# Can quantum theory be characterized in terms of information-theoretic constraints?

Chris Heunen and Aleks Kissinger

***Abstract*—Does information play a significant role in the foundations of physics? We investigate whether information-theoretic constraints characterize quantum theory. In a C*-algebraic framework, this is known to hold via three equivalences: no broadcasting and noncommutativity; no bit commitment and nonlocality; no signalling and kinematic independence. But this complex linear framework could be said to build in quantum theory from the start. We show that the first two equivalences break, and the third holds, in a framework of generalized, possibly nonlinear, C*-algebras. This uncovers a hierarchy of notions of when (quantum) information is classical.**

## I. Introduction

Does information play a significant role in the foundations of physics? This question, often abbreviated 'it from bit' after John Wheeler, has received significant attention, and lies at the root of quantum information theory. Recent work [14], [20], [21], [17] suggests the answer to the title of this paper is a resounding 'yes'. Notably, when the traditional formalism is generalized to operational probabilistic theories by retaining only the probabilistic data in the form of convex structure, it is possible to precisely capture quantum theory via certain information theoretic axioms [5]. Instead, one can retain only the algebraic structure of interaction between classical and quantum systems. Within this approach, we analyse the seminal work by Clifton, Bub, and Halvorson [7], which isolates quantum theory according to the following three information-theoretic constraints:

- It is impossible to *broadcast* an unknown state;
- It is impossible to implement secure *bit commitment*;
- It is impossible to *signal* information faster than light.

by proving each of them equivalent, respectively, to the following algebraic conditions, characteristic of quantum theory:

- There exist *noncommuting* observables;
- There exist *entangled*, or *nonlocal*, states;
- Distinct systems are *kinematically independent*.

A criticism often raised against this result is that a C*-algebraic framework, including complex numbers and linearity, is assumed from the start [26], [4], [3]. In the words of one of the authors himself [23, page 204]:

> The characterization theorem we proved assumes a C*-algebraic framework for physical theories, which I would now regard as not sufficiently general in the relevant sense, even though it includes a broad class of classical and quantum theories, including field theories, and hybrid theories with superselection rules.

To a lesser extent, operational probabilistic theories also retains some linearity in the form of convexity. To find out to what extent this criticism is valid, we employ the CP*-construction [8]. This framework allows us to consider C*-algebras in various categories, with the category of Hilbert spaces recovering traditional C*-algebras. Of special interest is the nonstandard model of possibilistic quantum theory, as captured by the category of relations. We phrase the above six statements in categorical terms, and then investigate the implications between them in various categories. Our main result proves that the first two equivalences break in categories of relations, but the third holds in any category:

| *information theory* | | *quantum theory* |
|---|---|---|
| no broadcasting | $\not\rightrightarrows$ | noncommutativity |
| no bit commitment | $\not\not\rightleftarrows$ | nonlocality |
| no signalling | $\Leftrightarrow$ | kinematic independence |

This sheds light on the foundational nature of information theory. For example, we discover a hierarchy of operational notions of what it means for (quantum) information to be classical, that just happen to coincide in the traditional formalism. The above nonequivalences also raise the interesting question of what could be the 'weakest' information-theoretic notion that is equivalent to nonlocality in this generality; ideally such a constraint takes the form of an information-theoretic primitive more practical than, say, the GHZ game [13].

The rest of this article is laid out as follows. Section II sets up our general categorical framework of process theories. Sections III–V then investigate one equivalence each: Section III broadcasting; Section IV bit commitment; and Section V signalling.

## II. The CP*-construction

The *CP*-construction* transforms one compact dagger category into another in a way that mirrors the passage from finite-dimensional Hilbert spaces and linear maps to finite-dimensional C*-algebras and completely positive linear maps. This section summarizes the construction. We refer to [19] for the basics of monoidal categories, and start by briefly recalling the notion of a compact category.

Chris Heunen, University of Edinburgh, Scotland, chris.heunen@ed.ac.uk, supported by EPSRC Fellowship EP/L002388/1.

Aleks Kissinger, Radboud University Nijmegen, the Netherlands, aleks@cs.ru.nl.

We thank Katriel Cohn–Gordon and Mariami Gachechiladze for their MSc thesis work, supervised by Chris Heunen, on parts of this topic [10], [11].

**Definition II.1.** A *compact category* is a symmetric monoidal category $\mathbf{C}$ such that for every object $A$ in $\mathbf{C}$ has a *dual* object $A^*$ and morphisms

$$\varepsilon\colon A \otimes A^* \to I \qquad \eta\colon I \to A^* \otimes A$$

satisfying

$$(\varepsilon \otimes 1) \circ (1 \otimes \eta) = 1_A \qquad 1_{A^*} = (1 \otimes \varepsilon) \circ (\eta \otimes 1). \tag{1}$$

We will use the graphical calculus to reason about morphisms; for more information we refer to the survey [25]. Objects are depicted as wires with upward directed arrows, and their duals as downward wires:


The morphisms $\varepsilon$ and $\eta$ are called *caps* and *cups*, drawn as:

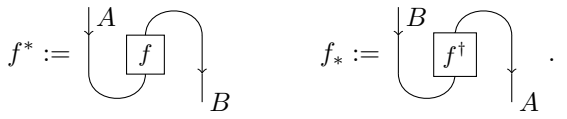

They model entanglement: the cap a completely mixed state, and the cup the accompanying Bell measurement. Equations (1) become:


and embody quantum teleportation [1].

A *dagger category* is a category equipped with a contravariant functor $(-)^\dagger\colon \mathbf{C} \to \mathbf{C}\mathrm{op}$ that satisfies $A^\dagger = A$ on objects and $f^{\dagger\dagger} = f$ on morphisms. An isomorphism in a dagger category is *unitary* if $f^{-1} = f^\dagger$. A category that is both a compact category and a dagger category is a *compact dagger category* when the coherence isomorphisms (associators, unitors, and swap maps) are unitary, and $\varepsilon_A^\dagger = \eta_{A^*}$. Key examples are the category $\mathbf{FHilb}$ of finite-dimensional Hilbert spaces and linear maps and linear maps, and the category $\mathbf{Rel}$ of sets and relations, where the composition of $R \subseteq A \times B$ and $S \subseteq B \times C$ is given by $S \circ R = \{(a, c) \mid \exists b \in B\colon (a, b) \in R, (b, c) \in C\}$.

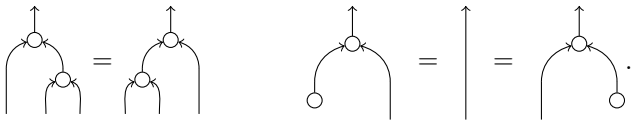
In a compact dagger category, a morphism can take four forms: the morphism itself and its dagger


and its *transpose* and its *conjugate*

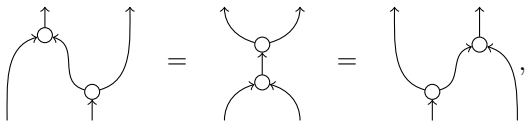



The CP*-construction lets us build a new category whose objects are 'abstract C*-algebras' in the original category and whose morphisms are abstract completely positive maps. The following notion makes the notion of 'abstract C*-algebra' we use more precise.

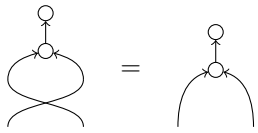
**Definition II.2.** A *monoid* in a compact dagger category is an object $A$ together with a morphism $\curlyvee\colon A \otimes A \to A$ such that there is a morphism $\hat{\bigcirc}\colon I \to A$ satisfying



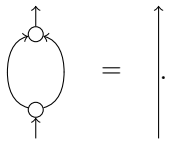

A *dagger Frobenius structure* is a monoid satisfying




where $\curlyvee := (\curlyvee)^\dagger$ and $\curlyvee := (\hat{\bigcirc})^\dagger$. It is *symmetric* when




and *special* when

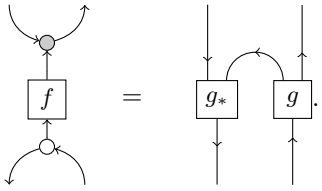



We are now ready to define the CP*-construction. The following definition uses the *action* and *coaction* morphisms of a dagger Frobenius structure:


**Definition II.3.** For a compact dagger category $\mathbf{C}$, the category $\mathrm{CP}^*[\mathbf{C}]$ has as objects special symmetric dagger Frobenius structures $(A, \curlyvee)$, and as morphisms

$$(A, \curlyvee) \to (B, \curlyvee)$$

morphisms $f\colon A \to B$ from $\mathcal{C}$ satisfying the *CP*-condition*: there exists a morphism $g\colon A \to X \otimes B$ in $\mathcal{C}$ satisfying




In other words, the morphisms are precisely those which, once sandwiched between an action and a coaction, satisfy the 'CPM condition' [24]. If $\mathbf{C}$ is a compact dagger category, then so is $\mathrm{CP}^*[\mathbf{C}]$ [8, Theorem 3.3]. However, whereas the cup in $\mathbf{C}$ models entanglement, the cup in $\mathrm{CP}^*[\mathbf{C}]$ models probabilistic mixture.

Our key examples become:

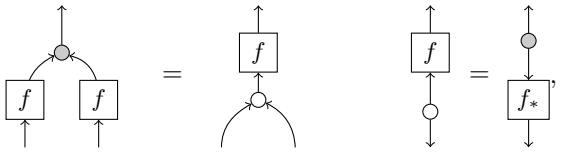
- The category $\mathrm{CP}^*[\mathbf{FHilb}]$ is equivalent to the category of finite-dimensional C*-algebras and completely positive maps [8, Proposition 3.5].

- The category $\mathrm{CP}^*[\mathbf{Rel}]$ is equivalent to the category of groupoids and *inverse-respecting* relations [8, Proposition 5.3]. The latter are relations $R \subseteq G \times H$ between the sets of morphisms of two groupoids satisfying:

$$(g,h) \in R \iff (g^{-1}, h^{-1}) \in R, \tag{2}$$

$$(g,h) \in R \implies (1_{\mathrm{dom}(g)}, 1_{\mathrm{dom}(h)}) \in R. \tag{3}$$

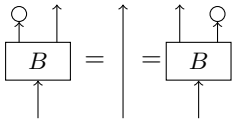
We will use two more facts about the CP*-construction, and for more details refer to [8]. First, there is a functor $\mathcal{B}\colon \mathbf{C} \to \mathrm{CP}^*[\mathbf{C}]$ by [8, Theorem 4.3]. Finally, $*$*-homomorphisms* are always morphisms $(A, \delta) \to (B, \delta)$ in $\mathrm{CP}^*[\mathbf{C}]$ by [8, Lemma 3.8]: these are morphisms $f\colon A \to B$ in $\mathbf{C}$ satisfying




where $\delta := \mu \circ (1 \otimes \eta)\colon A^* \to A$.

## III. Broadcasting

**Definition III.1.** A *broadcasting map* for an object $(A, \delta)$ of $\mathrm{CP}^*[\mathbf{C}]$ is a morphism $B\colon A \to A \otimes A$ in $\mathrm{CP}^*[\mathbf{C}]$ satisfying the following equation.

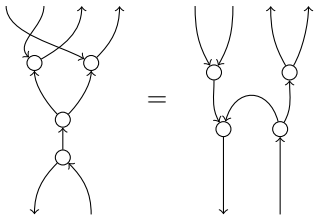



(4)

The object $(A, \delta)$ is called *broadcastable* when there exists a broadcasting map.

**Lemma III.2.** *Let* $\mathbf{C}$ *be a compact dagger category. Commutative dagger Frobenius structures in* $\mathbf{C}$ *are broadcastable objects in* $\mathrm{CP}^*[\mathbf{C}]$.

*Proof.* Suppose that $(A, \delta)$ is commutative. We will show that $\mu$ is a broadcasting map. It clearly satisfies (4), so it suffices to show that it is a well-defined morphism in $\mathrm{CP}^*[\mathbf{V}]$. Using the spider theorem for commutative dagger Frobenius structures [9, Lemma 3.1], we obtain the following:




Therefore $\mu \circ \mu \circ \delta$ is a composition of the identity on $(A, \delta)$ in $\mathrm{CP}^*[\mathbf{V}]$ and the image of $\mu$ under the functor $\mathcal{B}$. Since these are both completely positive, so is their composition. Thus $\mu$ is a well-defined morphism in $\mathrm{CP}^*[\mathbf{V}]$. □

**Lemma III.3.** *Broadcastable objects in* $\mathrm{CP}^*[\mathbf{FHilb}]$ *are precisely commutative finite-dimensional C*-algebras.*

*Proof.* Unpacking Definition III.1 in $\mathbf{C} = \mathbf{FHilb}$, the counit $\varepsilon$ of $\mathcal{B}(H)$ is given by the trace, and so $\mathrm{Tr}_1 = \varepsilon \otimes 1_A\colon A \otimes A \to A$ is precisely the partial trace over the first system. Therefore, a map $B\colon A \to A \otimes A$ is broadcasting precisely when $\mathrm{Tr}_1(B(\rho)) = \rho = \mathrm{Tr}_2(B(\rho))$ for any density matrix $\rho$. In other words, thanks to [8, Proposition 7.5], the previous definition coincides with the standard notion of broadcastability in $\mathbf{FHilb}$. The no-broadcasting theorem, see [3], says that if a broadcasting map for $(A, \delta)$ exists, then $\delta$ must be commutative. Lemma III.2 gives the converse. □

A category is *totally disconnected* when its only morphisms are endomorphisms.

**Lemma III.4.** *Broadcastable objects in* $\mathrm{CP}^*[\mathbf{Rel}]$ *are precisely totally disconnected groupoids.*

*Proof.* Let $\mathbf{G}$ be a totally disconnected (small) groupoid, and write $G$ for its set of morphisms. We will show that the morphism $B\colon G \to G \times G$ in $\mathbf{Rel}$ given by

$$\begin{aligned} B = &\{(f, (1_{\mathrm{dom}(f)}, f)) \mid f \in \mathrm{Mor}(\mathbf{G})\} \\ &\cup \{(f, (f, 1_{\mathrm{dom}(f)})) \mid f \in \mathrm{Mor}(\mathbf{G})\} \end{aligned}$$

is a broadcasting map. First of all, $B$ is readily seen to respect inverses [8, Definition 5.2], so it is a well-defined morphism in $\mathrm{CP}^*[\mathbf{Rel}]$ [8, Proposition 5.3]. When interpreted in $\mathbf{Rel}$, the broadcastability equation (4) reads

$$\begin{aligned} \{(g,g) \mid g \in G\} &= \{(f,g) \mid (f, (1_{\mathrm{cod}(g)}, g)) \in B\} \\ &= \{(f,g) \mid (f, (g, 1_{\mathrm{dom}(g)})) \in B\}. \end{aligned} \tag{$*$}$$

It is satisfied because $\mathbf{G}$ is totally disconnected, so $B$ is a broadcasting map for $\mathbf{G}$.

Conversely, suppose that a small groupoid $\mathbf{G}$ is broadcastable. Then there is a morphism $B$ in $\mathbf{Rel}$ that respects inverses, and satisfies $(*)$. Let $f \in \mathrm{Mor}(\mathbf{G})$. By (2), there is an object $C$ of $\mathbf{G}$ such that $(f, (1_C, f)) \in B$. Next, $(*)$ and (2) give $(1_{\mathrm{dom}(f)}, (1_C, 1_{\mathrm{dom}(f)})) \in B$ and $C = \mathrm{dom}(f)$. But by (3) also $(f^{-1}, (1_C, f^{-1})) \in B$. So, using $(*)$ and (2) again, we also have $(1_{\mathrm{cod}(f)}, (1_C, 1_{\mathrm{cod}(f)}))$ and $C = \mathrm{cod}(f)$. Hence $\mathrm{dom}(f) = \mathrm{cod}(f)$. Thus $\mathbf{G}$ is totally disconnected. □

**Theorem III.5.** *In* $\mathrm{CP}^*[\mathbf{C}]$ *for general* $\mathbf{C}$*:*

*no broadcasting* $\Rightarrow\not\Leftarrow$ *noncommutativity*

*Proof.* The implication is Lemma III.2. The other implication does not hold in $\mathrm{CP}^*[\mathbf{Rel}]$ by Lemma III.4, as not all groupoids are totally disconnected. □

**Remark III.6.** In $\mathbf{C} = \mathbf{FHilb}$, commutativity and broadcastability are equivalent. They also coincide with a third notion of classicality, namely that a C*-algebra is a direct sum of 1-dimensional C*-algebras. This can be phrased for general compact dagger categories $\mathbf{C}$ with biproducts, for in that case $\mathrm{CP}^*[\mathbf{C}]$ inherits biproducts [15]. We can refine Theorem III.5 as follows:

biproduct of unit $\Rightarrow\not\Leftarrow$ commutative $\Rightarrow\not\Leftarrow$ broadcastable

Any abelian group of order 2 or more gives a counterexample showing that being a biproduct of units in $\mathbf{C} = \mathbf{Rel}$ is a coarser notion of classicality.

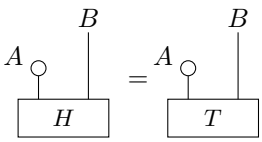
## IV. Bit commitment

Briefly, *bit commitment* is the following two-party protocol. Alice claims to know something, and Bob wants to verify that Alice indeed has that knowledge, but Alice doesn't want to reveal her secret yet. Let's say the information is a single bit; Bob wants Alice to commit to either 'heads' or 'tails' now, and wants to be able to verify her committed value later. Alice could cheat by changing the value she committed to later on; if this is impossible the protocol is *binding*. Bob could cheat by learning the value Alice committed to before she is ready to unveil it; if this is impossible the protocol is *concealing*. A *secure* bit commitment protocol is one where neither cheat is possible. Secure bit commitment is possible clasically, but impossible in the presence of entanglement. We can model it categorically as follows.

**Definition IV.1.** A *bit commitment protocol* on a compact dagger category $\mathbf{C}$ is:

- two states $H, T\colon I \to A \otimes B$ of $\mathrm{CP}^*[\mathbf{C}]$;
- a monomorphism unveil$\colon A \otimes B \to A \otimes B$ in $\mathrm{CP}^*[\mathbf{C}]$;
- a classical structure $(A \otimes B, \overset{\uparrow}{\curlyvee}, \overset{\uparrow}{\bigcirc})$ in $\mathbf{C}$, with copyable states $\widehat{H} \neq \widehat{T}$.

It is *sound* when unveil $\circ\, H = \widehat{H}$ and unveil $\circ\, T = \widehat{T}$. It is *concealing* when:




(5)

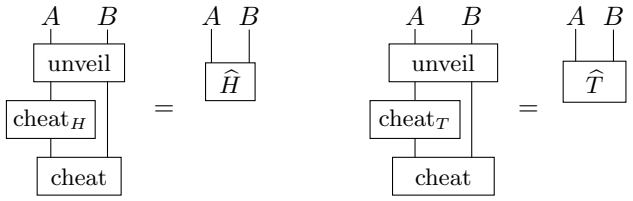
It is *binding* when $(u \otimes 1_B) \circ H \neq T$ for all $u\colon A \to A$ in $\mathrm{CP}^*[\mathbf{C}]$. Finally, it is *secure* when it is sound, concealing, and binding.

We do not consider the possibility of protocols that are *approximately* concealing in this article, to stay as closed as possible to [7]. Equation 5 only prevents Bob from gaining perfect information about the committed value, but says nothing about whether he can guess with high probability. To model such approximation, one might think to use categories enriched in topological spaces, or even metric spaces.

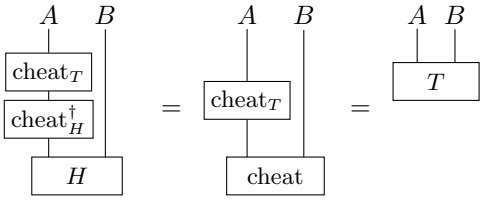
Alice can prepare $H$ or $T$ depending on her bit, giving her two (possibly entangled) quantum systems. By sending the latter to Bob she performs the commitment. Sending the former to Bob later allows him to verify it. Bob can apply unveil and perform a measurement in the specified classical structure; since $\widehat{H}$ and $\widehat{T}$ are copyable states Bob can identify them with certainty. This explains soundness.

The above definition of concealment is straightforward: if Alice discards her half of the system, Bob cannot extract any information that enables him to determine Alice's bit. Note that $\overset{\uparrow}{\bigcirc}$ indeed plays the role of discarding [8, Example 2.2].

Finally, consider the above definition of binding. Intuitively, it says that Alice cannot change her commitment afterwards: there is no state she can prepare that allows her, by means of a local operation, to delay choosing her committed bit until the unveil phase. More precisely, suppose there were a state cheat and isometries $\mathrm{cheat}_H, \mathrm{cheat}_T\colon A \to A$ in $\mathrm{CP}^*[\mathbf{C}]$ satisfying:




The fact that unveil is a monomorphism says that it loses no information; typically it takes the form of a unitary from $\mathbf{C}$. It follows that:




But then $\mathrm{cheat}_T \circ \mathrm{cheat}_H^\dagger$ would be a morphism allowing Alice to change from $H$ to $T$, contradicting the above definition of binding. Conversely, if a protocol is not binding because $(u \otimes 1_B) \circ H = T$ then Alice can cheat using cheat $= H$, $\mathrm{cheat}_H = 1_A$, and $\mathrm{cheat}_T = u$.

We have quantified the morphisms $\mathrm{cheat}_H, \mathrm{cheat}_T\colon A \to A$ over the set $\{H, T\}$ here. We could quantify over a larger index set; this would not make a difference to the permissiveness of the protocol, and so we opted for the minimal version.

**Lemma IV.2.** *Secure bit commitment is impossible in* $\mathrm{CP}^*[\mathbf{FHilb}]$.

*Proof.* In the category $\mathbf{FHilb}$, Definition IV.1 precisely recaptures quantum bit commitment as usually considered, which is not possible securely; see [18], [22], [6]. □

**Lemma IV.3.** *Secure bit commitment is possible in* $\mathrm{CP}^*[\mathbf{Rel}]$.

*Proof.* Recall that the objects of $\mathrm{CP}^*[\mathbf{Rel}]$ of the from $\mathcal{B}(A)$ are the *indiscrete groupoids* on $A$ [8, Proposition 5.4]: those groupoids with objects $A$ that have a unique arrow between any two objects. Take $A$ to be the indiscrete groupoid on $\{0,1,2\}$, and $B$ to be the indiscrete groupoid on $\{x,y\}$. Then $A \otimes B$ in $\mathrm{CP}^*[\mathbf{Rel}]$ is the indiscrete groupoid on $\{0,1,2\} \times \{x,y\}$. Next, recall that the tensor unit $I$ in $\mathrm{CP}^*[\mathbf{Rel}]$ is the indiscrete groupoid on a fixed singleton set, so by (2) and (3) we may identify states with subsets that are closed under inverses and identities:

$$H = \big\{(0,x),(1,y),(2,y)\big\} \subseteq A \otimes B,$$
$$T = \big\{(1,y),(0,x),(2,x)\big\} \subseteq A \otimes B,$$

are well-defined morphisms $I \to A \otimes B$ in $\mathrm{CP}^*[\mathbf{Rel}]$. Observe that $H$ and $T$ are disjoint subsets that partition $\{0,1,2\} \times \{x,y\}$. Let the classical structure $\overset{\uparrow}{\curlyvee}$ correspond to the abelian groupoid with two objects, one with endohomset $H \cong \mathbb{Z}_3$ made into an abelian group, and one with endohomset $T \cong \mathbb{Z}_3$; then $\widehat{H} = H$ and $\widehat{T} = T$ are its (unique) distinct copyable states.

Setting unveil $= 1_{A\otimes B}$ clearly makes this protocol sound. It is also concealing:

$$\begin{aligned}
&(\text{♀} \times 1) \circ H \\
&= \big(\{(0,0),(1,1),(2,2)\} \times 1\big) \circ \{(0,x),(1,y),(2,y)\}^2 \\
&= \{(x,x),(y,y)\} \\
&= \big(\{(0,0),(1,1),(2,2)\} \times 1\big) \circ \{(0,y),(1,x),(2,x)\}^2 \\
&= (\text{♀} \times 1) \circ T.
\end{aligned}$$

Finally, observe that the unitaries in $\mathbf{Rel}$ are (graphs of) bijections, and in particular (graphs of) functions. But

$$\begin{aligned}
&(f \times 1) \circ H \\
&= \big\{\big(f(0,0),(x,x)\big), \big(f(0,1),(x,y)\big), \big(f(0,2),(x,y)\big), \\
&\quad \big(f(1,0),(y,x)\big), \big(f(1,1),(y,y)\big), \big(f(1,2),(y,y)\big), \\
&\quad \big(f(2,0),(y,x)\big), \big(f(2,1),(y,y)\big), \big(f(2,2),(y,y)\big)\big\}
\end{aligned}$$

can never equal

$$\begin{aligned}
T = \big\{&\big((0,0),(y,y)\big), \big((0,1),(y,x)\big), \big((0,2),(y,x)\big), \\
&\big((1,0),(x,y)\big), \big((1,1),(x,x)\big), \big((1,2),(x,x)\big), \\
&\big((2,0),(x,y)\big), \big((2,1),(x,x)\big), \big((2,2),(x,x)\big)\big\}
\end{aligned}$$

for any function $f\colon \{0,1,2\}^2 \to \{0,1,2\}^2$. Thus the protocol is binding. □

Nonlocality means that "spacelike separated systems must at least sometimes occupy entangled states" [7], where 'entangled' means 'not mixed'. There are other operational ways to model non-locality categorically [2], [12], but for our present purposes we stay as close as possible to [7].

**Definition IV.4.** Let $\mathbf{C}$ be a compact dagger category. An object $A$ in $\mathrm{CP}^*[\mathbf{C}]$ *admits entanglement* if there exists another object $B$ and a state $I \to A \otimes B$ that is not of the form $(f \otimes g) \circ \psi$ for $\psi\colon I \to A' \otimes B'$, where $A'$ and $B'$ are broadcastable. We say the category $\mathbf{C}$ is *nonlocal* when every object admits entanglement.

The category $\mathrm{CP}^*[\mathbf{FHilb}]$ is clearly nonlocal, because there exist bipartite density matrices that are not probabilistic mixtures of product states.

**Example IV.5.** The category $\mathrm{CP}^*[\mathbf{Rel}]$ is nonlocal.

*Proof.* Let $\mathbf{A}$ and $\mathbf{B}$ be groupoids. Taking names, we see that the nonexistence of entangled states $I \to \mathbf{A} \times \mathbf{B}$ is equivalent to the following: any morphism $R\colon \mathbf{A} \to \mathbf{B}$ factors through a totally disconnected groupoid. Suppose that $R = T \circ S$ for $S\colon \mathbf{A} \to \mathbf{T}$ and $T\colon \mathbf{T} \to \mathbf{B}$ with $\mathbf{T}$ totally disconnected. Then

$$\begin{aligned}
(a,b) \in R &\iff \exists c \in \mathrm{Mor}(\mathbf{T})\colon (a,c) \in S \wedge (c,b) \in T \\
&\implies \exists c \in \mathrm{Mor}(\mathbf{T})\colon (1_{\mathrm{dom}(a)}, 1_{\mathrm{dom}(c)}) \in S \\
&\qquad\qquad \wedge (1_{\mathrm{cod}(c)}, 1_{\mathrm{cod}(b)}) \in T \\
&\implies \exists c \in \mathrm{Mor}(\mathbf{T})\colon (1_{\mathrm{dom}(a)}, 1_{\mathrm{cod}(c)}) \in S \\
&\qquad\qquad \wedge (1_{\mathrm{cod}(c)}, 1_{\mathrm{cod}(b)}) \in T \\
&\implies (1_{\mathrm{dom}(a)}, 1_{\mathrm{cod}(b)}) \in R.
\end{aligned}$$

Let $A$ be any set, and let $\mathbf{A} = \mathbf{B}$ be the indiscrete groupoid with objects $A$ and precisely one morphism between every two objects. Applying the above to the identity relation $R\colon \mathbf{A} \to \mathbf{B}$ gives $a = b$ for all $a, b \in A$. This is a contradiction, and so no morphism (between nontrivial objects) in $\mathrm{CP}^*[\mathbf{Rel}]$ factors through a totally disconnected groupoid. That is, $\mathrm{CP}^*[\mathbf{Rel}]$ is nonlocal. □

**Theorem IV.6.** *In* $\mathrm{CP}^*[\mathbf{C}]$ *for general* $\mathbf{C}$*:*

$$\textit{no bit commitment} \;\not\Rightarrow\; \not\Leftarrow\; \textit{nonlocality}$$

*Proof.* It follows immediately from Lemma IV.3 and Example IV.5 that $\mathbf{C} = \mathbf{Rel}$ is nonlocal but does support secure bit commitment.

For the converse, consider the category of relations $\mathbf{Rel}(\mathbf{Gp})$ over the regular category of groups. Any classical structure in that compact dagger category can have at most one copyable state [16, Proposition 5.10]. Hence the clause $\widehat{H} \neq \widehat{T}$ of Definition IV.1 cannot be fulfilled. That is, $\mathbf{Rel}(\mathbf{Gp})$ cannot support secure bit commitment. However, the proof of Example IV.5 uses only regular logic and hence holds in $\mathbf{Rel}(\mathbf{Gp})$, too. Therefore, this category is local. □

## V. Signalling

This section focuses on the relationship between signalling and kinematic independence. To be able to model these in categories of the form $\mathrm{CP}^*[\mathbf{C}]$, we first have to introduce the notion of subsystem.

**Definition V.1.** Let $(C,$  $)$ be a dagger Frobenius structure in a monoidal dagger category. A *subsystem* is another dagger Frobenius structure $(A,$  $)$ together with a morphism $f\colon A \to C$ satisfying $f^\dagger \circ f = 1_A$ that is a unital $*$-homomorphism. We call $f$ the *inclusion* of the subsystem, and depict it as  . When  is broadcastable, we speak of a *classical context*.

In general, if $(C,$  $) = (A \otimes B,$  $)$, then $(A,$  $)$ and $(B,$  $)$ are subsystems with inclusions  $= 1_A \otimes$  and  $=$  $\otimes 1_B$. But there can also be subsystems that are not tensor factors. For $\mathbf{C} = \mathbf{FHilb}$, subsystems correspond precisely to C*-subalgebras. The following lemma characterises subsystems for $\mathbf{C} = \mathbf{Rel}$. Recall that a subcategory is *wide* when it encompasses all objects.

**Lemma V.2.** *In* $\mathbf{Rel}$*, a subsystem of a groupoid* $\mathbf{G}$ *is a wide subgroupoid.*

*Proof.* Let a groupoid $\mathbf{H}$ and relation $R \subseteq G \times H$ form a subsystem of $\mathbf{G}$. As any relation, we may regard $R$ as a function $R\colon G \to \mathcal{P}(H)$. Isometry then says that $R(g) \neq \emptyset$, and that $R(g) \cap R(g') = \emptyset$ when $g \neq g'$. That is, $R$ is a multi-valued function. In these terms, $R$ being a unital $*$-homomorphism translates into

$$R(g^{-1}) = R(g)^{-1}, \tag{6}$$

$$R(g \circ g') = R(g) \circ R(g'), \tag{7}$$

$$\bigcup_{x \in \mathbf{G}} R(1_x) = \bigcup_{y \in \mathbf{H}} \{1_y\}. \tag{8}$$

Now if $h, h' \in R(g)$, then

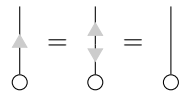
$$\begin{aligned} h^{-1} \circ h' \in R(g)^{-1} \circ R(g) &= R(g^{-1} \circ g) \\ &= R(1_{\mathrm{dom}(g)}) \\ &\subseteq \{1_y \mid y \in \mathbf{H}\}, \end{aligned}$$

so $h = h'$. That is, $R$ is in fact a single-valued function. Furthermore, isometry says that $g \neq g'$ implies $R(g) \neq R(g')$, that is, $R$ is an injective function. Since $*$-homomorphisms are completely positive [8, Lemma 3.8], $R$ satisfies (3), that is, $R(1_{\mathrm{dom}(g)}) = 1_{\mathrm{dom}(R(g))}$. Hence $R$ is a faithful functor that is injective on objects, and we may identify the subsystem with a subgroupoid of $\mathbf{G}$. Finally, because of (8), $R$ is surjective on objects, that is, the subgroupoid is wide. □

In general, inclusions of subsystems are physical maps: by [8, Lemma 3.8] they are completely positive, and the following lemma shows that they are trace-preserving.

**Lemma V.3.** *Subsystem inclusions* $\downarrow : (B, \curlyvee) \to (A, \curlyvee)$ *preserve counits.*

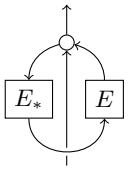
*Proof.* Use that classical contexts are monoid homomorphisms and dagger monic:



Taking daggers now shows that $\downarrow$ preserves counits. □

The other notion that we will need to model signalling and kinematic independence is that of measurement. Here we can no longer be agnostic of the details of the CP*-construction. We can speak about measurements of $\mathcal{B}(H)$ as completely positive maps $\mathbb{C}^n \to \mathcal{B}(H)$ without diving into their structure [8]. But it is completely unclear what local measurements on a subsystem should look like in such general terms, precisely because we may not assume that the subsystem is a simple tensor factor of the system in quantum field theory settings. See also [27].

**Definition V.4.** Let $(A, \curlyvee)$ be a dagger Frobenius structure in a compact dagger category $\mathbf{C}$. A *measurement* on $(A, \curlyvee)$ is a morphism $A \to A$ in $\mathbf{C}$ of the form




(9)

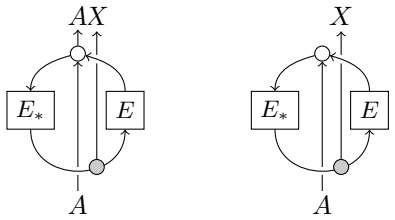
for some object $X$ and some morphism $E\colon X \to A$ in $\mathbf{C}$ satisfying $E^\dagger \circ E = 1_X$.

Observe that, by its form, a measurement is automatically a well-defined morphism $(A, \curlyvee) \to (A, \curlyvee)$ of $\mathrm{CP}^*[\mathbf{C}]$. For $\mathbf{C} = \mathbf{FHilb}$, the previous definition coincides precisely with the usual notion of positive-operator valued measure.

**Example V.5.** In $\mathbf{C} = \mathbf{Rel}$, the measurements on a groupoid $\mathbf{G}$ are 'conjugacy classes': relations of the form $\{(g, g^{-1} \circ f \circ h) \mid g, h \in E_i\}$ for disjoint families $\{E_i\}$ of nonempty subsets of $\mathrm{Mor}(\mathbf{G})$.

*Proof.* For $i \in X$, define $E_i = \{f \in \mathrm{Mor}(\mathbf{G}) \mid (i, f) \in E\}$. Because $E^\dagger \circ E = 1_X$, the subsets $E_i$ are disjoint and nonempty. We also see that $(g, h) \in E \circ E^\dagger$ if and only if $g, h \in E_i$ for some $i \in X$. Now (9) relates morphisms $f$ and $k$ of $\mathbf{G}$ if and only if there exist $g, h \in \mathrm{Mor}(\mathbf{G})$ such that $(g, h) \in E \circ E^\dagger$ and $g \circ k = f \circ h$. But this comes down precisely to $k = g^{-1} \circ f \circ h$ for $g, h \in E_i$ for some $i \in X$. □

If we want to extract classical information during a measurement, we can write a measurement as on the left below, where $(X, \curlyvee)$ is broadcastable. Notice that we are using the weakest notion of classicality examined in Section III. We can even forget the resulting state of the measured system, and get a destructive measurement as on the right below.



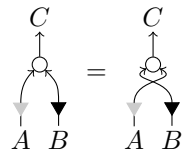

(10)

This destructive form of a measurement should give a result with certainty, because in the category $\mathbf{FHilb}$ it gives an answer with probability one. Remembering that $\curlyvee$ is the trace on $\mathcal{B}(H)$ in $\mathbf{FHilb}$, we can phrase this as follows.

**Definition V.6.** Let $(A, \curlyvee)$ and $(B, \curlyvee)$ be dagger Frobenius structures in a monoidal dagger category. A morphism $f\colon A \to B$ is *causal* when $\curlyvee \circ f = \curlyvee$. A measurement is causal when its destructive form, *i.e.* the right picture in (10) above, is causal.

We can now introduce the two concepts of interest. First, we consider kinematic independence. This concept formalises when systems controlled by Alice and Bob do not influence each other's kinematics. The following definition formalizes this in such a way that for $\mathbf{C} = \mathbf{FHilb}$ it coincides with the usual notion [7].

**Definition V.7.** Let $(C, \curlyvee)$ be a dagger Frobenius structure in a compact dagger category $\mathbf{C}$. We say that two subsystems $(A, \curlyvee)$ and $(B, \curlyvee)$, with inclusions $\downarrow_A^C$ and $\downarrow_B^C$, are *kinematically independent* when the following equation is satisfied.




(11)

Notice that if $(C, \curlyvee)$ decomposes into a tensor product $(A \otimes B, \curlyvee)$, then the subsystems $(A, \curlyvee)$ and $(B, \curlyvee)$ are always kinematically independent. In other words, kinematic independence is a notion that essentially concerns subsystems that are not tensor factors. For example, in $\mathbf{C} = \mathbf{FHilb}$, kinematic independence means that the C*-subalgebras $A$ and $B$ commute.

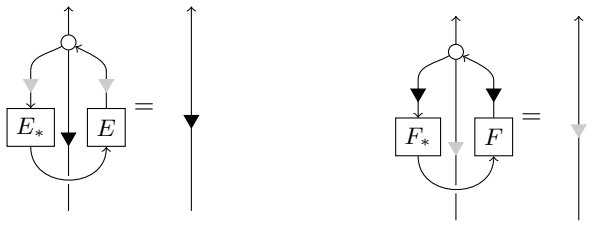
**Example V.8.** Let $\mathbf{G}$ be a groupoid. Wide subgroupoids $\mathbf{A}$ and $\mathbf{B}$ are kinematically independent if and only if they are totally

disconnected and $a \circ b = b \circ a$ for all endomorphisms $a$ in $\mathbf{A}$ and $b$ in $\mathbf{B}$ on the same object. This means that the subgroups $A_x = \mathbf{A}(x,x)$ and $B_x = \mathbf{B}(x,x)$ of $\mathbf{G}(x,x)$ commute for each object $x \in \mathrm{Ob}(\mathbf{G})$, in the sense that they have trivial commutator $\{1_x\} = [A_x, B_x] = \{a^{-1}b^{-1}ab \mid a \in A_x, b \in B_x\}$.

*Proof.* Suppose $\mathbf{A}$ and $\mathbf{B}$ are kinematically independent. If $a\colon x \to y$ is a morphism in $\mathbf{A}$ and $b\colon y \to z$ in $\mathbf{B}$, then $a \circ b = b \circ a$ by plugging the states $\{a\}\colon 1 \to \mathbf{A}$ and $\{b\}\colon 1 \to \mathbf{B}$ into Definition V.7. It follows that $x = y = z$, so $\mathbf{A}$ and $\mathbf{B}$ are totally disconnected. The converse is easy. □

The next notion we consider prohibits superliminal information transfer. It says that when Alice and Bob both control a system, any data that Alice extracts from her system (through measurement) cannot instantaneously influence Bob's system. We formalise this as follows.

**Definition V.9.** Let $(C, \bigcirc)$ be a dagger Frobenius structure in a compact dagger category. Two subsystems $\downarrow : (A, \bigcirc) \to (C, \bigcirc)$ and $\downarrow : (B, \bullet) \to (C, \bigcirc)$ are *no signalling* when



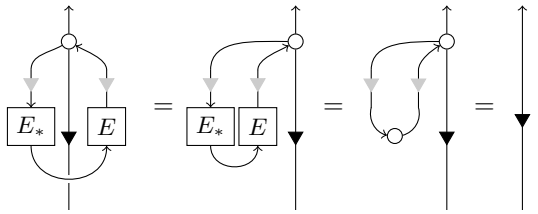

for all causal measurements $E$ on $A$ and $F$ on $B$.

Again, tensor factors are automatically no signalling, making this notion essentially about subsystems that are not tensor factors: if $(C, \bigcirc)$ decomposes into a tensor product $(A \otimes B, \bigcirc\bullet)$, then the subsystems $(A, \bigcirc)$ and $(B, \bullet)$ are always no signalling. For $\mathbf{C} = \mathbf{FHilb}$, our definition of no signalling comes down to the usual one employed in [7].

**Lemma V.10.** *In* $\mathrm{CP}^*[\mathbf{C}]$ *for a compact dagger category* $\mathbf{C}$*:*

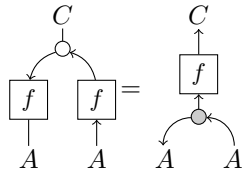
*no signalling* $\Leftarrow$ *kinematic independence*

*Proof.* Use Definition V.7, the causality of Definition V.9, and unitality:

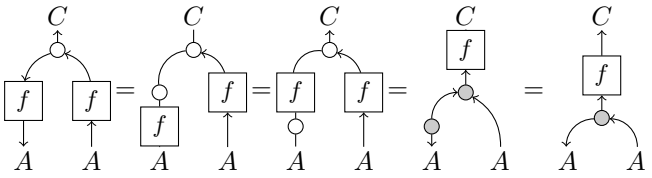



This finishes the proof. □

**Lemma V.11.** *Any* $*$*-homomorphism* $f\colon (A, \bigcirc) \to (C, \bigcirc)$ *satisfies:*




*Proof.* First use that $f$ preserves involution, then that it preserves multiplication:

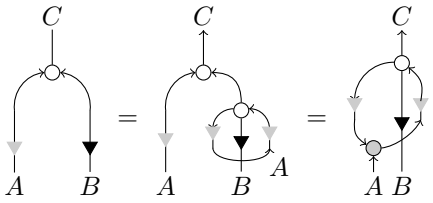



This finishes the proof. □

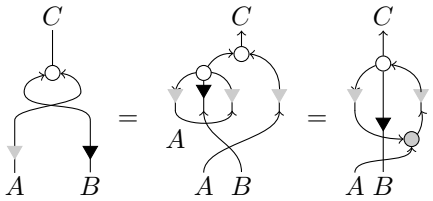
**Theorem V.12.** *In* $\mathrm{CP}^*[\mathbf{C}]$ *for a compact dagger category* $\mathbf{C}$*:*

*no signalling* $\Leftrightarrow$ *kinematic independence*

*Proof.* One direction is Lemma V.10. For the other, unfold the left-hand side of (11) using $E = 1_A$:




These equalities used, respectively, Definition V.9, associativity, and Lemma V.11. Similarly, the right-hand side of (11) becomes:




Hence these two morphisms are equal. □

Finally, we remark that the causality restriction in Definion V.9 is necessary for the previous theorem. The following example together with Example V.8 shows that otherwise, a counterexample would be given by taking $\mathbf{A} = \mathbf{B} = \mathbf{G}$ the discrete groupoid on two elements.

**Example V.13.** For $\mathbf{C} = \mathbf{Rel}$, two wide subgroupoids $\mathbf{A}$ and $\mathbf{B}$ of a groupoid $\mathbf{G}$ are no signalling if and only if $\mathbf{G}$ is a group and $\mathbf{A}$ and $\mathbf{B}$ are the trivial subgroup.

*Proof.* Using Example V.5 to unpack Definition V.9 shows that no signalling means the following: for all disjoint families $\{E_i\}$ of nonempty subsets of $\mathrm{Mor}(\mathbf{A})$ and for all morphisms $b$ in $\mathbf{B}$ and $g$ in $\mathbf{G}$

$$b = g \quad \iff \quad \exists i\, \exists h, k \in E_i \colon g = h^{-1} \circ b \circ k \qquad (12)$$

as well as the symmetric condition in which $\mathbf{A}$ and $\mathbf{B}$ swap roles. Taking $E_i = \{a\}$ and $g = b$, it follows from $\Rightarrow$ that each morphism $b$ in $\mathbf{B}$ must commute with each morphism $a$ in $\mathbf{A}$. In particular, both wide subgroupoids can only have a

single object; hence $\mathbf{G}$ is a group and $\mathbf{A}$ and $\mathbf{B}$ are subgroups. Taking $E_i = \{a, a'\}$, the direction $\Leftarrow$ shows that if $g = a^{-1}a'b$ then $g = b$. That is, $a = a'$, and $\mathbf{A}$ must be the trivial group. The same holds for $\mathbf{B}$. The converse is easy. □

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