_B = 1_B$, $p_A \circ i_B = 0$, and $p_B \circ i_A = 0$. A category has *dagger biproducts* when it has a zero object and biproducts of any pair of objects such that $p_A = i_A^\dagger$ and $p_B = i_B^\dagger$.

Categories with biproducts are automatically enriched over commutative monoids: $f + g = [1,1] \circ (f \oplus g) \circ \langle 1,1 \rangle$. This means that morphisms between biproducts of objects can be handled using a matrix calculus. We will also write Δ_A for the diagonal tuple $\langle 1,1 \rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : A \to A \oplus A$.

In a compact category $\mathbf{C}$, the functor $- \otimes A \colon \mathbf{C} \to \mathbf{C}$ is both left and right adjoint to the functor $- \otimes A^*$. If $\mathbf{C}$ has a zero object, it follows directly that $A \otimes 0 \cong 0$ for any object A. Consequently, if f is any morphism, then $f \otimes 0$ factors through $\mathrm{dom}(f) \otimes 0$ and must therefore equal the zero morphism.

The adjunctions also imply that $- \otimes A$ preserves both limits and colimits. So if $\mathbf{C}$ has biproducts, then $\otimes$ distributes over $\oplus$. Consequently, the following morphisms are each other's inverse.

$$\begin{array}{c} (A \oplus B) \otimes (C \oplus D) \\ \begin{pmatrix} i_A \otimes i_C & i_A \otimes i_D & i_B \otimes i_C & i_B \otimes i_D \end{pmatrix} \Big\uparrow \ \Big\downarrow \begin{pmatrix} p_A \otimes p_C \\ p_A \otimes p_D \\ p_B \otimes p_C \\ p_B \otimes p_D \end{pmatrix} \\ (A \otimes C) \oplus (A \otimes D) \oplus (B \otimes C) \oplus (B \otimes D) \end{array}$$

It follows that $f \otimes (g+h) = (f \otimes g) + (f \otimes h)$ and $(f+g) \otimes h = (f \otimes h) + (g \otimes h)$. Also

$$\begin{aligned} \mathbf{C}((A \oplus B)^*, C) &\cong \mathbf{C}(I, (A \oplus B) \otimes C) \\ &\cong \mathbf{C}(I, (A \otimes C) \oplus (B \otimes C)) \\ &\cong \mathbf{C}(I, A \otimes C) \times \mathbf{C}(I, B \otimes C) \\ &\cong \mathbf{C}(A^*, C) \times \mathbf{C}(B^*, C) \\ &\cong \mathbf{C}(A^* \oplus B^*, C), \end{aligned}$$

so by the Yoneda lemma $(A \oplus B)^* \cong A^* \oplus B^*$. Tracing through the steps carefully, we may in fact choose the following unit and counit for compactness:

$$\begin{aligned} \varepsilon_{A \oplus B} &= (\varepsilon_A \circ (p_A \otimes p_{A^*})) + (\varepsilon_B \circ (p_B \otimes p_{B^*})) \colon (A \oplus B) \otimes (A^* \oplus B^*) \to I, \\ \eta_{A \oplus B} &= ((i_{A^*} \otimes i_A) \circ \eta_A) + ((i_{B^*} \otimes i_B) \circ \eta_B) \colon I \to (A^* \oplus B^*) \otimes (A \oplus B). \end{aligned}$$

Lemma 3.1. *If (A,m_A,u_A) and (B,m_B,u_B) are normal dagger Frobenius algebras in a dagger compact category with dagger biproducts, then*

$$m_{A\oplus B} = \begin{pmatrix} m_A\circ(p_A\otimes p_A) \\ m_B\circ(p_B\otimes p_B)\end{pmatrix} : (A\oplus B)\otimes(A\oplus B)\to(A\oplus B)$$
$$u_{A\oplus B} = \begin{pmatrix} u_A \\ u_B \end{pmatrix} : I\to A\oplus B$$

make $A\oplus B$ into a normal dagger Frobenius algebra. Furthermore, 0 is uniquely made into a normal dagger Frobenius algebra by

$$m_0 = 0\colon 0\otimes 0\to 0, \qquad u_0 = 0\colon I\to 0.$$

Proof. Verifying the required properties is a matter of equational rewriting of matrices. For example, to show that $(A\oplus B, m_{A\oplus B}, u_{A\oplus B})$ is normal:

$$\begin{aligned}&\varepsilon_{A\oplus B}\circ(1_{A^*\oplus B^*}\otimes m_{A\oplus B})\circ(\eta_{A\oplus B}\otimes 1_{A\oplus B})\\ &=[\varepsilon_A\circ(1_{A^*}\otimes m_A)\circ(\eta_A\otimes 1_A), \varepsilon_B\circ(1_{B^*}\otimes m_B)\circ(\eta_B\otimes 1_B)]\\ &=[u_A^\dagger, u_B^\dagger] = u_{A\oplus B}^\dagger.\end{aligned}$$

One similarly verifies associativity and the Frobenius law. Because **V** is compact, unitality then follows automatically [2, Proposition 7]. As for $(0,m_0,u_0)$: all required diagrams commute because they are in fact equal to the zero morphism, and hence the multiplication m_0 is unique. □

Theorem 3.2. *If* **V** *is dagger compact with dagger biproducts, so is* $\mathrm{CP}^*[\mathbf{V}]$.

Proof. Because dagger biproducts are preserved under dagger equivalences, it suffices to prove that $\mathrm{CP}^*_\mathrm{n}[\mathbf{V}]$ has dagger biproducts by Remark 1.3. We claim that the objects defined in Lemma 3.1 in fact form dagger biproducts in $\mathrm{CP}^*_\mathrm{n}[\mathbf{V}]$. We prove this according to the following strategy. Any object A of $\mathrm{CP}^*_\mathrm{n}[\mathbf{V}]$ gives morphisms $0\colon A\to 0$, $i_A\colon A\to A\oplus 0$, $p_A\colon A\oplus 0\to A$, and $\Delta_A\colon A\to A\oplus A$ in **V**. We will show that these are all *-homomorphisms (see [4, Definition 3.6]), and hence well-defined morphisms in $\mathrm{CP}^*_\mathrm{n}[\mathbf{V}]$ by [4, Lemma 3.7]. Furthermore, it is easy to see that if f and g are morphisms in $\mathrm{CP}^*_\mathrm{n}[\mathbf{V}]$, then so is $f\oplus g$. Observing that all coherence isomorphisms for $(\mathbf{V},\oplus,0)$ and their inverses are built by composition from the above maps and their daggers, these are also well-defined morphisms in $\mathrm{CP}^*_\mathrm{n}[\mathbf{V}]$. Thus we may conclude that $\mathrm{CP}^*_\mathrm{n}[\mathbf{V}]$ has a symmetric monoidal structure $(\oplus,0)$, under which every object has a unique comonoid structure. Therefore, the monoidal product is in fact a product [6, Theorem 2.1]. Moreover, because $\mathrm{CP}^*_\mathrm{n}[\mathbf{V}]$ is compact by [4, Theorem 3.4], products are biproducts [7]. Finally, these biproducts are dagger biproducts because they are so in **V**.

First consider $0\colon A\to 0$. Regardless of the multiplication m_A, the morphism $0\colon (A,m_A)\to(0,m_0)$ is trivially a *-homomorphism.

Next, consider $i_A\colon A\to A\oplus 0$. Lemma 3.1 shows $m_{A\oplus 0} = i_A\circ m_A\circ(p_A\otimes p_A)$ and $u_{A\oplus 0} = i_A\circ u_A$. Therefore $m_{A\oplus 0}\circ(i_A\otimes i_A) = i_A\circ m_A\colon A\otimes A\to A\oplus 0$. Writing $s_A = \lambda_A\circ(\varepsilon_A\otimes 1_A)\circ(1_{A^*}\otimes m_A^\dagger)\circ(1_{A^*}\otimes u_A)\circ\rho_{A^*}^{-1}\colon A^*\to A$ for the involution and $\lambda_A\colon I\otimes A\to A$ and $\rho_A\colon A\otimes I\to A$ for the coherence isomorphisms, one can verify that $s_{A\oplus 0} = s_A\oplus 0\colon A^*\oplus 0\to A\oplus 0$. Hence $s_{A\oplus 0}\circ(i_A)_* = s_{A\oplus 0}\circ i_{A^*} = i_A\circ s_A$, making i_A into a *-homomorphism.

As to $p_A\colon A\oplus 0\to A$, the above gives $p_A\circ s_{A\oplus 0} = s_A\circ p_{A^*} = s_A\circ(p_A)_*$ and $p_A\circ m_{A\oplus 0} = m_A\circ(p_A\otimes p_A)$. Hence also p_A is a *-homomorphism.

Finally, we turn to $\Delta_A\colon A\to A\oplus A$. It follows from Lemma 3.1 that $m_{A\oplus A}\circ(\Delta_A\otimes\Delta_A) = \Delta_A\circ m_A$. Furthermore, one verifies $s_{A\oplus A}\circ(\Delta_A)_* = \Delta_A\circ s_A$. So Δ is a *-homomorphism, completing the proof. □

Write $\mathbf{C}^{\oplus}$ for the biproduct completion of a category $\mathbf{C}$.

Corollary 3.3. *If* $\mathbf{V}$ *is a dagger compact category with dagger biproducts, there is a full and faithful functor* $\mathrm{CPM}[\mathbf{V}]^{\oplus} \to \mathrm{CP}^*[\mathbf{V}]$.

Proof. [4, Theorem 4.3] gives a full and faithful functor $\mathscr{B}\colon \mathrm{CPM}[\mathbf{V}] \to \mathrm{CP}^*[\mathbf{V}]$. Theorem 3.2 shows that $\mathrm{CP}^*[\mathbf{V}]$ has biproducts. Thus the universal property of $\mathrm{CPM}[\mathbf{V}]^{\oplus}$ guarantees that L lifts to a functor $\mathrm{CPM}[\mathbf{V}]^{\oplus} \to \mathrm{CP}^*[\mathbf{V}]$ that is full and faithful. □

Example 3.4. By [4, Proposition 3.5], $\mathrm{CP}^*[\mathbf{FHilb}]$ is the category of finite-dimensional C*-algebras and completely positive maps. Similarly, $\mathrm{CPM}[\mathbf{FHilb}]$ can be identified with the full subcategory of finite-dimensional C*-*factors*, i.e. matrix algebras $\mathbb{M}_n$. Because any finite-dimensional C*-algebra is a direct sum of matrix algebras [5, Theorem III.1.1], the functor of the previous corollary is an equivalence between the categories $\mathrm{CP}^*[\mathbf{FHilb}]$ and $\mathrm{CPM}[\mathbf{FHilb}]^{\oplus}$.

Example 3.5. By [4, Proposition 5.3], $\mathrm{CP}^*[\mathbf{Rel}]$ is the category of (small) groupoids and relations respecting inverses. Similarly, by [4, Proposition 5.4], $\mathrm{CPM}[\mathbf{FHilb}]$ can be identified with the full subcategory of indiscrete (small) groupoids. But there exist groupoids that are not isomorphic to a disjoint union of indiscrete ones in $\mathrm{CP}^*[\mathbf{Rel}]$. For example, groupoids isomorphic to $\mathbb{Z}_2$ in $\mathrm{CP}^*[\mathbf{Rel}]$ must have a single object and two morphisms, and therefore cannot be a disjoint union of indiscrete groupoids. Hence the functor $\mathrm{CPM}[\mathbf{Rel}]^{\oplus} \to \mathrm{CP}^*[\mathbf{Rel}]$ of the previous corollary is not an equivalence, and in fact there cannot be an equivalence between $\mathrm{CPM}[\mathbf{Rel}]^{\oplus}$ and $\mathrm{CP}^*[\mathbf{Rel}]$.

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