

The general impossibility of jointly implementing arbitrary sets of measurements is a fundamental feature of quantum theory. Indeed, it is a key ingredient that enables a demonstration of the nonclassicality of quantum theory in proofs of Bell’s theorem [1] and the Kochen-Specker theorem [2]. A finite set of measurements is called jointly measurable or compatible if there exists a single measurement whose various coarse-grainings recover the original measurements. The problem of characterizing the joint measurability of observables has been studied in the literature [3, 4], and at least the joint measurability of binary qubit observables has been completely characterized [5, 6]. The connection between Bell inequality violations and the joint measurability of observables has also been quantitatively studied [7, 8].

A natural question that arises when thinking about the (in)compatibility of observables is the following: given a set of (in)compatibility relations on a set of vertices representing observables, do they admit a quantum realization? That is, can one write down a positive-operator valued measure (POVM) for each vertex such that the (in)compatibility relations among the vertices are realized by the assigned POVMs? After formally defining these notions, we answer this question in the affirmative by providing an explicit construction of POVMs for any set of (in)compatibility relations. This is our main result. We will use the terms ‘(not) jointly measurable’ and ‘(in)compatible’ interchangeably in this paper. Our motivation for studying this question arose from the simplest example of joint measurability relations that is realizable with POVMs but does not admit a realization in quantum theory with projective measurements. This joint measurability scenario, referred to as Specker’s scenario [9, 11, 12], involves three binary measurements that can be jointly measured pairwise but not triplewise: that is, for the set of binary measurements \{M_1, M_2, M_3\}, the (in)compatibility relations are given by the collection of compatible subsets \{\{M_1, M_2\}, \{M_2, M_3\}, \{M_1, M_3\}\}. The remaining nontrivial subset (with at least two observables), namely \{M_1, M_2, M_3\}, is incompatible. This can be pictured as a hypergraph (Fig. 1).

Speckner’s scenario has been exploited to violate a generalized noncontextuality inequality using a set of three qubit POVMs realizing this scenario [10–12]. This novel demonstration of contextuality in quantum theory raises the question whether there exist other contextuality scenarios—for example in an observable-based hypergraph approach as in [13, 14]—that do not admit a proof of quantum contextuality using projective measurements, but do admit such a proof using POVMs. A necessary first step towards answering this question is to figure out what compatibility scenarios are realizable in quantum theory. One can then ask whether these scenarios allow nontrivial correlations that rule out generalized noncontextuality [10]. We take this first step by proving that, in principle, all joint measurability hypergraphs are realizable in quantum theory. The realizability of all joint measurability graphs via projective measurements has been shown recently [15]. This prompted our question whether all joint measurability hypergraphs are realizable via POVMs. Our positive answer includes joint measurability hypergraphs that do not admit a realization using projective measurements. We allow any amount of POVMs on Hilbert spaces of any dimension, but focus on observables with discrete spectrum and (in)compatibility relations involving finitely many of them to prevent measure-theoretic technicalities.

We start with a more detailed discussion of the relevant concepts.
**POVMs.** A positive operator valued measure (POVM) on a Hilbert space $\mathcal{H}$ is a mapping $x \mapsto M(x)$ from an outcome set $X$ to the set of positive semidefinite operators, 

$$M(x) \in \mathcal{B}(\mathcal{H}), \quad M(x) \geq 0,$$

such that the POVM elements $M(x)$ sum to the identity operator,

$$\sum_{x \in X} M(x) = I.$$

If $M(x)^2 = M(x)$ for all $x \in X$, then the POVM becomes a “projection valued measure”, or simply a projective measurement.

**Joint measurability of POVMs.** A finite set of POVMs

$$\{M_1, \ldots, M_N\},$$

where measurement $M_i$ has outcome set $X_i$, is said to be jointly measurable or compatible if there exists a POVM $M$ with outcome set $X_1 \times X_2 \times \cdots \times X_N$ that marginalizes to each $M_i$ with outcome set $X_i$, meaning that

$$M_i(x_i) = \sum_{x_1, \ldots, x_N \in X_1 \times \cdots \times X_N} M(x_1, \ldots, x_N)$$

for all outcomes $x_i \in X_i$.

**Joint measurability hypergraphs.** A hypergraph consists of a set of vertices $V$, and a family $E \subseteq \{e \mid e \subseteq V\}$ of subsets of $V$ called edges. We think of each vertex as representing a POVM, while an edge models joint measurability of the POVMs it links. Since every subset of a set of compatible measurements should also be compatible, a joint measurability hypergraph should have the property that any subset of an edge is also an edge,

$$e \in E, \ e' \subseteq e \implies e' \in E.$$

Additionally, we focus on the case where each edge $e$ is a finite subset of $V$. This makes a joint measurability hypergraph into an abstract simplicial complex.

Every set of POVMs on $\mathcal{H}$ has such an associated joint measurability hypergraph. Hence characterizing joint measurability of quantum observables comes down to figuring out their joint measurability hypergraph. Our main result solves the converse problem. Namely, every abstract simplicial complex arises from the joint measurability relations of a set of quantum observables.

**Theorem.** Every joint measurability hypergraph admits a quantum realization with POVMs.

**Proof.** We begin by proving a necessary and sufficient criterion for the joint measurability of $N$ binary POVMs $M_k := \{E_k^+, E_k^-\}$ of the form

$$E_k^\pm := \frac{1}{2} (I \pm \eta \Gamma_k),$$

where the $\Gamma_k$ are generators of a Clifford algebra as in the Appendix. The variable $\eta \in [0, 1]$ is a purity parameter. Since $\Gamma_k^2 = I$, the eigenvalues of $\Gamma_k$ are $\pm 1$, so that $E_k^\pm$ is indeed positive. The following derivation of a joint measurability criterion is adapted from a proof first obtained in [11], and subsequently revised in [12], for the joint measurability of a set of noisy qubit POVMs.

Because $\Gamma_k$ is traceless by (9), we can recover the purity parameter $\eta$ as

$$\text{Tr}(\Gamma_k E_k^\pm) = \pm \frac{\eta}{2} d,$$

so that

$$\eta = \frac{1}{Nd} \sum_{k=1}^{N} \sum_{x_k \in X_k} \text{Tr}(x_k \Gamma_k E_k^\pm),$$

where we have introduced one separate outcome $x_k \in X_k := \{+1, -1\}$ for each measurement $M_k$.

If all $M_k = \{E_k^+, E_k^\pm\}$ together are jointly measurable, then there exists a joint POVM $M = \{E_{x_1, \ldots, x_N}\}$ satisfying

$$E_{x_k}^k = \sum_{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N} E_{x_1, \ldots, x_N}.$$

Writing $\vec{x} := (x_1, \ldots, x_N)$ and $\vec{\Gamma} := (\Gamma_1, \ldots, \Gamma_N)$, this assumption together with (2) implies that

$$\eta = \frac{1}{N^d} \sum_{\vec{x}} \text{Tr} \left[ \left( \sum_{k=1}^{N} x_k \Gamma_k \right) E_{x_1, \ldots, x_N} \right]$$

$$\leq \frac{1}{N^d} \sum_{\vec{x}} ||\vec{x} \cdot \vec{\Gamma}|| \text{Tr}[E_{\vec{x}}]$$

$$= \frac{1}{N} ||\vec{x} \cdot \vec{\Gamma}||,$$

where the last step used the normalization $\sum_{\vec{x}} E_{\vec{x}} = I$. Since $(\vec{x} \cdot \vec{\Gamma})^2 = \sum_k X_k^2 = N \cdot I$ by (10), we have $||\vec{x} \cdot \vec{\Gamma}|| = \sqrt{N}$, and therefore

$$\eta \leq \frac{1}{\sqrt{N}},$$

a necessary condition for joint measurability of $M_k$.

To show that this condition is also sufficient, we consider the joint POVM $M = \{E_{\vec{x}}\}$ given by

$$E_{x_1, \ldots, x_N} := \frac{1}{2N} \left( I + \eta \vec{x} \cdot \vec{\Gamma} \right),$$

(3)

We start by showing that this indeed defines a POVM,

$$E_{x_1, \ldots, x_N} \geq 0, \quad \sum_{x_1, \ldots, x_N} E_{x_1, \ldots, x_N} = I.$$

Positivity follows again from noting that the eigenvalues of $\vec{x} \cdot \vec{\Gamma}$ are $\pm \sqrt{N}$ by (10), and normalization from $\sum_{\vec{x}} \vec{x} \cdot \vec{\Gamma} = 0$. Since

$$\sum_{x_1, \ldots, x_N} E_{x_1, \ldots, x_N} = \frac{1}{2} (I + \eta x_k \Gamma_k)$$
measurements among the labels and observing that the above reasoning did not rely on any specific ordering of the labels and also that the above reasoning did not rely on any specific ordering of the labels. We conclude that any $N-1$ measurements among the $M_1, \ldots, M_N$ are compatible.

What we have established so far is that, if we are given any $N$-vertex joint measurability hypergraph where every subset of $N-1$ vertices is compatible (i.e., belongs to a common edge), but the $N$-vertex set is incompatible, then the above construction provides us with a quantum realization of it. These “Specker-like” hypergraphs are crucial to our construction. For example, for $N = 3$, we obtain a simple realization of Specker’s scenario (Fig. 1). For $N = 2$, we simply obtain a pair of incompatible observables. Given an arbitrary joint measurability hypergraph, the procedure to construct a quantum realization is now the following:

1. Identify the minimal incompatible sets of vertices in the hypergraph. A minimal incompatible set is an incompatible set of vertices such that any of its proper subsets is compatible. In other words, it is a Specker-like hypergraph embedded in the given joint measurability hypergraph.

2. For each minimal incompatible set, construct a quantum realization as above. Vertices that are outside this minimal incompatible set can be assigned a trivial POVM in the identity operator $I$. Let $H_i$ denote the Hilbert space on which the minimal incompatible set is realized, where $i$ indexes the minimal incompatible sets.

3. Having thus obtained a quantum representation of each minimal incompatible set, we simply “stack” these together in a direct sum over the Hilbert spaces on which each of the minimal incompatible sets are realized. On this larger direct sum Hilbert space $H = \bigoplus_i H_i$, we then have a quantum realization of the joint measurability hypergraph we started with.

For any edge $e \in E$, the associated measurements are compatible on every $H_i$, and therefore also on $H$. On the other hand, every $e' \subseteq V$ that is not an edge is contained in some minimal incompatible set (or is itself already minimal), and therefore the associated POVMs are incompatible on some $H_i$, and hence also on $H$.

A simple example. To illustrate these ideas, we construct a POVM realization of a simple joint measurability hypergraph that does not admit a representation with projective measurements (Fig. 2). This hypergraph can be decomposed into three minimal incompatible sets of vertices (Fig. 3). Two of these are Specker scenarios for $\{M_1, M_2, M_4\}$ and $\{M_2, M_3, M_4\}$, and the third one is a pair of incompatible vertices $\{M_1, M_3\}$. For the minimal incompatible set $\{M_1, M_2, M_4\}$, we construct a set of three binary POVMs, $A_k \equiv \{A^k_x, A^k_y\}$ with $k \in \{1, 2, 4\}$ on a qubit Hilbert space $H_1$ given by

$$A^k_{\pm} := \frac{1}{2} \left( I \pm \frac{1}{\sqrt{2}} \Gamma_k \right),$$

where the matrices $\{\Gamma_1, \Gamma_2, \Gamma_4\}$ can be taken to be the Pauli matrices,

$$\Gamma_1 = \sigma_z, \quad \Gamma_2 = \sigma_x, \quad \Gamma_4 = \sigma_y,$$
similar to (8). The remaining vertex \( M_3 \) can be taken to be the
trivial POVM \( A_3 = \{0, I\} \) on \( \mathcal{H}_1 \). A similar construction
works for the second Specker scenario \( \{M_2, M_3, M_4\} \) by setting
\( B_k := \{B_k^+, B_k^-\} \) with \( k \in \{2, 3, 4\} \) to be
\[
B_k^\pm := \frac{1}{2} \left( I \pm \frac{1}{\sqrt{2}} \Gamma_k \right),
\]
where
\[
\Gamma_2 = \sigma_z, \quad \Gamma_3 = \sigma_x, \quad \Gamma_4 = \sigma_y
\]
act on another qubit Hilbert space \( \mathcal{H}_2 \). The remaining
vertex \( M_1 \) can be assigned the trivial POVM, \( B_1 = \{0, I\} \). The
third minimal incompatible set \( \{M_1, M_3\} \) can similarly be ob-
tained on another qubit Hilbert space \( \mathcal{H}_3 \) as \( C_k := \{C_k^+, C_k^-\} \),
with \( k \in \{1, 3\} \), given by
\[
C_k^\pm := \frac{1}{2} (I \pm \Gamma_k),
\]
where now e.g. \( \Gamma_1 = \sigma_z \) and \( \Gamma_3 = \sigma_x \). The remaining vertices
\( M_2 \) and \( M_4 \) can both be assigned the trivial POVM \( C_4 := \{0, I\} \) on \( \mathcal{H}_3 \).

In the direct sum Hilbert space \( \mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \), we
then have a POVM realization of the joint measurability hyper-
graph of Fig. 2, given by
\[
M_k := A_k^+ \oplus B_k^+ \oplus C_k^+.
\]

**Discussion.** The main result of this paper settles an impor-
tant question: do there exist joint measurability hypergraphs
that quantum theory does not admit? We have shown that this
is not the case. Quantum theory allows enough freedom for
any conceivable set of (in)compatibility relations between an
arbitrary number of observables to be realized in the theory.
Our simple construction is probably not the most efficient
one for a given joint measurability hypergraph: for Fig. 2,
our representation lives on a six-dimensional Hilbert space.
It remains open what the most efficient construction—requiring
the smallest Hilbert space dimension—for a given joint
measurability hypergraph is. Concerning quantum contextu-
tality, it should be interesting to see whether our sets of
POVMs can lead to nonclassical correlations in the scenarios
associated to the underlying joint measurability hypergraphs.
On the theoretical side, our result opens the door to the use in
quantum contextuality of homology theory, matroid theory,
and other powerful combinatorial machinery that relies on
hypergraphs, and vice versa.

**APPENDIX: CLIFFORD ALGEBRAS**

A **Clifford algebra** consists of a finite set of hermitian ma-
trices \( \Gamma_1, \ldots, \Gamma_N \) satisfying the relations\(^1\)

\[
\Gamma_j \Gamma_k + \Gamma_k \Gamma_j = 2\delta_{jk}I, \quad (7)
\]

Clifford algebras are the mathematical structure behind the
definition of spinors and the Dirac equation [16]. They
can be constructed recursively as follows [16, Sec. 16.3].
Given \( \Gamma_1, \ldots, \Gamma_N \) living on a Hilbert space \( \mathcal{H}_N \), one obtains
\( \Gamma_1, \ldots, \Gamma_N, \Gamma_{N+1} \) on \( \mathcal{H}_N \otimes \mathbb{C}^2 \) by the following rules.

1. For each \( i = 1, \ldots, N \), substitute
\[
\Gamma_i \rightarrow \Gamma_i \otimes \sigma_z.
\]

2. Further, define
\[
\Gamma_{N+1} := I \otimes \sigma_x, \quad \Gamma_{N+2} := I \otimes \sigma_y.
\]

It is easy to show that if the original \( \Gamma_1 \) satisfy (7), then
so do the new ones. One can simply start the recursion with
\( \Gamma_1 = 1 \) on the one-dimensional Hilbert space \( \mathcal{H}_1 := \mathbb{C} \), and
then apply the construction as often as necessary to obtain any
finite number of matrices satisfying (7). For example, a single
iteration gives the Pauli matrices
\[
\Gamma_1 = \sigma_z, \quad \Gamma_2 = \sigma_x, \quad \Gamma_3 = \sigma_y, \quad (8)
\]
while after two iterations one has
\[
\Gamma_1 = \sigma_z \otimes \sigma_z, \quad \Gamma_2 = \sigma_x \otimes \sigma_z, \quad \Gamma_3 = \sigma_y \otimes \sigma_z, \quad \Gamma_4 = I \otimes \sigma_z, \quad \Gamma_5 = I \otimes \sigma_y.
\]

The Clifford algebra relations (7) have many interesting
consequences. For example for \( N \geq 2 \), one has for any \( k \)
and \( j \neq k \),
\[
\text{Tr}(\Gamma_k) = \text{Tr}(\Gamma_k \Gamma_j \Gamma_j) = -\text{Tr}(\Gamma_j \Gamma_k \Gamma_j) = -\text{Tr}(\Gamma_k \Gamma_j),
\]
so that
\[
\text{Tr}(\Gamma_k) = 0. \quad (9)
\]

Another consequence is that
\[
\left( \sum_k \Gamma_k X_k \right)^2 = \left( \sum_k X_k^2 \right) \cdot I \quad (10)
\]
for arbitrary real coefficients \( X_k \).

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\(^1\) Strictly speaking, this is a representation of a Clifford algebra, but the
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