All joint measurability structures are quantum realizable

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We prove that quantum theory realizes all possible (in)compatibility relations among sets of observables.

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The general impossibility of jointly implementing arbitrary sets of measurements is a fundamental feature of quantum theory. Indeed, it is a key ingredient that enables a demonstration of the nonclassicality of quantum theory in proofs of Bell's theorem [1] and the Kochen-Specker theorem [2]. A finite set of measurements is called *jointly measurable* or *compatible* if there exists a single measurement whose various coarse-grainings recover the original measurements. The problem of characterizing the joint measurability of observables has been studied in the literature [3, 4], and at least the joint measurability of binary qubit observables has been completely characterized [5, 6]. The connection between Bell inequality violations and the joint measurability of observables has also been quantitatively studied [7, 8].

A natural question that arises when thinking about the (in)compatibility of observables is the following: given a set of (in)compatibility relations on a set of vertices representing observables, do they admit a quantum realization? That is, can one write down a positive-operator valued measure (POVM) for each vertex such that the (in)compatibility relations among the vertices are realized by the assigned POVMs? After formally defining these notions, we answer this question in the affirmative by providing an explicit construction of POVMs for any set of (in)compatibility relations. This is our main result. We will use the terms '(not) jointly measurable' and '(in)compatible' interchangeably in this paper. Our motivation for studying this question arose from the simplest example of joint measurability relations that is realizable with POVMs but does not admit a realization in quantum theory with projective measurements. This joint measurability scenario, referred to as Specker's scenario [9, 11, 12], involves three binary measurements that can be jointly measured pairwise but not triplewise: that is, for the set of binary measurements $\{M_1, M_2, M_3\}$, the (in)compatibility relations are given by the collection of compatible subsets $\{\{M_1, M_2\}, \{M_2, M_3\}, \{M_1, M_3\}\}$. The remaining nontrivial subset (with at least two observables), namely $\{M_1, M_2, M_3\}$, is incompatible. This can be pictured as a hypergraph (Fig. 1).

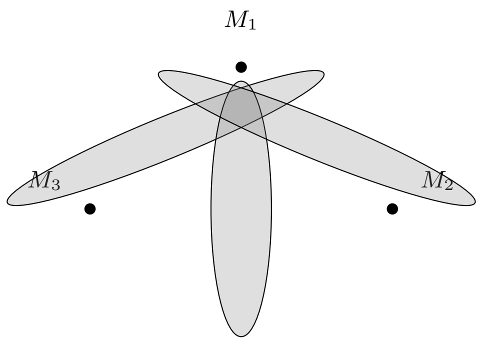

FIG. 1. Specker's scenario.

Specker's scenario has been exploited to violate a generalized noncontextuality inequality using a set of three qubit POVMs realizing this scenario [10–12]. This novel demonstration of contextuality in quantum theory raises the question whether there exist other contextuality scenarios—for example in an observable-based hypergraph approach as in [13, 14]—that do not admit a proof of quantum contextuality using projective measurements, but do admit such a proof using POVMs. A necessary first step towards answering this question is to figure out what compatibility scenarios are realizable in quantum theory. One can then ask whether these scenarios allow nontrivial correlations that rule out generalized noncontextuality [10]. We take this first step by proving that, in principle, all joint measurability hypergraphs are realizable in quantum theory. The realizability of all joint measurability graphs via projective measurements has been shown recently [15]. This prompted our question whether all joint measurability hypergraphs are realizable via POVMs. Our positive answer includes joint measurability hypergraphs that do not admit a realization using projective measurements. We allow any amount of POVMs on Hilbert spaces of any dimension, but focus on observables with discrete spectrum and (in)compatibility relations involving finitely many of them to prevent measure-theoretic technicalities.

We start with a more detailed discussion of the relevant concepts.

POVMs. A positive operator valued measure (POVM) on a Hilbert space $\mathcal{H}$ is a mapping $x \mapsto M(x)$ from an outcome set X to the set of positive semidefinite operators,

$$M(x) \in \mathcal{B}(\mathcal{H}), \quad M(x) \geq 0,$$

such that the POVM elements $M(x)$ sum to the identity operator,

$$\sum_{x \in X} M(x) = I.$$

If $M(x)^2 = M(x)$ for all $x \in X$, then the POVM becomes a "projection valued measure", or simply a projective measurement.

Joint measurability of POVMs. A finite set of POVMs

$$\{M_1, \ldots, M_N\},$$

where measurement M_i has outcome set X_i, is said to be *jointly measurable* or *compatible* if there exists a POVM M with outcome set $X_1 \times X_2 \times \cdots \times X_N$ that marginalizes to each M_i with outcome set X_i, meaning that

$$M_i(x_i) = \sum_{x_1, \ldots, \cancel{x_i}, \ldots, x_N} M(x_1, \ldots, x_N)$$

for all outcomes $x_i \in X_i$.

Joint measurability hypergraphs. A *hypergraph* consists of a set of vertices V, and a family $E \subseteq \{e \mid e \subseteq V\}$ of subsets of V called *edges*. We think of each vertex as representing a POVM, while an edge models joint measurability of the POVMs it links. Since every subset of a set of compatible measurements should also be compatible, a joint measurability hypergraph should have the property that any subset of an edge is also an edge,

$$e \in E,\ e' \subseteq e \implies e' \in E.$$

Additionally, we focus on the case where each edge e is a finite subset of V. This makes a joint measurability hypergraph into an abstract simplicial complex.

Every set of POVMs on $\mathcal{H}$ has such an associated joint measurability hypergraph. Hence characterizing joint measurability of quantum observables comes down to figuring out their joint measurability hypergraph. Our main result solves the converse problem. Namely, every abstract simplicial complex arises from the joint measurability relations of a set of quantum observables.

Theorem. *Every joint measurability hypergraph admits a quantum realization with POVMs.*

Proof. We begin by proving a necessary and sufficient criterion for the joint measurability of N binary POVMs $M_k := \{E_+^k, E_-^k\}$ of the form

$$E_\pm^k := \frac{1}{2}\left(I \pm \eta \Gamma_k\right), \tag{1}$$

where the $\Gamma_k$are generators of a Clifford algebra as in the Appendix. The variable $\eta \in [0, 1]$ is a purity parameter. Since $\Gamma_k^2 = I$, the eigenvalues of Γ_k are ± 1, so that $E_\pm^k$ is indeed positive. The following derivation of a joint measurability criterion is adapted from a proof first obtained in [11], and subsequently revised in [12], for the joint measurability of a set of noisy qubit POVMs.

Because Γ_k is traceless by (9), we can recover the purity parameter η as

$$\mathrm{Tr}(\Gamma_k E_\pm^k) = \pm \frac{\eta}{2} d,$$

so that

$$\eta = \frac{1}{Nd} \sum_{k=1}^{N} \sum_{x_k \in X_k} \mathrm{Tr}(x_k \Gamma_k E_{x_k}^k), \tag{2}$$

where we have introduced one separate outcome $x_k \in X_k := \{+1, -1\}$ for each measurement M_k.

If all $M_k = \{E_+^k, E_-^k\}$ together are jointly measurable, then there exists a joint POVM $M = \{E_{x_1 \ldots x_N}\}$ satisfying

$$E_{x_k}^k = \sum_{x_1, \ldots \cancel{x_k}, \ldots x_N} E_{x_1 \ldots x_N}.$$

Writing $\vec{x} := (x_1, \ldots, x_N)$ and $\vec{\Gamma} := (\Gamma_1, \ldots, \Gamma_N)$, this assumption together with (2) implies that

$$\begin{aligned} \eta &= \frac{1}{Nd} \sum_{\vec{x}} \mathrm{Tr}\left[\left(\sum_{k=1}^{N} x_k \Gamma_k\right) E_{x_1 \ldots x_N}\right] \\ &\leq \frac{1}{Nd} \sum_{\vec{x}} \|\vec{x} \cdot \vec{\Gamma}\| \, \mathrm{Tr}\left[E_{\vec{x}}\right] \\ &= \frac{1}{N} \|\vec{x} \cdot \vec{\Gamma}\|, \end{aligned}$$

where the last step used the normalization $\sum_{\vec{x}} E_{\vec{x}} = I$. Since $(\vec{x} \cdot \vec{\Gamma})^2 = \sum_k X_k^2 = N \cdot I$ by (10), we have $\|\vec{x} \cdot \vec{\Gamma}\| = \sqrt{N}$, and therefore

$$\eta \leq \frac{1}{\sqrt{N}},$$

a necessary condition for joint measurability of M_k.

To show that this condition is also sufficient, we consider the joint POVM $M = \{E_{\vec{x}}\}$ given by

$$E_{x_1 \ldots x_N} := \frac{1}{2^N}\left(I + \eta \, \vec{x} \cdot \vec{\Gamma}\right). \tag{3}$$

We start by showing that this indeed defines a POVM,

$$E_{x_1 \ldots x_N} \geq 0, \quad \sum_{x_1, \ldots, x_N} E_{x_1 \ldots x_N} = I.$$

Positivity follows again from noting that the eigenvalues of $\vec{x} \cdot \vec{\Gamma}$ are $\pm\sqrt{N}$ by (10), and normalization from $\sum_{\vec{x}} \vec{x} \cdot \vec{\Gamma} = 0$. Since

$$\sum_{x_1, \ldots, \cancel{x_k}, \ldots, x_N} E_{x_1 \ldots x_N} = \frac{1}{2}\left(I + \eta x_k \Gamma_k\right)$$

coincides with (1), we have indeed found a joint POVM marginalizing to the given M_k.

Thus $\eta \leq \frac{1}{\sqrt{N}}$ is a necessary and sufficient condition for the joint measurability of $M_1, \ldots, M_N$.

For arbitrary N, then, we can construct N POVMs on a Hilbert space of appropriate dimension such that any $N-1$ of them are compatible, whereas all N together are incompatible: simply take $M_1, \ldots, M_N$ from (1) for any purity parameter η satisyfing

$$\frac{1}{\sqrt{N}} < \eta \leq \frac{1}{\sqrt{N-1}}.$$

For example, $\eta = 1/\sqrt{N-1}$ will work. The above reasoning guarantees that all N of them together are not compatible, and also that the $M_1, \ldots, M_{N-1}$ are compatible. By permuting the labels and observing that the above reasoning did not rely on any specific ordering of the Γ_k, we conclude that *any* $N-1$ measurements among the $M_1, \ldots, M_N$ are compatible.

What we have established so far is that, if we are given any N-vertex joint measurability hypergraph where every subset of $N-1$ vertices is compatible (*i.e.* belongs to a common edge), but the N-vertex set is incompatible, then the above construction provides us with a quantum realization of it. These "Specker-like" hypergraphs are crucial to our construction. For example, for $N = 3$, we obtain a simple realization of Specker's scenario (Fig. 1). For $N = 2$, we simply obtain a pair of incompatible observables. Given an arbitrary joint measurability hypergraph, the procedure to construct a quantum realization is now the following:

1. Identify the minimal incompatible sets of vertices in the hypergraph. A minimal incompatible set is an incompatible set of vertices such that any of its proper subsets *is* compatible. In other words, it is a Specker-like hypergraph embedded in the given joint measurability hypergraph.

2. For each minimal incompatible set, construct a quantum realization as above. Vertices that are outside this minimal incompatible set can be assigned a trivial POVM in which one outcome is deterministic, represented by the identity operator I. Let $\mathcal{H}_i$ denote the Hilbert space on which the minimal incompatible set is realized, where i indexes the minimal incompatible sets.

3. Having thus obtained a quantum representation of each minimal incompatible set, we simply "stack" these together in a direct sum over the Hilbert spaces on which each of the minimal incompatible sets are realized. On this larger direct sum Hilbert space $\mathcal{H} = \oplus_i \mathcal{H}_i$, we then have a quantum realization of the joint measurability hypergraph we started with.

For any edge $e \in E$, the associated measurements are compatible on every $\mathcal{H}_i$, and therefore also on $\mathcal{H}$. On the other hand,

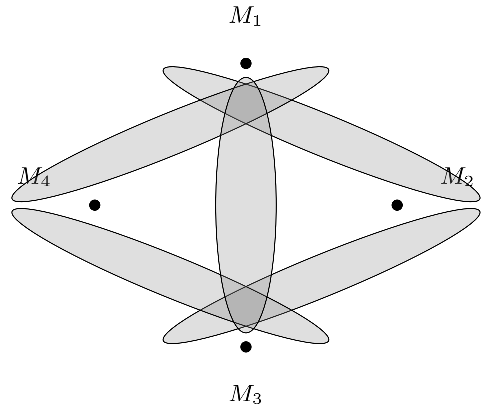

FIG. 2. A joint measurability hypergraph for $N = 4$.

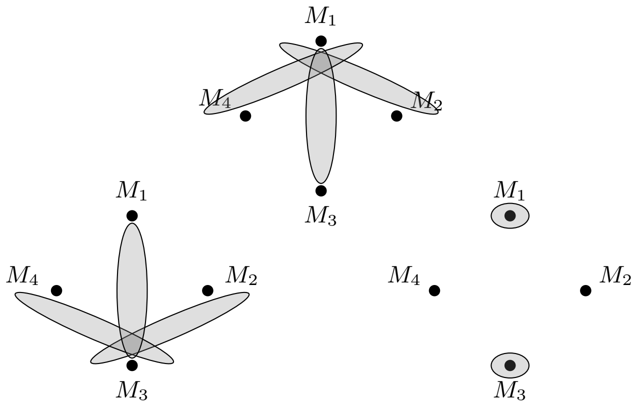

FIG. 3. Minimal incompatible sets for the joint measurability hypergraph in Fig. 2.

every $e' \subseteq V$ that is not an edge is contained in some minimal incompatible set (or is itself already minimal), and therefore the associated POVMs are incompatible on some $\mathcal{H}_i$, and hence also on $\mathcal{H}$. $\square$

A simple example. To illustrate these ideas, we construct a POVM realization of a simple joint measurability hypergraph that does not admit a representation with projective measurements (Fig. 2). This hypergraph can be decomposed into three minimal incompatible sets of vertices (Fig. 3). Two of these are Specker scenarios for $\{M_1, M_2, M_4\}$ and $\{M_2, M_3, M_4\}$, and the third one is a pair of incompatible vertices $\{M_1, M_3\}$. For the minimal incompatible set $\{M_1, M_2, M_4\}$, we construct a set of three binary POVMs, $A_k \equiv \{A_+^k, A_-^k\}$ with $k \in \{1, 2, 4\}$ on a qubit Hilbert space $\mathcal{H}_1$ given by

$$A_\pm^k := \frac{1}{2}\left(I \pm \frac{1}{\sqrt{2}}\Gamma_k\right), \tag{4}$$

where the matrices $\{\Gamma_1, \Gamma_2, \Gamma_4\}$ can be taken to be the Pauli matrices,

$$\Gamma_1 = \sigma_z, \quad \Gamma_2 = \sigma_x, \quad \Gamma_4 = \sigma_y,$$

similar to (8). The remaining vertex M_3 can be taken to be the trivial POVM $A_3 = \{0, I\}$ on $\mathcal{H}_1$. A similar construction works for the second Specker scenario $\{M_2, M_3, M_4\}$ by setting $B_k := \{B^k_+, B^k_-\}$ with $k \in \{2, 3, 4\}$ to be

$$B^k_\pm := \frac{1}{2}\left(I \pm \frac{1}{\sqrt{2}}\Gamma_k\right), \tag{5}$$

where

$$\Gamma_2 = \sigma_z, \quad \Gamma_3 = \sigma_x, \quad \Gamma_4 = \sigma_y$$

act on another qubit Hilbert space $\mathcal{H}_2$. The remaining vertex M_1 can be assigned the trivial POVM, $B_1 = \{0, I\}$. The third minimal incompatible set $\{M_1, M_3\}$ can similarly be obtained on another qubit Hilbert space $\mathcal{H}_3$ as $C_k := \{C^k_+, C^k_-\}$, with $k \in \{1, 3\}$, given by

$$C^k_\pm := \frac{1}{2}(I \pm \Gamma_k), \tag{6}$$

where now *e.g.* $\Gamma_1 = \sigma_z$ and $\Gamma_3 = \sigma_x$. The remaining vertices M_2 and M_4 can both be assigned the trivial POVM $C_2 = C_4 := \{0, I\}$ on $\mathcal{H}_3$.

In the direct sum Hilbert space $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, we then have a POVM realization of the joint measurability hypergraph of Fig. 2, given by

$$M^k_\pm := A^k_\pm \oplus B^k_\pm \oplus C^k_\pm.$$

Discussion. The main result of this paper settles an important question: do there exist joint measurability hypergraphs that quantum theory does not admit? We have shown that this is not the case. Quantum theory allows enough freedom for any conceivable set of (in)compatibility relations between an arbitrary number of observables to be realized in the theory. Our simple construction is probably not the most efficient one for a given joint measurability hypergraph: for Fig. 2, our representation lives on a six-dimensional Hilbert space. It remains open what the most efficient construction—requiring the smallest Hilbert space dimension—for a given joint measurability hypergraph is. Concerning quantum contextuality, it should be interesting to see whether our sets of POVMs can lead to nonclassical correlations in the scenarios associated to the underlying joint measurability hypergraphs. On the theoretical side, our result opens the door to the use in quantum contextuality of homology theory, matroid theory, and other powerful combinatorical machinery that relies on hypergraphs, and vice versa.

APPENDIX: CLIFFORD ALGEBRAS

A *Clifford algebra* consists of a finite set of hermitian matrices $\Gamma_1, \ldots, \Gamma_N$ satisfying the relations[1]

$$\Gamma_j\Gamma_k + \Gamma_k\Gamma_j = 2\delta_{jk}I, \tag{7}$$

Clifford algebras are the mathematical structure behind the definition of spinors and the Dirac equation [16]. They can be constructed recursively as follows [16, Sec. 16.3]. Given $\Gamma_1, \ldots, \Gamma_N$ living on a Hilbert space $\mathcal{H}_N$, one obtains $\Gamma_1, \ldots, \Gamma_{N+2}$ on $\mathcal{H}_N \otimes \mathbb{C}^2$ by the following rules.

1. For each $i = 1, \ldots, N$, substitute
$$\Gamma_i \to \Gamma_i \otimes \sigma_z.$$
2. Further, define
$$\Gamma_{N+1} := I \otimes \sigma_x, \quad \Gamma_{N+2} := I \otimes \sigma_y.$$

It is easy to show that if the original Γ_i satisfy (7), then so do the new ones. One can simply start the recursion with $\Gamma_1 = 1$ on the one-dimensional Hilbert space $\mathcal{H}_1 := \mathbb{C}$, and then apply the construction as often as necessary to obtain any finite number of matrices satisfying (7). For example, a single iteration gives the Pauli matrices

$$\Gamma_1 = \sigma_z, \quad \Gamma_2 = \sigma_x, \quad \Gamma_3 = \sigma_y, \tag{8}$$

while after two iterations one has

$$\Gamma_1 = \sigma_z \otimes \sigma_z, \quad \Gamma_2 = \sigma_x \otimes \sigma_z,$$
$$\Gamma_3 = \sigma_y \otimes \sigma_z, \quad \Gamma_4 = I \otimes \sigma_x, \quad \Gamma_5 = I \otimes \sigma_y.$$

The Clifford algebra relations (7) have many interesting consequences. For example for $N \geq 2$, one has for any k and $j \neq k$,

$$\begin{aligned}\mathrm{Tr}(\Gamma_k) &= \mathrm{Tr}(\Gamma_k\Gamma_j\Gamma_j) = -\mathrm{Tr}(\Gamma_j\Gamma_k\Gamma_j) \\ &= -\mathrm{Tr}(\Gamma_k\Gamma_j\Gamma_j) = -\mathrm{Tr}(\Gamma_k),\end{aligned}$$

so that

$$\mathrm{Tr}(\Gamma_k) = 0. \tag{9}$$

Another consequence is that

$$\left(\sum_k X_k\Gamma_k\right)^2 = \left(\sum_k X_k^2\right) \cdot I \tag{10}$$

for arbitrary real coefficients X_k.

[1] Strictly speaking, this is a *representation* of a Clifford algebra, but the difference between algebras and their representations is not relevant here.

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