Domains of Commutative C*-subalgebras

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***Abstract*—Operator algebras provide uniform semantics for deterministic, reversible, probabilistic, and quantum computing, where intermediate results of partial computations are given by commutative subalgebras. We study this setting using domain theory, and show that a given operator algebra is scattered if and only if its associated partial order is, equivalently: continuous (a domain), algebraic, atomistic, quasi-continuous, or quasi-algebraic. In that case, conversely, we prove that the Lawson topology, modelling information approximation, allows one to associate an operator algebra to the domain.**

I. Introduction

The venerable field of *operator algebras* has recently been used to give semantics for computing systems. The idea is to consider the algebra of possible observations one could make of a system's behaviour, rather than use some internal state space. This is especially useful when direct access to an internal state space is unavailable, as in quantum computing. There, a state of a computation with n quantum bits is a vector in some 2^n-dimensional complex Hilbert space, but measuring its value collapses it to a basis vector. To handle intermediate results of partial computations, it is therefore better to track observables, given by 2^n-by-2^n complex matrices [1]. Classical reversible computing is the special case using only permutation matrices as computation steps. Probabilistic computing, too, naturally leads to operator-algebraic semantics. Labelled probabilistic transition systems similarly give rise to matrices to model both computation steps and observations. Such labelled Markov processes are commonly used to model communication and concurrency, in particular for continuous state spaces or continuous time evolution [2]. Classical deterministic computing is a special case. Thus operator-algebraic semantics put deterministic, reversible, probabilistic, and quantum computing on equal footing.

All these settings are captured if we model observables of a system as *C*-algebras*. There is an inherent notion of *coarse graining*, or *approximation*. Observables are compatible when we can learn their joint value simultaneously, meaning that they commute as operators. Thus a measurement of the intermediate result of a partial computation is a *commutative C*-subalgebra*. Larger measurements, involving more observables, give us more information, leading us to use the partial order of inclusion: if $C \subseteq D$, then D contains more observables, and hence provides more information. Thus we can model approximation of quantum computations by classical ones. Performing a measurement halfway terminates a quantum computation and results in classical information, which is the only way to access quantum data. The later the measurement, the closer the approximation.

This sort of informational approximation is more commonly studied by *domain theory* [3]. As we are speaking of a continuous amount of observables, but in practice only have access to a discrete number of them, we are most interested in partial orders where every element can be approximated by empirically accessible ones. In domain-theoretic terms, such partial orders are called *continuous*, or a *domain*. If every element is approximable by finite ones, the domain is *algebraic*. There are also weaker versions called quasi-continuity and quasi-algebraicity. For nice enough partial orders, there is also a weaker version called meet-continuity. Another practically accessible notion of approximation in partial orders is that they be *atomistic*, meaning that the computation proceeds in indivisible steps. Finally, one can also endow domains with a topology, whose notion of limits then models approximation, such as the *Scott topology* and the *Lawson topology* [4]. In operator-algebraic terms, one might expect so-called *approximately finite-dimensional* C*-algebras, in which every observable can be approximated by observables with a finite number of outcomes [5]. This article studies the various relationships between these notions of approximation.

A. Contributions

We prove all these notions of approximations equivalent:

- the domain is continuous;
- the domain is algebraic;
- the domain is atomistic;
- the domain is quasi-continuous;
- the domain is quasi-algebraic;
- the underlying C*-algebra is *scattered*.

The latter is an established but not very well-known notion, intimately related to approximate finite-dimensionality. Additionally, these notions imply meet-continuity of the domain, and we prove a partial converse. Our results thus make precise exactly 'how much' approximate finite-dimensionality on the algebraic side is required for these desirable notions of approximation on the domain-theoretic side, and epitomize the robustness of operator-algebraic semantics.

Conversely, when the above properties hold, the domain itself becomes the spectrum of a commutative C*-algebra in its own right if we equip it with the Lawson topology.

B. Related work

Commutative C*-algebras provide relatively standard semantics for labelled Markov processes, albeit not often phrased that way, and bisimulations can be expressed algebraically [6], [7], [8], [9], [10]. But also noncommutative approximately finite-dimensional C*-algebras have been used as operational semantics of probabilistic languages [11], [12]. Furthermore, C*-algebras find applications in computer science in minimization of automata [13], and via graph theory: any directed graph gives rise to a C*-algebra which contains almost all information about the graph [14]. Additionally, C*-algebras give semantics for linear logic [15] and geometry of interaction [16]. A domain-theoretic study is new, however. Domains have played a role in labelled Markov processes [17], [18], but not in terms of C*-algebras. As far as we know, the only work similar to the current one is in quantum computing: modeling weakest preconditions but with a different notion of approximation [19], and giving semantics in a category of C*-algebras enriched over domains [20].

This article fits into a wider programme: its associated partial order determines a C*-algebra to a great extent [21], [22], and has therefore become popular as a substitute [23], [24], [25]. In the case of deterministic computing it can be axiomatized [26]. For related work in the case of probabilistic computing, see also [27], [28]. Finally, the special case of von Neumann algebras has been studied domain-theoretically [29], but forms a somewhat degenerate setting: the domain-theoretic notions listed above are not equivalent there, and come down to finite-dimensionality, which is relatively uninteresting from the perspective of information approximation; see Section IX for a detailed comparison.

C. Overview

Section II starts by definiting domains and C*-algebras. The following sections discuss one of the listed domain-theoretic properties each: Section III concerns algebraicity, Section IV continuity, Section V atomicity, and Section VI quasi-continuity and quasi-algebraicity. Section VIII then equates them to scatteredness of the C*-algebra, and considers the Lawson topology. Finally, Section IX compares our results to the degenerate special case of von Neumann algebras. The Appendix contains some point-set topology lemmas.

II. Domains and C*-algebras

This section recalls the main objects of study: domains, C*-algebras, and approximately finite-dimensional C*-algebras.

A. Domains

For detailed information about domain theory, we refer to [3], [4]; here we briefly recall the ingredients we need. Let $\mathcal{C}$ be a partially ordered set. We think of its elements as partial computations or observations, and the partial order $C \leq D$ as "D provides more information about the eventual outcome than C". With this interpretation, it is harmless to consider *downsets*, or *principal ideals*, instead of $C \in \mathcal{C}$:

$$\downarrow C = \{D \in \mathcal{C} \mid D \leq C\}.$$

Dually, it is also of interest to consider *upsets*, or *principal filters*, consisting of all possible expansions of the information contained in $C \in \mathcal{C}$:

$$\uparrow C = \{D \in \mathcal{C} \mid D \geq C\}.$$

This extends to subsets $\mathcal{D} \subseteq \mathcal{C}$ as:

$$\downarrow \mathcal{D} = \bigcup_{D \in \mathcal{D}} \downarrow D, \qquad \uparrow \mathcal{D} = \bigcup_{D \in \mathcal{D}} \uparrow D.$$

If $\mathcal{D}$ has a least upper bound in $\mathcal{C}$, it is denoted by $\bigvee \mathcal{D}$. Furthermore, $\mathcal{D}$ is called *directed* if for each $D_1, D_2 \in \mathcal{D}$ there is a $D_3 \in \mathcal{D}$ such that $D_1, D_2 \leq D_3$. This can be interpreted as saying that the partial computations or observations in $\mathcal{D}$ can always be compatibly continued without leaving $\mathcal{D}$. We write $C = \bigvee^{\uparrow} \mathcal{D}$ when $\mathcal{D}$ is directed and has C as a least upper bound. Similarly, we write $\bigwedge \mathcal{D}$ for a greatest lower bound, when it exists. For two-element sets $\mathcal{D}$ we just write the *meet* $\bigwedge\{D_1, D_2\}$ as $D_1 \wedge D_2$.

Definition II.1. A partially ordered set $\mathcal{C}$ is:

- a *directed-complete partial order* (dcpo) if each directed subset of $\mathcal{C}$ has a least upper bound;
- a *semilattice* if meets of any two elements exist;
- a *complete semilattice* if it is a dcpo in which every nonempty subset has a greatest lower bound.

Consider elements B, C of a dcpo $\mathcal{C}$. The element C could contain so much information that it is practically unobtainable. What does it mean for B to approximate C empirically? One answer is: whenever C is the eventual observation of increasingly fine-grained experiments $\mathcal{D}$, then all information in B is already contained in one of the approximants in $\mathcal{D}$. More precisely: we say that B is *way below* C and write $B \ll C$ if for each directed $\mathcal{D} \subseteq \mathcal{C}$ the inequality $C \leq \bigvee \mathcal{D}$ implies that $B \leq D$ for some $D \in \mathcal{D}$. Just like for down- and upsets, define:

$$\twoheaddownarrow C = \{B \in \mathcal{C} \mid B \ll C\}, \qquad \twoheaduparrow C = \{B \in \mathcal{C} \mid C \ll B\}.$$

With this interpretation, C is empirically accessible precisely when $C \ll C$. Such elements are called *compact*, and the subset they form is denoted by $\mathcal{K}(\mathcal{C})$:

$$\mathcal{K}(\mathcal{C}) = \{C \in \mathcal{C} \mid C \ll C\}.$$

Definition II.2. A dcpo is *continuous* when each element satisfies $C = \bigvee^{\uparrow} \twoheaddownarrow C$; continuous dcpos are also called *domains*. A domain is *algebraic* when each element satisfies $C = \bigvee^{\uparrow}\big(\mathcal{K}(\mathcal{C}) \cap \twoheaddownarrow C\big)$.

This definition captures situations where eventual outcomes can be approximated by empirically accessible partial observations. There are also some weaker notions, which we now describe. Generalize the way below relation of a dcpo $\mathcal{C}$ to nonempty subsets: write $\mathcal{G} \leq \mathcal{H}$ when $\uparrow \mathcal{H} \subseteq \uparrow \mathcal{G}$. This is a pre-order, and we can talk about *directed* families of nonempty subsets. A nonempty subset $\mathcal{G}$ is *way below* another one $\mathcal{H}$, written $\mathcal{G} \ll \mathcal{H}$, when $\bigvee^{\uparrow} \mathcal{D} \in \uparrow \mathcal{H}$ implies $D \in \uparrow \mathcal{G}$ for some $D \in \mathcal{D}$. Observe that $\{B\} \ll \{C\}$ if and only if $B \ll C$, so we may abbreviate $\mathcal{G} \ll \{C\}$ to $\mathcal{G} \ll C$, and $\{C\} \ll \mathcal{H}$ to $C \ll \mathcal{H}$.

Definition II.3. For an element C in a dcpo $\mathcal{C}$, define

$$\mathrm{Fin}(C) = \{\mathcal{F} \subseteq \mathcal{C} \mid \mathcal{F} \text{ is finite, nonempty, and } \mathcal{F} \ll C\},$$
$$\mathrm{KFin}(C) = \{\mathcal{F} \in \mathrm{Fin}(C) \mid \mathcal{F} \ll \mathcal{F}\}.$$

The dcpo is *quasi-continuous* if each $\mathrm{Fin}(C)$ is directed, and $C \not\leq D$ implies $D \notin \uparrow \mathcal{F}$ for some $\mathcal{F} \in \mathrm{Fin}(C)$. It is *quasi-algebraic* if each $\mathrm{KFin}(C)$ is directed, and $C \not\leq D$ implies $D \notin \uparrow F$ for some $\mathcal{F} \in \mathrm{KFin}(C)$. It is *meet-continuous* when it is a semilattice, and

$$C \wedge \bigvee{}^{\uparrow}\mathcal{D} = \bigvee\{C \wedge D \mid D \in \mathcal{D}\} \tag{II.4}$$

for each element C and directed subset $\mathcal{D}$.

Intuitively, quasi-continuity and quasi-algebraicity relax continuity and algebraicity to allow the information in approximants to be spread out over finitely many observations rather than be concentrated in a single one. Meet-continuity relaxes continuity by dropping the requirement that downsets are complete semilattices.

Proposition II.5. *Algebraic dcpos are continuous. Continuous dcpos that are semilattices are meet-continuous.*

Proof. See [4, I-4.3 and I-1.8]. □

Finally, the notion of approximation in domains can be turned into topological convergence.

Definition II.6. The *Scott topology* declares subsets $\mathcal{U}$ of a dcpo to be open if $\uparrow\mathcal{U} = \mathcal{U}$, and $D \cap \mathcal{U} \neq \emptyset$ when $\bigvee{}^{\uparrow}\mathcal{D} \in \mathcal{U}$. The *Lawson topology* has as basic open subsets $\mathcal{U} \setminus \uparrow \mathcal{F}$ for a Scott open subset $\mathcal{U}$ and a finite subset $\mathcal{F}$.

These topologies capture approximation in the following sense. A function f between partially ordered sets is *monotone* when $C \leq D$ implies $f(C) \leq f(D)$; an *order isomorphism* is a monotone bijection with a monotone inverse. A monotone function between dcpos is Scott continuous precisely when $\bigvee f[\mathcal{D}] = f\big(\bigvee{}^{\uparrow}\mathcal{D}\big)$ for directed subsets $\mathcal{D}$ [4, II-2.1]. A monotone function between complete semilattices is Lawson continuous precisely when additionally $\bigwedge f[\mathcal{D}] = f(\bigwedge \mathcal{D})$ for nonempty subsets $\mathcal{D}$ [4, III-1.8].

B. C-algebras*

For detailed information about C*-algebras, we refer to [30], [31]; here we briefly recall the ingredients we need. To introduce the idea intuitively, consider *transition systems* with n states. These can be represented as n-by-n matrices with entries 0 or 1. Linking two transitions becomes matrix multiplication (over the Boolean semiring $\{0, 1\}$, so with maximum instead of addition), and reversing transitions becomes matrix transpose. *Labelled transition systems* have transition matrices for each action in a whole set of labels. More generally, *probabilistic transition systems* can be represented as matrices with nonnegative real entries. Linking transitions is again matrix multiplication, reversing transitions is matrix transpose, and *labelled Markov systems* have different such probability matrices for transitions between states for a whole set of labels [2]. More generally still, *quantum systems* replace probabilities by complex numbers, that now model the amplitude of one computational state evolving into another [32]. (Taking the absolute square of the amplitudes recovers the probabilistic case.) Linking transitions is still matrix multiplication, reversing transitions is conjugate transpose. Again, one can have different transition matrices for different computation steps. This leads to the algebra of all 2^n-by-2^n complex matrices; the transition matrices together generate a subalgebra. C*-algebras define such operational semantics for situations with possibly infinitely many states rather than $n < \infty$. The star refers to the reversal operation. Here are the definitions.

Definition II.7. A *norm* on a complex vector space V is a function $\| - \|\colon V \to [0, \infty)$ satisfying

- $\|v\| = 0$ if and only if $v = 0$;
- $\|\lambda v\| = |\lambda|\|v\|$ for $\lambda \in \mathbb{C}$;
- $\|v + w\| \leq \|v\| + \|w\|$.

A *Banach space* is a normed complex vector space that is complete in the metric $d(v, w) = \|v - w\|$.

Example II.8. An *inner product* on a complex vector space V is a map $\langle - \mid - \rangle\colon V \times V \to \mathbb{C}$ that:

- is linear in the second variable;
- is conjugate symmetric: $\langle v \mid w\rangle = \overline{\langle w \mid v\rangle}$;
- satisfies $\langle v \mid v\rangle \geq 0$ with equality only when $v = 0$.

An inner product space V is a *Hilbert space* when the norm $\|v\| = \sqrt{\langle v \mid v\rangle}$ makes it a Banach space. For example, $\mathbb{C}^n$ with its usual inner product is a Hilbert space.

Definition II.9. A complex vector space A is a (unital) *algebra* when it carries a bilinear associative multiplication $A \times A \to A$ with a unit $1 \in A$ satisfying $1a = a = a1$. It is *commutative* when $ab = ba$ for all $a, b \in A$. A $*$-algebra is an algebra with an anti-linear map $*\colon A \to A$ satisfying:

- $(a^*)^* = a$;
- $(ab)^* = b^*a^*$.

A *C*-algebra* is a $*$-algebra that is simultaneously a Banach space with:

- $\|ab\| \leq \|a\|\|b\|$;
- $\|a^*a\| = \|a\|^2$.

Example II.10. As mentioned, the set of all n-by-n complex matrices is a C*-algebra, with the star being conjugate transpose. The infinite-dimensional version is this: the space $B(H)$ of all continuous linear maps $a\colon H \to H$ on a Hilbert space H is a C*-algebra as follows. Addition and scalar multiplication are defined componentwise by $a + b\colon v \mapsto a(v) + b(v)$, multiplication is composition by $ab\colon v \to a(b(v))$, and 1 is the identity map $v \mapsto v$. The star of $a\colon H \to H$ is its *adjoint*, *i.e.*, the unique map satisfying $\langle v \mid a(w)\rangle = \langle a^*(v) \mid w\rangle$ for each $v, w \in H$. The norm is given by $\|a\| = \sup\{\|a(v)\| \mid v \in H, \|v\| = 1\}$. Notice that this C*-algebra is noncommutative (unless H is one-dimensional).

The previous example is in fact prototypical, as the following theorem shows. A linear map $f\colon A \to B$ between C*-algebras is a unital $*$-*homomorphism* when $f(ab) = f(a)f(b)$, $f(a^*) = f(a)^*$, and $f(1) = 1$. If f is bijective we call it a $*$-*isomorphism*, and write $A \cong B$. Every $*$-homomorphism is automatically continuous, and is even an isometry when it is injective [31, 4.1.8]. A C*-algebra B is a *C*-subalgebra* of a C*-algebra A when $B \subseteq A$, and the inclusion $B \to A$ is a unital *-homomorphism. Since the inclusion must be an isometry, it follows that every C*-subalgebra of A is a closed subset of A.

Theorem II.11 (Gelfand–Neumark)**.** *Any C*-algebra is $*$-isomorphic to a C*-subalgebra of $B(H)$ for a Hilbert space H.*

Proof. See *e.g.* [33, 9.18]. □

The above C*-algebra $B(H)$ is noncommutative. Here is an example of a commutative one.

Example II.12. The vector space $\mathbb{C}^n$ is a commutative C*-algebra under pointwise operations. It sits inside the algebra $B(\mathbb{C}^n)$ of n-by-n matrices as the subalgebra of diagonal ones, illustrating Theorem II.11. The infinite version is as follows. Write $C(X)$ for the set of all continuous functions $f\colon X \to \mathbb{C}$ on a compact Hausdorff space X. It becomes a commutative C*-algebra as follows: addition and scalar multiplication are pointwise, *i.e.*, $f + g\colon x \mapsto f(x) + g(x)$, multiplication is componentwise $fg\colon x \mapsto f(x)g(x)$, the unit is the function $x \mapsto 1$, the star is $f^*\colon x \mapsto \overline{f(x)}$, and the norm is $\|f\| = \sup_{x\in X} |f(x)|$.

The above example is again prototypical for commutative C*-algebras, as the following theorem shows.

Theorem II.13 (Gelfand duality)**.** *Any commutative C*-algebra is $*$-isomorphic to $C(X)$ for some compact Hausdorff space X, called its* spectrum*.*

Proof. See *e.g.* [33, 1.3.10, 1.4.4, 1.4.5]. □

The previous theorem in fact extends to a categorical duality, but we will only need the following fact.

Proposition II.14. *Let A be a commutative C*-algebra with spectrum X. If $X \to Y$ is a continuous surjection onto a compact Hausdorff space Y, then Y is homeomorphic to the spectrum of a C*-subalgebra of A. Conversely, if a C*-subalgebra of A has spectrum Y, there is a continuous surjection $X \to Y$.*

Proof. If $q\colon X \to Y$ is a continuous surjection, then $B = \{f \circ q \mid f \in C(Y)\}$ is a C*-subalgebra of A. Conversely, if B is a C*-subalgebra of A, define a equivalence relation $\sim_B$ on X by setting $x \sim_B y$ if and only if $b(x) = b(y)$ for each $b \in B$. The quotient $Y = X/{\sim_B}$ is a compact Hausdorff space and it follows that $C(Y)$ is *-isomorphic to B. For details, see [34, 5.1.3]. □

We now come to our main object of study. The usual way of approximating labelled transition systems by simpler ones is to identify some *bisimilar* states, that is, to take a certain quotient of the (topological) space of states [2], [17]. In the commutative case, this comes down to considering C*-subalgebras by the previous proposition. When generally describing (quantum) systems C*-algebraically, observables become *self-adjoint* elements $a = a^* \in A$. These correspond [33, 4.6] to injective $*$-homomorphisms $C(\sigma(a)) \to A$ via the *spectrum* of a, the compact Hausdorff space

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1_A \text{ is not invertible}\},$$

linking observables to commutative C*-subalgebras. The following definition captures the main structure of approximation on the algebraic side.

Definition II.15. For a C*-algebra A, define

$$\mathcal{C}(A) = \{C \subseteq A \mid C \text{ is a commutative C*-subalgebra}\},$$

partially ordered by inclusion: $C \leq D$ when $C \subseteq D$. We call this the *dcpo corresponding to* A; the name will be justified in Proposition II.17 below.

A unital $*$-homomorphism $f\colon A \to B$ maps (commutative) C*-subalgebras $C \subseteq A$ to (commutative) C*-subalgebras $f[C] \subseteq B$ [31, 4.1.9]. Thus $\mathcal{C}$ extends to a functor from the category of C*-algebras and unital $*$-homomorphisms to that of partially ordered sets and monotone functions by $\mathcal{C}(f)\colon C \mapsto f[C]$.

The partially ordered set $\mathcal{C}(A)$ is of interest because it determines the C*-algebra A itself to a great extent. The following theorem illustrates this; see also [22], [25], [23], [21].

Theorem II.16. *Let A and B be commutative C*-algebras. Any order isomorphism $\psi\colon \mathcal{C}(A) \to \mathcal{C}(B)$ allows a $*$-isomorphism $f\colon A \to B$ with $\mathcal{C}(f) = \psi$ that is unique unless A is two-dimensional.*

Proof. See [21, 3.4]. Also compare [35]. □

It follows that for arbitrary C*-algebras A, the partial order on $\mathcal{C}(A)$ determines the C*-algebra structure of each individual element of $\mathcal{C}(A)$. Indeed, if $C \in \mathcal{C}(A)$, then $\downarrow C$ is order isomorphic to $\mathcal{C}(C)$, and since C is a commutative C*-algebra, it follows that the partially ordered set $\downarrow C$ determines the C*-algebra structure of C.

We now come to the first elementary connection between C*-algebras and domain theory.

Proposition II.17. *If A is any C*-algebra, $\mathcal{C}(A)$ is a dcpo.*

Proof. See [24]. The least upper bound $\bigvee \mathcal{D}$ of a directed subset $\mathcal{D} \subseteq \mathcal{C}(A)$ is the closure $\overline{\bigcup \mathcal{D}}$ of $\bigcup \mathcal{D}$. In particular, if A is finite-dimensional, then $\bigvee \mathcal{D} = \bigcup \mathcal{D}$. □

This assignment $A \mapsto \mathcal{C}(A)$ also extends to functions.

Proposition II.18. *If $f\colon A \to B$ is a unital $*$-homomorphism between C*-algebras, $\mathcal{C}(f) : \mathcal{C}(A) \to \mathcal{C}(B)$ is Scott continuous:*

$$\bigvee \mathcal{C}(f)[\mathcal{D}] = \mathcal{C}(f)\big(\bigvee \mathcal{D}\big) \tag{II.19}$$

for any directed subset $\mathcal{D} \subseteq \mathcal{C}(A)$.

Proof. Since $\mathcal{C}(f)$ is monotone, $\mathcal{C}(f)[\mathcal{D}]$ is directed. Now

$$f\left[\bigvee \mathcal{D}\right] = f\left[\overline{\bigcup \mathcal{D}}\right] \subseteq \overline{f\left[\bigcup \mathcal{D}\right]} = \overline{\bigcup\{f[D] \mid D \in \mathcal{D}\}},$$

where the inclusion holds because unital $*$-homomorphisms are continuous. Conversely,

$$\begin{aligned}\overline{\bigcup\{f[D] \mid D \in \mathcal{D}\}} &= \overline{f\left[\bigcup \mathcal{D}\right]} \subseteq \overline{f\left[\overline{\bigcup \mathcal{D}}\right]} \\ &= \overline{f\left[\bigvee \mathcal{D}\right]} = f\left[\bigvee \mathcal{D}\right],\end{aligned}$$

where the last equality holds because C*-subalgebras are closed. Hence

$$\bigvee\{f[D] \mid D \in \mathcal{D}\} = \overline{\bigcup\{f[D] \mid D \in \mathcal{D}\}} = f\left[\bigvee \mathcal{D}\right],$$

which is exactly (II.19). □

C. Approximately finite-dimensional C*-algebras

In practice, within finite time one can only measure or compute up to finite precision, and hence can only work with (sub)systems described by finite-dimensional C*-subalgebras. Therefore one might think that the natural counterpart of the notions in Section II-A is for the finite-dimensional C*-subalgebras to be dense in the whole C*-algebra.

Definition II.20. A C*-algebra A is *approximately finite-dimensional* when for each $a_1, \dots, a_n \in A$ and $\varepsilon > 0$ there exist a finite-dimensional C*-subalgebra $B \subseteq A$ and $b_1, \dots, b_n \in B$ such that $\|a_i - b_i\| < \varepsilon$ for any $i = 1, \dots, n$.

Let us point out that we do not, as some authors do, restrict approximately finite-dimensional C*-algebras to have countable dimension. It turns out that a countably-dimensional C*-algebra A is approximately finite-dimensional precisely when $A = \overline{\bigcup_{n\in\mathbb{N}} D_n}$ for a sequence $D_1 \subseteq D_2 \subseteq \cdots$ of finite-dimensional C*-subalgebras. These can be classified in several ways, for instance by means of Bratteli diagrams [5] or by K-theory [36].

Lemma A.3 in the Appendix proves that a commutative C*-algebra is approximately finite-dimensional if and only if its spectrum is a *totally disconnected* space: that is, when its connected components are exactly the singletons. Countable-dimensionality would come be the additional requirement that the spectrum be second-countable [37, Proposition 3.1].

Example II.21. Let X be the Cantor set. Then $C(X)$ is an approximately finite-dimensional commutative C*-algebra of countable dimension. Since there exists a continuous surjection $X \to [0,1]$, there is a C*-subalgebra of $C(X)$ that is $*$-isomorphic to $C([0,1])$ by Proposition II.14. This C*-subalgebra is not approximately finite-dimensional because $[0,1]$ is not totally disconnected.

The following useful lemma explains the terminology 'approximately' by linking the topology of a C*-algebra to approximating C*-subalgebras.

Lemma II.22. *Let A be a C*-algebra and $\mathcal{D}$ a directed family of C*-subalgebras with $A = \overline{\bigcup \mathcal{D}}$. For each $a \in A$ and $\varepsilon > 0$, there exist $D \in \mathcal{D}$ and $x \in D$ satisfying $\|a - x\| < \varepsilon$. If a is a* projection*, i.e., $a = a^2 = a^*$, then x can be chosen to be a projection as well.*

Proof. This is a special case of [36, Proposition L.2.2]. □

It will turn out that approximate finite-dimensionality of A does not correspond to nice domain-theoretic properties of $\mathcal{C}(A)$. We will need the following more subtle notion. In general, we will use quite some point-set topology of totally disconnected spaces, as covered *e.g.* in [4].

Definition II.23. A topological space is called *scattered* if every nonempty closed subset has an isolated point.

Scattered topological spaces are always totally disconnected, so commutative C*-algebras with scattered spectrum are always approximately finite-dimensional.

Example II.24. Any discrete topological space is scattered, and any finite discrete space is additionally compact Hausdorff, but there are more interesting examples.

The one-point compactification of the natural numbers is scattered, as well as compact Hausdorff. This is homeomorphic to the subspace $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ of $\mathbb{R}$ under the usual Euclidean topology.

More generally, any ordinal number α is scattered under the order topology. A basis for this topology is given by the intervals $\{\delta \mid \beta < \delta < \gamma\}$ for ordinals $\beta, \gamma \leq \alpha$. If α is a limit ordinal, then $\alpha + 1$ is furthermore compact Hausdorff [38, Counterexample 43].

There is also a notion of scatteredness in general C*-algebras A, which can be defined as follows [39]. A *positive functional* on A is a continuous linear map $f\colon A \to \mathbb{C}$ satisfying $f(a^*a) \geq 0$. The space of positive functionals of unit norm forms a convex set, whose extremal points are called *pure*. A C*-algebra is called *scattered* when each positive functional can be written as the countable sum of pure positive functionals. The following theorem connects the notions of approximately finite-dimensional algebras, scattered topological spaces, and scattered C*-algebras.

Example II.25. An operator $f \in B(H)$ on a Hilbert space H is *compact* when it is a limit of operators of finite rank. If H is infinite-dimensional, the compact operators form a proper ideal $K(H) \subseteq B(H)$, and all self-adjoint elements of $K(H)$ have countable spectrum [40, VII.7.1]. It follows that the C*-algebra $K(H) + \mathbb{C}1_H$ is scattered [41].

Theorem II.26. *The following are equivalent for C*-algebras:*

1) *A is scattered;*
2) *each $C \in \mathcal{C}(A)$ is approximately finite-dimensional;*
3) *each $C \in \mathcal{C}(A)$ has totally disconnected spectrum;*
4) *each maximal $C \in \mathcal{C}(A)$ has scattered spectrum.*

Proof. The equivalence between (1) and (2) follows from [42, 2.2], and the equivalence between (2) and (3) is proven in Lemma A.3. Assume that all maximal commutative C*-subalgebras have scattered spectrum. Since by Zorn's lemma every commutative C*-subalgebra is contained in a maximal one, it follows from [43, 12.24] that all commutative C*-subalgebras have scattered and hence totally disconnected spectra. Conversely, suppose that every commutative C*-subalgebra has totally disconnected spectra and let C be a maximal commutative C*-subalgebra. Since every C*-subalgebra of C has totally disconnected spectrum, there can be no C*-subalgebra of C with spectrum homeomorphic to $[0,1]$. It now follows that C has scattered spectrum [44, 8.5.4], which establishes the equivalence of (3) and (4). $\square$

III. Algebraicity

In this section we characterize C*-algebras A for which $\mathcal{C}(A)$ is an algebraic domain. We start by identifying the compact elements of $\mathcal{C}(A)$. Intuitively, these are the observations about the system in question that we can make in practice, *i.e.*, by finite means. For some notation: if X is a compact Hausdorff space, write $\mathcal{O}(x)$ the set of all open neighbourhoods of $x \in X$; if $K \subseteq X$ is a closed subspace, define

$$C_K = \{f \in C(X) \mid f \text{ is constant on } K\},$$

which is clearly a C*-subalgebra of $C(X)$.

Proposition III.1. *Let A be a C*-algebra. Then $C \in \mathcal{C}(A)$ is compact if and only if it is finite-dimensional.*

This is claimed in [24, Prop. 15] without complete proof of sufficiency; note that this direction does not always hold in the setting of von Neumann algebras (see Section IX) since the adaptation of Lemma II.22 to the relevant topology of von Neumann algebras fails.

Proof. Suppose that C is compact. Theorem II.13 then gives a compact Hausdorff space with $C \cong C(X)$. Let $x \in X$ and consider

$$\mathcal{D} = \{C_{\overline{U}} \mid U \in \mathcal{O}(x)\}.$$

It follows from Lemma A.4 that $\mathcal{D}$ is directed and $C(X) = \bigvee^{\uparrow}\mathcal{D}$. Because C is compact, it must equal some element $C_{\overline{U}}$ of $\mathcal{D}$. Since $C(X)$ separates all points of X, so must $C_{\overline{U}}$. But as each $f \in C_{\overline{U}}$ is constant on $\overline{U}$, this can only happen when $\overline{U}$ is a singleton $\{x\}$. This implies $\{x\} = U$, so $\{x\}$ is open. Since $x \in X$ was arbitrary, X must be discrete. Being compact, it must therefore be finite. Hence C is finite-dimensional.

Conversely, assume that C has a finite dimension n. Then it is generated by a finite set $\{p_1, \dots, p_n\}$ of projections that is *orthogonal* in the sense that $p_i p_j = 0$ for $i \neq j$. Let $\mathcal{D} \subseteq \mathcal{C}(A)$ be a directed family satisfying $C \subseteq \bigvee \mathcal{D}$. Since each projection p_i is contained in $\bigvee \mathcal{D}$, using Lemma II.22 we may approximate $\|p_i - p\| < \frac{1}{2}$ with some $D \in \mathcal{D}$ and some projection $p \in D$. Since projections $p \colon X \to \mathbb{C}$ can only take the value 0 or 1, $p \neq p_i$ implies $\|p_i - p\| = 1$, so we must have $p = p_i$. Hence there are $D_1, \dots, D_n \in \mathcal{D}$ such that $p_i \in D_i$. Since $\mathcal{D}$ is directed, there must be some $D \in \mathcal{D}$ with $D_1, \dots, D_n \subseteq D$. So $p_1, \dots, p_n \in D$, which implies that $C \subseteq D$. We conclude that C is compact. $\square$

We can now prove our first main result, characterizing algebraicity of $\mathcal{C}(A)$.

Theorem III.2. *A C*-algebra A is scattered if and only if $\mathcal{C}(A)$ is algebraic.*

Proof. By Proposition III.1 and Lemma A.3, the dcpo $\mathcal{C}(A)$ is algebraic if and only if each $C \in \mathcal{C}(A)$ is approximately finite-dimensional. By Theorem II.26, this is equivalent with scatteredness of A. $\square$

IV. Continuity

In this section we characterize C*-algebras A for which $\mathcal{C}(A)$ is continuous. These will turn out to be precisely the same ones for which $\mathcal{C}(A)$ is algebraic. We start by characterizing the way-below relation on $\mathcal{C}(A)$ completely in operator-algebraic notions.

Proposition IV.1. *The following are equivalent for a C*-algebra A and $B, C \in \mathcal{C}(A)$:*

1. *$B \ll C$;*
2. *$B \in \mathcal{K}(\mathcal{C})$ and $B \subseteq C$;*
3. *B is finite-dimensional and $B \subseteq C$.*

Proof. By Proposition III.1, B is finite-dimensional if and only if B is compact, which proves the equivalence between (2) and (3). It is almost trivial that (2) implies (1) by unfolding definitions. For (1) $\Rightarrow$ (3), assume $B \subseteq C$ but B infinite-dimensional. We may assume that $C = C(X)$ for the spectrum X of C by Theorem II.13. Lemma A.2 gives $p \in X$ with $B \not\subseteq C_{\overline{U}}$ for each $U \in \mathcal{O}(p)$. Set

$$\mathcal{D} = \{C_{\overline{U}} \mid U \in \mathcal{O}(p)\}.$$

By Lemma A.4, this is a directed family such that $\bigvee^{\uparrow}\mathcal{D} = C(X)$. However, B is not contained in any $D \in \mathcal{D}$, so B cannot be way below $C = C(X)$. $\square$

The following theorem is our second main result, characterizing continuity of $\mathcal{C}(A)$.

Theorem IV.2. *For a C*-algebra A, the dcpo $\mathcal{C}(A)$ is continuous if and only if it is algebraic.*

Proof. Let $C \in \mathcal{C}(A)$. It follows from Proposition IV.1 that $\twoheaddownarrow C = \mathcal{K}(\mathcal{C}) \cap \downarrow C$, whence $C = \bigvee^{\uparrow}\mathcal{K}(\mathcal{C}) \cap \downarrow C$ if and only if $C = \bigvee^{\uparrow} \twoheaddownarrow C$. Thus continuity and algebraicity coincide. $\square$

V. Atomicity

Let $\mathcal{C}$ be a partially ordered set with least element 0. An *atom* is an element $C \in \mathcal{C}$ such that $0 < C$, and there is nothing between 0 and C in the sense that $B = C$ whenever $0 < B \leq C$. A partially ordered set is called *atomistic* if each element is the least upper bound of some collection of atoms. For a system modeled by a C*-algebra A, intuitively, atoms of $\mathcal{C}(A)$ are the smallest nontrivial observations one can make, and $\mathcal{C}(A)$ is atomistic when any chain of increasing observations proceeds in indivisible steps. In this section we characterize those C*-algebras for which this is the case. It will turn out these are precisely the ones for which $\mathcal{C}(A)$ is algebraic (and/or continuous).

We begin by identifying the atoms in $\mathcal{C}(A)$. Write $C^*(S) \subseteq A$ for the smallest C*-subalgebra containing $S \subseteq A$, and say that $C^*(S)$ is *generated* by S. For example, $C^*(\{p\})$ is just the linear span $\mathrm{Span}\{p, 1-p\}$ for projections $p^2 = p^* = p \in A$; this is two-dimensional unless p is *trivial*, *i.e.* 0 or 1, in which case it collapses to the one-dimensional least element $\mathbb{C}1_A$ of $\mathcal{C}(A)$.

Lemma V.1. *[21, 3.1]. Let A be a C*-algebra. Then C is an atom in $\mathcal{C}(A)$ if and only if it is generated by a nontrivial projection.*

Proof. Clearly two-dimensional C are atoms in $\mathcal{C}(A)$. Conversely, assume that C is an atom of $\mathcal{C}(A)$. By Theorem II.13, $C \cong C(X)$ for a compact Hausdorff space X. If X contains three distinct point x, y, z, then $C(X)$ contains a proper subalgebra $\{f \in C(X) \mid f(x) = f(y)\}$ with dimensional at least two, which contradicts atomicity of C. Hence X must contain exactly two points x and y. Using the *-isomorphism between C and $C(X)$, let $p \in C$ be the element corresponding to the element of $C(X)$ given by $x \mapsto 1$ and $y \mapsto 0$ for $y \neq x$. It follows that $C = \mathrm{Span}\{p, 1-p\}$. □

To characterize atomicity we will need two auxiliary results. The first deals with least upper bounds of subalgebras in terms of generators.

Lemma V.2. *Let A be a C*-algebra and $C \in \mathcal{C}(A)$. If $\{S_i\}_{i \in I}$ is a family of subsets of C, then each $C^*(S_i)$ is in $\mathcal{C}(A)$, and $C^*\left(\bigcup_{i \in I} S_i\right) = \bigvee_{i \in I} C^*(S_i)$.*

Proof. For any $i \in I$, clearly $C^*(S_i)$ is a commutative C*-subalgebra of A, and hence an element of $\mathcal{C}(A)$.

Writing $S = \bigcup_{i \in I} S_i$, we have $S_j \subseteq C^*(S)$, and so $C^*(S_j) \subseteq C^*(C^*(S)) = C^*(S)$. Therefore, $\bigvee_{i \in I} C^*(S_i)$ is contained in $C^*(S)$. For the inclusion in the other direction, notice that clearly $S \subseteq \bigvee_{i \in I} C^*(S_i)$, whence

$$C^*(S) \subseteq C^*\left(\bigvee_{i \in I} C^*(S_i)\right) = \bigvee_{i \in I} C^*(S_i).$$

This finishes the proof. □

The second auxiliary result deals with subalgebras generated by projections. It shows that projections are the building blocks for C*-algebras A whose dcpos $\mathcal{C}(A)$ are atomistic. This explains why mere approximate finite-dimensionality is not good enough to characterize algebraicity and/or continuity. See also Section IX below.

Proposition V.3. *For a C*-algebra A, a C*-subalgebra C is the least upper bound of a collection of atoms of $\mathcal{C}(A)$ if and only if it is generated by projections.*

Proof. Let $C = C^*(P)$, where $P \subseteq A$ is a collection of projections. Then $P \subseteq C$, so that $C^*(\{p\}) \in \downarrow C$ for each $p \in P$. Since $P = \bigcup_{p \in P}\{p\}$, it follows from Lemma V.2 that

$$C = C^*(P) = \bigvee_{p \in P} C^*(\{p\}).$$

It now follows from Lemma V.1 that C is the least upper bound of a collection of atoms in $\mathcal{C}(A)$.

Conversely, if $C = \bigvee \mathcal{D}$ for a collection $\mathcal{D}$ of atoms in $\mathcal{C}(A)$, we must have $\mathcal{D} = \{C^*(\{p\}) \mid p \in P\}$ for some family $P \subseteq C$ of projections. Hence

$$C = \bigvee_{p \in P} C^*(\{p\}) = C^*\Big(\bigcup_{p \in P}\{p\}\Big) = C^*(P),$$

where the second equality used Lemma V.2. Thus C is generated by projections. □

We can now prove our third main result, characterizing atomicity of $\mathcal{C}(A)$.

Theorem V.4. *For a C*-algebra A, the dcpo $\mathcal{C}(A)$ is algebraic if and only if it is atomistic.*

Proof. Assume that $\mathcal{C}(A)$ is algebraic and let $C \in \mathcal{C}(A)$. If $C = \mathbb{C}1_A$, then C is the least upper bound of the empty set, which is a subset of the set of atoms. Assume that $C \neq \mathbb{C}1_A$. Since $\mathcal{C}(A)$ is algebraic, it follows from Proposition III.1 that C is the least upper bound of all its finite-dimensional C*-subalgebras. Since every finite-dimensional C*-algebra is generated by a finite set of projections, it follows from Proposition V.3 that each element $D \in K(\mathcal{C}(A)) \cap \downarrow C$ can be written as the least upper bound of atoms in $\mathcal{C}(A)$. Hence C is a least upper bound of atoms, so $\mathcal{C}(A)$ is atomistic.

Conversely, assume $\mathcal{C}(A)$ is atomistic and let $C \in \mathcal{C}(A)$. Since, if C is finite dimensional, C clearly is a least upper bound of $K(\mathcal{C}(A)) \cap \downarrow C$, we may assume C is infinite-dimensional. By Lemma V.1, $C = \bigvee_{p \in P} C^*(\{p\})$ for some collection P of projections in A. As we must have $P \subseteq C$, all projections in P commute. We may replace P by the set of all projections of C, which we will denote by P as well; then we still have $C = \bigvee_{p \in P} C^*(\{p\})$. Write $\mathcal{F}$ for the collection of all finite subsets of P, and set

$$\mathcal{D} = \{C^*(F) \mid F \in \mathcal{F}\}.$$

If $F \in \mathcal{F}$, then $C^*(F)$ is finite-dimensional, and since finite-dimensional C*-algebras are generated by a finite number of projections, it follows that $\mathcal{D} = K(\mathcal{C}(A)) \cap \downarrow C$. Now let $F_1, F_2 \in \mathcal{F}$. By Lemma V.2, $C^*(F_1) \vee C^*(F_2) = C^*(F_1 \cup F_2)$, making $\mathcal{D}$ directed. Then:

$$\begin{aligned} C &= \bigvee_{p \in P} C^*(\{p\}) = \bigvee_{F \in \mathcal{F}}^{\uparrow} \bigvee_{p \in F} C^*(\{p\}) \\ &= \bigvee_{F \in \mathcal{F}}^{\uparrow} C^*(F) = \bigvee{}^{\uparrow} \mathcal{D}, \end{aligned}$$

where the third equality used Lemma V.2. Hence $\mathcal{C}(A)$ is algebraic. □

VI. Quasi-continuity and quasi-algebraicity

In this section we show that for dcpos $\mathcal{C}(A)$ of C*-algebras A, the notions of quasi-continuity and quasi-algebraicity, which are generally weaker than continuity and algebraicity, are in fact equally strong. We start by analyzing the way below relation generalized to finite subsets.

Lemma VI.1. *Let A be a C*-algebra, $C \in \mathcal{C}(A)$ and $\mathcal{F} \subseteq \mathcal{C}(A)$. Then $\mathcal{F} \in \mathrm{Fin}(C)$ if and only if $\mathcal{F}$ contains finitely many elements and $F \ll C$ for some $F \in \mathcal{F}$.*

Proof. Let $\mathcal{F}$ contain finitely many elements and assume that $F \ll C$ for some $F \in \mathcal{F}$. Let $\mathcal{D}$ be a directed subset of $\mathcal{C}(A)$ such that $C \subseteq \bigvee \mathcal{D}$. Since $F \ll C$, we have $F \subseteq D$ for some $D \in \mathcal{D}$, so $D \in \uparrow \mathcal{F}$. Thus $\mathcal{F} \in \mathrm{Fin}(C)$.

Conversely, $\mathcal{F} \in \mathrm{Fin}(C)$. Then $\mathcal{F} \ll C$ and $\mathcal{F}$ is nonempty and finite. Setting $\mathcal{D} = \{C\}$ gives a directed subset with $C \subseteq \bigvee^{\uparrow} \mathcal{D}$, so there is some $F \in \mathcal{F}$ such that $F \subseteq C$. Let $\{F_1, \ldots, F_n\}$ be the subset of $\mathcal{F}$ of all elements contained in C, and assume that each F_i has infinite dimension. Write X for the spectrum of C, so $C \cong C(X)$. Lemma A.2 now guarantees the existence of points $p_1, \ldots, p_n \in X$ with $F_j \not\subseteq C_{\overline{U_j}}$ for each $U_j \in \mathcal{O}(p_j)$. In particular, $F_j \not\subseteq \bigcap_{i=1}^{n} C_{\overline{U_i}}$ for each $i = 1, \ldots, n$ and $U_i \in \mathcal{O}(p_i)$. Define

$$\mathcal{D} = \Big\{ \bigcap_{i=1}^{n} C_{\overline{U_i}} \mid U_i \in \mathcal{O}(p_i),\ i = 1, \ldots, n \Big\}.$$

By Lemma A.4, $\mathcal{D}$ is directed and $\bigvee^{\uparrow} \mathcal{D} = C(X)$. However, $F_i \not\subseteq D$ for each $D \in \mathcal{D}$. If $F \in \mathcal{F}$ such that $F \not\subseteq C$, we cannot have $F \subseteq D$ for some $D \in \mathcal{D}$, since each D is contained in C by construction of $\mathcal{D}$, contradicting $\mathcal{F} \ll C$. We conclude that there must be a finite-dimensional $F \in \mathcal{F}$ such that $F \subseteq C$. From Proposition IV.1 it follows that $F \ll C$. □

Lemma VI.2. *Let A be a C*-algebra and let $C \in \mathcal{C}(A)$. If $F \ll C$, then $\{F\} \in \mathrm{KFin}(C)$. If $\mathcal{F} \in \mathrm{Fin}(C)$, then $\mathcal{F} \leq \mathcal{F}'$ for some $\mathcal{F}' \in \mathrm{KFin}(C)$.*

Proof. Let $F \ll C$. By Lemma VI.1, we have $\{F\} \in \mathrm{Fin}(C)$. By Lemma IV.1, we have $F \ll F$. Therefore $\{F\} \ll \{F\}$, and soc $\{F\} \in \mathrm{KFin}(C)$.

Let $\mathcal{F} \in \mathrm{Fin}(C)$. By Lemma VI.1, there is an $F \in \mathcal{F}$ such that $F \ll C$. The reasoning in the previous paragraph shows $\{F\} \in \mathrm{KFin}\, C$. Since $F \in \mathcal{F}$, we have $F \in \uparrow \mathcal{F}$, and so $\uparrow\{F\} \subseteq \uparrow \mathcal{F}$. We conclude that $\mathcal{F} \leq \mathcal{F}'$ for $\mathcal{F}' = \{F\}$. □

We are now ready for our fourth main result, characterizing quasi-continuity and quasi-algebraicity.

Theorem VI.3. *The following are equivalent for C*-algebras:*

- *$\mathcal{C}(A)$ is continuous;*
- *$\mathcal{C}(A)$ is quasi-algebraic;*
- *$\mathcal{C}(A)$ is quasi-continuous.*

Proof. Assume $\mathcal{C}(A)$ is continuous and let $C \in \mathcal{C}(A)$. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{KFin}(C)$. Since $\mathrm{KFin}(C) \subseteq \mathrm{Fin}(C)$, it follows from Lemma VI.1 that there are $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ such that $F_1, F_2 \ll C$. Hence $F_1, F_2 \in \twoheaddownarrow C$, and since $\twoheaddownarrow C$ is directed by continuity of $\mathcal{C}(A)$, it follows that there is some $F \in \twoheaddownarrow C$ such that $F_1, F_2 \subseteq F$. Setting $\mathcal{F} = \{F\}$, Lemma VI.2 shows $\mathcal{F} \in \mathrm{KFin}(C)$. Because $F_1, F_2 \subseteq F$, we obtain $\mathcal{F} = \{F\} \subseteq \uparrow \mathcal{F}_1 \cap \uparrow \mathcal{F}_2$, making $\mathrm{KFin}(C)$ directed.

Let $B \in \mathcal{C}(A)$ satisfy $C \not\subseteq B$. We have to show $B \notin \uparrow \mathcal{F}$ for some $\mathcal{F} \in \mathrm{KFin}(C)$. Assume for a contradiction that $B \in \uparrow \mathcal{F}$ for each $\mathcal{F} \in \mathrm{KFin}(C)$. Lemma VI.2 shows that $\{F\} \in \mathrm{KFin}(C)$ for each $F \in \twoheaddownarrow C$, whence $F \subseteq B$ for each $F \in \twoheaddownarrow C$. Therefore $\bigvee \twoheaddownarrow C \subseteq B$, and by continuity of $\mathcal{C}(A)$ we have $\bigvee \twoheaddownarrow C = C$, so $C \subseteq B$. This is clearly a contradiction, so $\mathcal{C}(A)$ must be quasi-algebraic.

Now assume that $\mathcal{C}(A)$ is quasi-algebraic and let $C \in \mathcal{C}(A)$. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Fin}(C)$. By Lemma VI.2, there exist elements $\mathcal{F}'_1, \mathcal{F}'_2 \in \mathrm{KFin}(C)$ such that $\mathcal{F}_i \leq \mathcal{F}'_i$. By quasi-algebraicity, $\mathrm{KFin}(C)$ is directed, so there is an $\mathcal{F} \in \mathrm{KFin}(C)$ such that $\mathcal{F}'_1, \mathcal{F}'_2 \leq \mathcal{F}$. Hence $\mathcal{F}_1, \mathcal{F}_2 \leq \mathcal{F}$. Since $\mathrm{KFin}(C) \subseteq \mathrm{Fin}(C)$, it then follows that $\mathrm{Fin}(C)$ is directed. Let $B \in \mathcal{C}(A)$ satisfy $C \not\subseteq B$. Assume that $B \in \uparrow \mathcal{F}$ for $\mathcal{F} \in \mathrm{Fin}(C)$. Lemma VI.2 provides $\mathcal{F}' \in \mathrm{KFin}(C)$ with $\mathcal{F} \leq \mathcal{F}'$. But this means that $\uparrow \mathcal{F} \subseteq \uparrow \mathcal{F}'$. Hence $B \in \uparrow \mathcal{F}'$, which contradicts quasi-algebraicity. Therefore we must have $B \notin \uparrow \mathcal{F}$ for each $\mathcal{F} \in \mathrm{Fin}(C)$, making $\mathcal{C}(A)$ quasi-continuous.

Finally, assume $\mathcal{C}(A)$ is quasi-continuous. Let $F_1, F_2 \in \twoheaddownarrow C$. By Lemma VI.1, we have $\{F_1\}, \{F_2\} \in \mathrm{Fin}(C)$, and since $\mathrm{Fin}(C)$ is directed, there is an $\mathcal{F} \in \mathrm{Fin}(C)$ such that $\mathcal{F} \subseteq \uparrow\{F_1\} \cap \uparrow\{F_2\}$. In other words, $F_1, F_2 \subseteq F$ for each $F \in \mathcal{F}$, and since $\mathcal{F} \in \mathrm{Fin}(C)$, Lemma VI.1 guarantees the existence of some F such that $F \ll C$, making $\twoheaddownarrow C$ directed. Let $B = \bigvee \twoheaddownarrow C$. Since $F \subseteq C$ for each $F \in \twoheaddownarrow C$, we have $B \subseteq C$. If $B \neq C$, then $C \not\subseteq B$, so by quasi-continuity there must be an $\mathcal{F} \in \mathrm{Fin}(C)$ with $B \notin \uparrow \mathcal{F}$. Hence $F \not\subseteq B$ for each $F \in \mathcal{F}$, and in particular, Lemma VI.1 implies the existence of some $F \in \mathcal{F}$ satisfying $F \ll C$, but $F \not\subseteq B$. By definition of B we have $F \subseteq B$ for each $F \ll C$, giving a contradiction. Thus $\mathcal{C}(A)$ is continuous. □

VII. Meet-continuity

This section considers meet-continuity, which is a weakening of continuity that is especially useful for complete semilattices. If $\{C_i\}_{i \in I}$ is a collection of commutative C*-subalgebras of a C*-algebra A, then $\bigcap_{i \in I} C_i$ is again a commutative C*-subalgebra. Hence $\mathcal{C}(A)$ is a complete semilattice. It is a standard result that in this case continuity implies meet-continuity.

Proposition VII.1. *Let $\mathcal{C}$ be a semilattice that is also a dcpo. If it is continuous, then it is meet-continuous.*

Proof. See [4, I-1.8]. □

We can also prove a partial converse.

Proposition VII.2. *Let A be a C*-algebra with a commutative C*-subalgebra whose spectrum is totally disconnected but not scattered. Then $\mathcal{C}(A)$ is not meet-continuous.*

Proof. Write C for the commutative C*-subalgebra in question, and X for its spectrum. Since X is totally disconnected,

C is approximately finite-dimensional. Hence there is some directed $\mathcal{D} \subseteq \mathcal{C}(A)$ consisting of finite-dimensional subalgebras of C with $\bigvee^{\uparrow}\mathcal{D} = C$. Since X is not scattered, there must be some C*-subalgebra $B \subseteq C$ that is not approximately finite-dimensional. If $\mathcal{C}(A)$ is meet-continous, then

$$B = B \cap C = B \cap \bigvee{}^{\uparrow}\mathcal{D} = \bigvee{}^{\uparrow}\{B \cap D \mid D \in \mathcal{D}\}.$$

Since $B \cap D$ is finite-dimensional as each $D \in \mathcal{D}$ is finite dimensional, it follows that B must be approximately finite-dimensional. This is a contradiction, hence $\mathcal{C}(A)$ cannot be meet-continuous. $\square$

We leave open the question whether non-scattered C*-algebras A can have dcpos $\mathcal{C}(A)$ that are not meet-continuous.

VIII. Scattered C*-algebras

In the previous sections, we have seen that a C*-algebra A has a dcpo $\mathcal{C}(A)$ that is nice – in the sense of being (quasi-)algebraic, (quasi-)continuous, or atomistic, which are all equivalent – precisely when it is scattered. In this case, we can turn the domain $\mathcal{C}(A)$ itself into the spectrum of another C*-algebra, which this section studies.

Proposition VIII.1. *If A is a scattered C*-algebra, then $\mathcal{C}(A)$ is a totally disconnected compact Hausdorff space in the Lawson topology.*

Proof. If A is scattered, then $\mathcal{C}(A)$ is both an algebraic domain and a complete semilattice. Therefore it is compact Hausdorff in the Lawson topology [4, III-1.11]. Moreover, it follows that $\mathcal{C}(A)$ is zero-dimensional [4, III-1.14], which for compact Hausdorff spaces is equivalent to being totally disconnected [45, 29.7]. $\square$

Our fifth main result follows from the previous proposition: any scattered C*-algebra A gives rise to another, commutative, C*-algebra $C(X)$ for $X = \mathcal{C}(A)$ with its Lawson topology. Thus we can speak about the domain of commutative C*-subalgebras entirely in the language of C*-algebras. The interpretation of this construction in terms of transition systems is unclear, but it might give rise to a similar construction as the Brzozowski minimization of an automaton [13], and hence provides interesting material for further study.

To that end, a first step might be to ask whether the construction from the previous proposition is functorial, that is, whether it respects unital $*$-homomorphisms. This turns out not to be the case, as it is in fact prohibited by rigorous no-go theorems [46].

A second question might be what happens when we iterate this construction. The following proposition shows that this only makes sense in the degenerate case of finite dimensions.

Proposition VIII.2. *Let A be a scattered C*-algebra. Its domain $\mathcal{C}(A)$ is scattered in the Lawson topology only if A is finite-dimensional.*

Proof. Recall from Definition II.6 that a basis for the Lawson topology on $\mathcal{C}(A)$ is given by sets of the form $\mathcal{U} \setminus \uparrow\mathcal{F}$ with $\mathcal{F} \subseteq \mathcal{C}(A)$ finite and $\mathcal{U}$ Scott open. Since A is scattered, $\mathcal{C}(A)$ is algebraic. It follows that a basis for the Scott topology is given by $\uparrow C$ for C compact [4, II-1.15]. Thus sets of the form $\uparrow C \setminus \uparrow \mathcal{F}$ with C compact and $\mathcal{F}$ finite form a basis for the Lawson topology.

Let A have finite dimension, so it is certainly scattered. Take a nonempty subset $\mathcal{S} \subseteq \mathcal{C}(A)$, and let M be a maximal element of $\mathcal{S}$, which exists by [25, 3.16]. Since M must be finite-dimensional too, it is compact by Proposition III.1. Hence $\uparrow M$ is Scott open and therefore Lawson open. Maximality of M in $\mathcal{S}$ now gives $\mathcal{S} \cap \uparrow M = \{M\}$, and since $\uparrow M$ is Lawson open, it follows that M is an isolated point of $\mathcal{S}$. Hence $\mathcal{C}(A)$ is scattered.

Conversely, assume A is infinite-dimensional. Then $\mathcal{C}(A)$ has a noncompact element C. Then $\mathcal{S} = \downarrow C$ contains an isolated point if $\mathcal{S} \cap \mathcal{U}$ is a singleton for some basic Lawson open $\mathcal{U}$. Hence $\downarrow C \cap \uparrow K \setminus \uparrow \mathcal{F}$ must be a singleton for some finite set $\mathcal{F} \subseteq \mathcal{C}(A)$ and some compact $K \in \mathcal{C}(A)$. In other words, $[K, C] \setminus \mathcal{F}$ is a singleton, where $[K, C]$ is the interval $\{D \in \mathcal{C}(A) \mid K \subseteq D \subseteq C\}$. Since C is infinite-dimensional and scattered (by Theorem II.26), there are infinitely many atoms in $[K, C]$: for C is atomistic by Theorem V.4 and hence has infinitely many atoms C_i, but K is finite-dimensional by Proposition III.1, so that $C_i \vee K$, excepting the finitely many $C_i \leq K$, give infinitely many atoms in $[K, C]$. Hence there is no finite subset $\mathcal{F} \subseteq \mathcal{C}(A)$ making $[K, C] \setminus \uparrow \mathcal{F}$ a singleton. We conclude that $\downarrow C$ has no isolated points, so $\mathcal{C}(A)$ cannot be scattered. $\square$

IX. Von Neumann algebras

In the infinite case, probabilistic transition systems are usually phrased in terms of measure theory rather than topology [2]. There is an analogue to Theorem II.13, saying that measure spaces are dual to commutative so-called *von Neumann algebras* [47], [34]. In this section we investigate the domain-theoretic properties of these special C*-algebras. Notice that nevertheless our formulation of the C*-algebra generated by the transition probability matrices still holds and has no need of von Neumann algebras. It will turn out that dcpos of von Neumann algebras only behave well domain-theoretically in degenerate cases.

Definition IX.1. If V is a Banach space, then so is its *dual space* V^* of continuous linear functions $V \to \mathbb{C}$. A *von Neumann algebra* is a C*-algebra of the form V^*, for some Banach space V.

We can turn any C*-algebra A into a von Neumann algebra by taking its double dual A^{**}, also called the *enveloping von Neumann algebra* [33]. This in fact gives an adjunction of categories showing that von Neumann algebras form a reflexive subcategory of C*-algebras [48, 3.2].

We can play the same game of approximation with commutative von Neumann subalgebras rather than C*-subalgebras. Indeed, this setting has been studied before [29]. However, notice that the reason we considered commutative C*-subalgebras – namely bisimilarity of states for probabilistic systems, and empirical accessibility for quantum systems –

had little to do with measure theory and does not suggest the use of von Neumann subalgebras.

Definition IX.2. A von Neumann subalgebra of a von Neumann algebra M is a C*-subalgebra $V \subseteq M$ that is a von Neumann algebra in its own right. Write $\mathcal{V}(M)$ for the partially ordered set of von Neumann subalgebras of M under inclusion.

We now also show that $\mathcal{V}(M)$ is a reflexive subcategory of $\mathcal{C}(M)$, a satisfying decategorification of the fact that von Neumann algebras form a reflexive subcategory of C*-algebras.

Proposition IX.3. *For a von Neumann algebra M there is a Galois correspondence*

$$\begin{tikzcd} \mathcal{V}(M) \arrow[r, hook, shift right=1.5] \arrow[r, phantom, "\perp"] & \mathcal{C}(M) \arrow[l, shift right=1.5] \end{tikzcd}$$

where the upper adjoint maps a C-subalgebra $C \in \mathcal{C}(M)$ to the smallest W*-subalgebra of M containing it.*

Proof. Write $C'' = \bigcap\{V \in \mathcal{V}(M) \mid C \subseteq V\}$ for the smallest W*-subalgebra of M containing $C \in \mathcal{C}(M)$. If $C \subseteq D$, then clearly $C'' \subseteq D''$. By construction we have $C'' \subseteq V$ if and only if $C \subseteq V$, for $C \in \mathcal{C}(M)$ and $V \in \mathcal{V}(M)$. Finally, notice that $V'' = V$ for $V \in \mathcal{V}(M)$. □

Our last main result shows that for von Neumann algebras M the dcpo $\mathcal{C}(M)$ is only interesting in degenerate cases.

Theorem IX.4. *The following are equivalent for a von Neumann algebra M:*

- *$\mathcal{C}(M)$ is continuous;*
- *$\mathcal{C}(M)$ is algebraic;*
- *$\mathcal{V}(M)$ is continuous;*
- *$\mathcal{V}(M)$ is algebraic;*
- *M is finite dimensional.*

Proof. The equivalence of the last three properties is proved in [29, 6.1].

If $\mathcal{C}(M)$ is algebraic or continuous, then M is scattered by Theorems III.2 and IV.2. Theorem II.26 then implies that all maximal commutative C*-subalgebras of M are scattered. But maximal C*-subalgebras are automatically von Neumann algebras by Proposition IX.3, and scattered commutative von Neumann algebras are finite-dimensional by Lemma A.5 in the Appendix. Since all maximal commutative C*-subalgebras of M are finite-dimensional, so is M itself [31, 4.12].

The converse is easy: any finite-dimensional C*-algebra is scattered. □

We conclude that, at least from a domain-theoretic perspective of approximating quantum computations by classical ones, von Neumann algebras are a lot less interesting than C*-algebras. There are many examples of C*-algebras A for which $\mathcal{C}(A)$ is continuous but $\mathcal{C}(A^{**})$ is not: any infinite-dimensional scattered C*-algebra will do, such as $C(X)$ for the infinite compact Hausdorff scattered spaces X of Example II.24.

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APPENDIX

In this appendix we prove five lemmas about point-set topology which would distract too much from the main text.

The first two lemmas concern the equivalence relation $\sim_B$ on a compact Hausdorff space X defined by $x \sim_B y$ if and only if $b(x) = b(y)$ for each element b of a C*-subalgebra $B \subseteq C(X)$, which was already used in the proof of Proposition II.14.

Lemma A.1. *Let X be a compact Hausdorff space and let $B \subseteq C(X)$ a C*-subalgebra. Consider the equivalence relation $\sim_B$ from Section IV. Each equivalence class $[x]_B$ is a closed subset of X.*

Proof. The proof of Proposition II.14 shows that the quotient $X/{\sim_B}$ is compact Hausdorff. If q is the quotient map, then $q(x) = [x]_B$, which is closed since q is a closed map, being a continuous function between compact Hausdorff spaces. □

Lemma A.2. *For a compact Hausdorff space X and C*-subalgebra $B \subseteq C(X)$:*

(i) *B is finite dimensional if and only if $[x]_B \subseteq X$ is open for each $x \in X$;*

(ii) *if X is connected, B is the (one-dimensional) subalgebra of all constant functions on X if and only if $[x]_B$ is open for some $x \in X$;*

(iii) *if B is infinite-dimensional, there are $x \in X$ and $p \in [x]_B$ such that $B \nsubseteq C_{\overline{U}}$ for each $U \in \mathcal{O}(p)$. If X is connected, this holds for all $x \in X$.*

Proof. Fix X and B.

(i) Let $q\colon X \to X/{\sim_B}$ be the quotient map. By definition of the quotient topology, $V \subseteq X/{\sim_B}$ is open if and only if its preimage $q^{-1}[V]$ is open in X. We can regard $[x]_B$ both as a subset of X and as point in $X/{\sim_B}$. Since $[x]_B = q^{-1}[\{[x]_B\}]$, we find that $\{[x]_B\}$ is open in $X/{\sim_B}$ if and only if $[x]_B$ is open in X. Hence $X/{\sim_B}$ is discrete if and only if $[x]_B$ is open in X for each $x \in X$. Now $X/{\sim_B}$ is compact, being a continuous image of a compact space. It is also Hausdorff by Proposition II.14. Hence $X/{\sim_B}$ is discrete if and only if it is finite. Thus each $[x]_B$ is open in X if and only if B is finite-dimensional.

(ii) An equivalence class $[x]_B$ is always closed in X (see Lemma A.1). Assume that it is also open. By connectedness $X = [x]_B$, so $f(y) = f(x)$ for each $f \in B$ and each $y \in X$. Hence B is the algebra of all constant functions on X, and since this algebra is spanned by the function $x \mapsto 1$, it follows that B is one dimensional. Conversely, if B is the one-dimensional subalgebra of all constant function on X, then for each $f \in B$ there is some $\lambda \in \mathbb{C}$ such that $f(x) = \lambda$ for each $x \in X$. Hence $f(x) = f(y)$ for each $x, y \in X$, whence for each $x \in X$ we have $[x]_B = X$, which is clearly open.

(iii) Assume that B is infinite-dimensional. By (i) there must be some $x \in X$ such that $[x]_B$ is not open. Hence there must be a point $p \in [x]_B$ such that $U \nsubseteq [x]_B$ for each $U \in \mathcal{O}(p)$. If X is connected, (ii) implies that $[x]_B$ is not open for any $x \in X$, so p can be chosen as an element of $[x]_B$ for each $x \in X$. In both cases, we have $U \nsubseteq [x]_B$ for $U \in \mathcal{O}(p)$, hence there is $q \in U$ such that $q \notin [x]_B$. We have $y \in [x]_B$ if and only if $f(x) = f(y)$ for each $f \in B$. So $p \in [x]_B$, and $q \notin [x]_B$ implies the existence of some $f \in B$ such that $f(p) \neq f(q)$. That is, there is some $f \in B$ such that f is not constant on U, so f is certainly not constant on $\overline{U}$. We conclude that for each $U \in \mathcal{O}(p)$ there is an $f \in B$ such that $f \notin C_{\overline{U}}$, so $B \nsubseteq C_{\overline{U}}$ for each $U \in \mathcal{O}(p)$. □

The following lemma shows that the different definitions of approximate finite-dimensionality of [5] and [42] coincide in the commutative case, as used in Sections II and III.

Lemma A.3. *The following are equivalent for a compact Hausdorff space X:*

- *$C(X)$ is approximately finite-dimensional;*

- *X is totally disconnected;*
- *$C(X) = \overline{\bigcup \mathcal{D}}$ for a directed set $\mathcal{D}$ of finite-dimensional C*-subalgebras $D \subseteq C(X)$ with $1_X \in D$.*

Proof. Assume $C(X)$ is approximately finite-dimensional and let $x, y \in X$ be distinct points. Urysohn's lemma gives $f \in C(X)$ with $f(x) = 1 \neq 0 = f(y)$. By Definition II.20, there exist a finite-dimensional C*-subalgebra B and $g \in B$ with $\|f - g\| < \frac{1}{2}$. This implies that $g(x) \neq g(y)$, and so $y \notin [x]_B$. Since B is finite-dimensional, Lemma A.2 makes $[x]_B$ clopen. Hence x and y cannot share a connected component, and X is totally disconnected.

Next, let X be totally disconnected. Distinct points $x, y \in X$ induce a clopen subset $C \subseteq X$ containing x but not y. Hence the characteristic function of C is continuous, and thus a projection. So the projections of $C(X)$ separate X. It follows from the Stone–Weierstrass theorem that the projections span an algebra B that is dense in $C(X)$. Let $\mathcal{D}$ be the family of algebras spanned by finitely many projections and 1_X. Then $\mathcal{D}$ is clearly directed and consists solely of finite-dimensional C*-subalgebras. Clearly $\bigcup \mathcal{D} = B$, so $\overline{\bigcup \mathcal{D}} = C(X)$.

Finally, if there is a directed set of finite-dimensional C*-subalgebras whose union lies dense in $C(X)$, it follows easily that $C(X)$ is approximately finite-dimensional. □

The following lemma gives a convenient way to construct directed subsets of $\mathcal{C}(A)$ used in Sections III, IV, and VI. It uses the notation $\mathcal{O}(x)$ for the set of all open neighbourhoods of a point x in a compact Hausdorff space X, already introduced in Section III, and C_K for the C*-subalgebra of $C(X)$ of functions that are constant on a closed subset $K \subseteq X$.

Lemma A.4. *Let A be a C*-algebra, and $C \subseteq A$ a commutative C*-subalgebra with spectrum X. If $P \subseteq X$ is finite,*

$$\mathcal{D} = \Big\{ \bigcap_{p \in P} C_{\overline{U_p}} \mid U_p \in \mathcal{O}(p) \Big\}$$

is a directed family in $\mathcal{C}(A)$ satisfying $\bigvee^{\uparrow} \mathcal{D} = C$.

Proof. If U_p and V_p are in $\mathcal{O}(p)$, then $U_p \cap V_p$ is an open neighbourhood of p. Moreover

$$\overline{U_p \cap V_p} \subseteq \overline{U_p} \cap \overline{V_p} \subseteq \overline{U_p},$$

so $C_{\overline{U_p}} \subseteq C_{\overline{U_p \cap V_p}}$. In a similar way, we find $C_{\overline{V_p}} \subseteq C_{\overline{U_p \cap V_p}}$. Hence if $D_1 = \bigcap_{p \in P} C_{\overline{U_p}}$ and $D_2 = \bigcap_{p \in P} C_{\overline{V_p}}$ are elements of $\mathcal{D}$, then they are both contained in $D = \bigcap_{p \in P} C_{\overline{U_p \cap V_p}}$, which is clearly an element of $\mathcal{D}$. Thus $\mathcal{D}$ is directed.

We use the Stone–Weierstrass Theorem [31, 3.4.14] to show that $\bigvee^{\uparrow} \mathcal{D} = C(X)$. First, $\bigvee^{\uparrow} \mathcal{D}$ clearly contains all constant functions, since $x \mapsto 1$ is in every element of $\mathcal{D}$. Second, to see that $f \in \bigvee^{\uparrow} \mathcal{D}$ implies $f^* \in \bigvee^{\uparrow} \mathcal{D}$, assume that $f \in D = \bigcap_{p \in P} C_{\overline{U_p}}$. Then f is constant on each $\overline{U_p}$. Since f^* is defined by $f^*(x) = \overline{f(x)}$ for each $x \in X$, we see that f^* is constant too on each $\overline{U_p}$, so $f^* \in D$. If $f \in \bigvee^{\uparrow} \mathcal{D}$, there is a sequence of $f_1, f_2, \ldots$ such that each $f_n \in D$ for some $D \in \mathcal{D}$. Hence each $f_n^* \in D$ for some $D \in \mathcal{D}$, and since $\lim_{n \to \infty} \|f_n^* - f^*\| = \lim_{n \to \infty} \|f_n - f\| = 0$, we find that $f^* \in \bigvee^{\uparrow} \mathcal{D}$.

Finally, let x and y be distinct points in X; we will show that $f(x) \neq f(y)$ for some $f \in \bigvee^{\uparrow} \mathcal{D}$ by distinguishing two cases. For the first case, suppose $x, y \in P$. Since P is finite, it is closed, as is $P \setminus \{x\}$. Hence $\{x\}$ and $P \setminus \{x\}$ are disjoint closed subsets in X, and since X is compact Hausdorff and hence normal, one can find open subsets U and V containing x and $P \setminus \{x\}$, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. Because $U \in \mathcal{O}(x)$ and $V \in \mathcal{O}(p)$ for each $p \in P \setminus \{x\}$, it follows that $C_{\overline{U}} \cap C_{\overline{V}} \in \mathcal{D}$. But since $\overline{U}$ and $\overline{V}$ are disjoint closed sets, Urysohn's lemma provides a function $f \in C(X)$ satisfying $f[\overline{U}] = \{0\}$ and $f[\overline{V}] = \{1\}$. Hence f is constant on $\overline{U}$ and on $\overline{V}$, so $f \in C_{\overline{U}} \cap C_{\overline{V}} \subseteq \bigvee^{\uparrow} \mathcal{D}$. Since $y \in P \setminus \{x\} \subseteq \overline{V}$, we find $f(x) = 0 \neq 1 = f(y)$.

For the second case, suppose $x \notin P$. We proceed in a similar way. Regardless of whether $y \in P$ or not, $\{x\}$ and $P \cup \{y\}$ are disjoint closed subsets, hence there are open sets U and V containing $\{x\}$ and $P \cup \{y\}$, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. Since $V \in \mathcal{O}(p)$ for each $p \in P$, we find that $C_{\overline{V}} \in \mathcal{D}$. Again Urysohn's lemma provides a function $f \in C(X)$ satisfying $f[\overline{U}] = \{0\}$ and $f[\overline{V}] = \{1\}$, and since f constant on $\overline{V}$, we find $f \in C_{\overline{V}} \subseteq \bigvee^{\uparrow} \mathcal{D}$. Again $f(x) \neq f(y)$.

We conclude that $\bigvee^{\uparrow} \mathcal{D}$ separates all points of X, so by the Stone–Weierstrass Theorem, $\bigvee^{\uparrow} \mathcal{D} = C(X)$. □

The following lemma shows that scattered von Neumann algebras must be finite-dimensional, as used in Section IX. A topological space is *Stonean*, or *extremally disconnected*, when the closure of an open set is still open. The spectrum of a commutative von Neumann algebra is always Stonean.

Lemma A.5. *If a compact Hausdorff space X is both scattered and Stonean, then it must be finite.*

Proof. Consider the open and discrete set

$$U = \{x \in X \mid \{x\} \text{ is closed and open}\}.$$

Assume that $X \setminus U \neq \emptyset$. By scatteredness, $\{x\}$ is open in $X \setminus U$ for some $x \in X \setminus U$. Therefore $X \setminus (U \cup \{x\}) = (X \setminus U) \setminus \{x\}$ is closed in $X \setminus U$ and hence closed in X. Thus both $\{x\}$ and $X \setminus (U \cup \{x\})$ are closed subsets. Since X is compact Hausdorff, there are disjoint open subsets V_1 and V_2 containing x and $X \setminus (U \cup \{x\})$, respectively. We may assume V_1 is closed because X is Stonean. Observe that V_1 is infinite; otherwise $V_1 \setminus \{x\}$ would be closed and $\{x\} = V_1 \setminus (V_1 \setminus \{x\})$ open, contradicting $x \notin U$. Hence $V_1 \setminus \{x\}$ is infinite, too. Pick two disjoint infinite subsets W_1, W_2 covering $V_1 \setminus \{x\}$. Since W_i is contained U, it must be open. If W_i were closed, then it is compact, contradicting that it is both discrete (as subset of U) and infinite. So $W_i \subsetneq \overline{W_i} \subseteq V_1$. Moreover, $\overline{W_i} \cap W_j = \emptyset$ for $i \neq j$, and $W_1 \cup W_2 = V_1 \setminus \{x\}$, so $\overline{W_i} = W_i \cup \{x\}$ whence $\overline{W_1} \cap \overline{W_2} = \{x\}$. Since X is Stonean, $\overline{W_i}$ is open, whence $\{x\}$ is open, contradicting $x \notin U$. Hence $X = U$, and since U is discrete and compact, it must be finite. □