COMPACT INVERSE CATEGORIES

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Abstract. The Ehresmann-Schein-Nambooripad theorem gives a structure theorem for inverse monoids: they are inductive groupoids. A particularly nice case due to Jarek is that commutative inverse monoids become semilattices of abelian groups. It has also been categorified by DeWolf-Pronk to a structure theorem for inverse categories as locally complete inductive groupoids. We show that in the case of compact inverse categories, this takes the particularly nice form of a semilattice of compact groupoids. Moreover, one-object compact inverse categories are exactly commutative inverse monoids. Compact groupoids, in turn, are determined in particularly simple terms of 3-cocycles by Baez-Lauda.

1. Introduction

Inverse monoids model partial symmetry [24], and arise naturally in many combinatorial constructions [8]. The easiest example of an inverse monoid is perhaps a group. There is a structure theorem for inverse monoids, due to Ehresmann-Schein-Nambooripad [9, 10, 27, 26], that exhibits them as inductive groupoids. The latter are groupoids internal to the category of partially ordered sets with certain extra requirements. By a result of Jarek [19], the inductive groupoids corresponding to commutative inverse monoids can equivalently be described as semilattices of abelian groups.

A natural typed version of an inverse monoid is an inverse category [22, 6]. This notion can for example model partial reversible functional programs [12]. The easiest example of an inverse category is perhaps a groupoid. DeWolf-Pronk have generalised the ESN theorem to inverse categories, exhibiting them as locally complete inductive groupoids. This paper investigates 'the commutative case', thus fitting in the bottom right cell of Figure 1.

objects	general case	commutative case
one	inductive groupoid [26]	semilattice of abelian groups [19]
many	locally inductive groupoid [7]	semilattice of compact groupoids

Figure 1. Overview of structure theorems for inverse categories.

However, let us emphasise two ways in which Figure 1 is overly simplified. First, the term 'commutative case' is misleading: we mean considering compact inverse categories. More precisely, we prove that compact inverse categories correspond to

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semilattices of compact groupoids. Compact inverse categories are only commutative in that their endohomset of scalars is always commutative. In particular, the categorical composition of the compact inverse category can be as noncommutative as you like. We expect that the tensor product also need not be symmetric. But compact categories are interesting in their own right: they model quantum entanglement [17]; they model linear logic [29]; and they naturally extend traced monoidal categories modelling feedback [20].

Second, our result is not a straightforward special case of DeWolf-Pronk [7], nor of Jarek [19], but instead rather a common categorification. We prove that one-object compact inverse categories are exactly commutative inverse monoids. Semilattices of groupoids are a purely categorical notion, whereas ordered groupoids have more ad hoc aspects. Compact groupoids are also known as 2-groups or crossed modules, and have fairly rigid structure themselves, due to work by Baez and Lauda [5]. We take advantage of this fact to ultimately show that there is a (weak) 2-equivalence of (weak) 2-categories of compact inverse categories, and semilattices of 3-cocycles.

Section 2 starts by recalling the ESN structure theorem for inverse monoids, and its special commutative case due to Jarek in a language that the rest of the paper will follow. Section 3 discusses the generalisation of the ESN theorem to inverse categories due to DeWolf and Pronk, and its relation to semilattices of groupoids. Section 4 is the heart of the paper, and considers additional structure on inverse categories that was hidden for inverse monoids. It shows that the construction works for compact inverse categories, and argues that this is the right generalisation of inverse monoids in this sense. After all this theory, Section 5 lists examples. We have chosen to treat examples after theory; that way they can illustrate not just compact inverse categories, but also the construction of the structure theorem itself. Section 6 then moves to a 2-categorical perspective, to connect to the structure theorem for compact groupoids due to Baez and Lauda. Finally, Section 7 discusses the many questions left open and raised in the paper.

2. Inverse monoids

An *inverse monoid* is a monoid where every element x has a unique element $x^\dagger$ satisfying $x = xx^\dagger x$ and $x^\dagger = x^\dagger x x^\dagger$ [24]. Equivalently, the monoid carries an involution $\dagger$ such that $x = xx^\dagger x$ and $xx^\dagger yy^\dagger = yy^\dagger xx^\dagger$ for all elements x and y. Inverse monoids and involution-respecting homomorphisms form a category **InvMon**, and commutative inverse monoids form a full subcategory **cInvMon**. This section recalls structure theorems for inverse monoids. In general they correspond to inductive groupoids by the Ehresmann-Schein-Nambooripad theorem [9, 10, 26, 27], that we now recall.

Definition 1. A *(bounded meet-)semilattice* is a partially ordered set with a greatest element $\top$, in which any two elements s and t have a greatest lower bound $s \wedge t$. A *morphism of semilattices* is a function f satisfying $f(\top) = \top$ and $f(s \wedge t) = f(s) \wedge f(t)$.

We regard a semilattice as a category by letting elements be objects and having a unique morphism $s \to t$ when $s \leq t$, that is, when $s \wedge t = s$. We will disregard size issues altogether; either by restricting to small categories throughout the article, or by allowing semilattices (and monoids) that are large – the only place it seems to matter is Lemma 23 below. Recall that a *groupoid* is a category whose every morphism is invertible.

Definition 2. An *ordered groupoid* is a groupoid internal to the category of partially ordered sets and monotone functions, together with a choice of *restriction* $(f|A)\colon A \to B$ for each $f\colon A' \to B$ and $A \leq A'$ satisfying $(f|A) \leq f$. Explicitly, the sets G_0 and G_1 of objects and arrows are partially ordered, and the functions

$$\begin{tikzcd} G_0 \arrow[r, "\mathrm{id}"] & G_1 \arrow[l, bend right, "\mathrm{dom}"'] \arrow[l, bend left, "\mathrm{cod}"] \arrow[loop above, "\mathrm{inv}"] & G_2 \arrow[l, "\mathrm{comp}"'] \end{tikzcd}$$

are all monotone, where $G_2 = \{(g,f) \in G_1^2 \mid \mathrm{dom}(g) = \mathrm{cod}(f)\}$ is ordered by $(g,f) \leq (g',f')$ when $g \leq g'$ and $f \leq f'$. An *inductive groupoid* is an ordered groupoid whose partially ordered set of objects forms a semilattice.

A morphism of ordered groupoids is a functor F that is monotone in morphisms, that is, $F(f) \leq F(g)$ when $f \leq g$. Inductive groupoids and their morphisms form a category **IndGpd**.

Theorem 3. *There is an equivalence* **InvMon** $\simeq$ **IndGpd**.

Proof sketch. See [24, Section 4.2] or [7] for details. An inverse monoid M turns into an inductive groupoid as follows. Objects are idempotents $ss^\dagger = s \in M$. Every element of M is a morphism $x\colon x^\dagger x \to xx^\dagger$. The identity on s is s itself, and composition is given by multiplication in M. Inverses are given by $x^{-1} = x^\dagger$. The order $x \leq y$ holds when $x = yx^\dagger x$. The restriction of $x\colon x^\dagger x \to xx^\dagger$ to $s^\dagger s = s \leq x^\dagger x$ is xs. □

Observe from the proof of the previous theorem that commutative inverse monoids correspond to inductive groupoids where every morphism is an endomorphism. Moreover, the endohomsets are abelian groups. Hence commutative inverse monoids correspond to a semilattice of abelian groups.

Definition 4. A *semilattice over* a subcategory $\mathbf{V}$ of $\mathbf{Cat}$ is a functor $F\colon \mathbf{S}^{\mathrm{op}} \to \mathbf{V}$ where $\mathbf{S}$ is a semilattice and all categories $F(s)$ have the same objects. A *morphism of semilattices* $F \to F'$ *over* $\mathbf{V}$ is a morphism of semilattices $\varphi\colon \mathbf{S} \to \mathbf{S}'$ together with a natural transformation $\theta\colon F \Rightarrow F' \circ \varphi$. Write $\mathbf{SLat}[\mathbf{V}]$ for the category of semilattices over $\mathbf{V}$ and their morphisms.

The ordinary category of semilattices can be recovered by choosing $\mathbf{V}$ to be the category containing as its single object the terminal category $\mathbf{1}$. In the commutative case, the ESN theorem simplifies, as worked out by Jarek [19]. The following formulation chooses $\mathbf{V} = \mathbf{Ab}$, regarding an abelian group as a one-object category.

Theorem 5. *If M is a commutative inverse monoid, then*

$$\mathbf{S} = \{s \in M \mid ss^\dagger = s\}, \qquad s \wedge t = st, \qquad \top = 1,$$

is a semilattice, and for each $s \in \mathbf{S}$,

$$F(s) = \{x \in M \mid xx^\dagger = s\}$$

is an abelian group with multiplication inherited from M and unit s, giving a semilattice of abelian groups $F\colon \mathbf{S} \to \mathbf{Ab}$ by $F(s \leq t)(x) \to sx$.

If $F\colon \mathbf{S} \to \mathbf{Ab}$ is a semilattice of abelian groups, then $M = \coprod_{s\in\mathbf{S}} F(s)$ is a commutative inverse monoid under

$$\begin{aligned} xy &= F(s\wedge t \leq s)(x)\cdot F(s\wedge t\leq t)(y) && \text{if } x\in F(s),\ y\in F(t),\\ x^\dagger &= x^{-1}\in F(s) && \text{if } x \in F(s),\\ 1 &= 1 \in F(\top). \end{aligned}$$

This gives an equivalence $\mathbf{cInvMon} \simeq \mathbf{SLat}[\mathbf{Ab}]$.

Proof. First, let M be an inverse monoid. To see that $\mathbf{S}$ is a semilattice, it suffices to show that it is a commutative idempotent monoid. Commutativity is inherited from M, and idempotence follows from the fact that M is an inverse monoid: $(xx^\dagger)^2 = xx^\dagger xx^\dagger = xx^\dagger$. Next we verify that each $F(s)$ is an abelian group. It is closed under multiplication: if $x, y \in F(s)$, then $(xy)(xy)^\dagger = xx^\dagger y^\dagger y = ss^\dagger = s$ so also $xy \in F(s)$. It has s as a unit: if $x \in F(s)$, then $sx = xx^\dagger x = x$. The inverse of $x \in F(s)$ is given by $x^\dagger$, because $xx^\dagger = s$ by definition. Furthermore, the diagram F is functorial: clearly $F(s \leq t) \circ F(r \leq s)(x) = Rx = F(r \leq t)(x)$, and $F(s \leq s)(x) = sx = xx^\dagger x = x$. It is also well-defined: if $s \leq t$ and $x \in F(t)$, then $sx(sx)^\dagger = sxx^\dagger s^\dagger = sts^\dagger = ss^\dagger = s$ so $sx \in F(t)$.

Now let $F \in \mathbf{SLat}[\mathbf{Ab}]$. Then $1 \in F(\top)$ acts as a unit in M: if $x \in F(s)$ then $x1 = F(s \leq s)(x) \cdot F(s \leq \top)(1) = x \cdot 1 = x \in F(s)$. The multiplication is clearly associative and commutative, so M is an abelian monoid. It is an inverse monoid because $xx^\dagger x = xx^{-1}x = x$ is computed within $F(s)$.

Next we move to morphisms. Given a morphism $f\colon M \to M'$ of commutative inverse monoids, define a morphism $F \to F'$ of their associated semilattices of abelian groups as follows: $\varphi\colon \mathbf{S} \to \mathbf{S}'$ is just $\varphi(s) = f(s)$, and $\theta_s\colon F(s) \to F'(f(s))$ is just $\theta_s(x) = f(x)$. This is clearly functorial $\mathbf{cInvMon} \to \mathbf{SLat}[\mathbf{Ab}]$.

Conversely, given a morphism $(\varphi, \theta)\colon F \to F'$ of semilattices of abelian groups, define a homomorphism $M \to M'$ of their associated commutative inverse monoids by $F(s) \ni x \mapsto \theta_s(x) \in F(\varphi(s))$. This is clearly functorial $\mathbf{SLat}[\mathbf{Ab}] \to \mathbf{cInvMon}$.

Finally, turning a commutative inverse monoid M into a semilattice of abelian groups and that in turn into a commutative inverse monoid ends up with the exact same monoid M. A semilattice of abelian groups $F\colon \mathbf{S} \to \mathbf{Ab}$ gets mapped to the inverse monoid $\coprod_s F(s)$, which in turn gets mapped to the following semilattice of abelian groups $G\colon \mathbf{T} \to \mathbf{Ab}$. The semilattice $\mathbf{T}$ is given by $\{t \in F(s) \mid s \in \mathbf{S}, t = tt^\dagger\} = \{t \in F(s) \mid s \in \mathbf{S}, t = tt^{-1} = 1\} = \{1 \in F(s) \mid s \in \mathbf{S}\}$; clearly $s \mapsto 1 \in F(s)$ is an isomorphism $\varphi\colon \mathbf{S} \to \mathbf{T}$. The abelian group $G(\varphi(s))$ is given by $\{x \mid xx^\dagger = s\} = \{x \mid 1 = xx^{-1} = s\} = \{x \in F(s)\}$; clearly $x \mapsto x$ is a natural isomorphism $\theta_s\colon F(s) \to G(\varphi(s))$. Thus $G \simeq F$, and the two functors implement an equivalence. □

3. Inverse categories

This section extends the previous one to a typed setting. A *dagger category* is a category with a contravariant involution † that acts as the identity on objects. A *dagger functor* is a functor between dagger categories satisfying $F(f^\dagger) = F(f)^\dagger$. An *inverse category* is a dagger category where $f = ff^\dagger f$ and $ff^\dagger gg^\dagger = gg^\dagger ff^\dagger$ for any pair of morphisms f and g with the same domain [6]. Equivalently, it is a category where every morphism $f\colon A \to B$ allows a unique morphism $f^\dagger\colon B \to A$ satisfying $f = ff^\dagger f$ and $f^\dagger = f^\dagger ff^\dagger$; thus every functor between inverse categories is in fact a dagger functor. Inverse categories and (dagger) functors form a category **InvCat**, and groupoids and functors form a full subcategory **Gpd**. The ESN theorem extends to inverse categories, as worked out by DeWolf and Pronk [7].

Definition 6. A *locally complete inductive groupoid* is an ordered groupoid with a partition of the semilattice G_0 of objects into semilattices $\{M_i\}$ such that two objects are comparable if and only if they are in the same semilattice M_i. Locally complete inductive groupoids form a subcategory **lcIndGpd** of **IndGpd** of those functors that preserve greatest lower bounds of objects.

Theorem 7. *There is an equivalence* $\mathbf{InvCat} \simeq \mathbf{lcIndGpd}$.

Proof sketch. See [7] for details. An inverse category $\mathbf{C}$ turns into a locally complete inductive groupoid as follows. Objects are idempotents $ff^\dagger$ for some endomorphism $f \colon A \to A$ in $\mathbf{C}$. These partition into the semilattices of idempotents on a fixed object A. Every morphism $f \colon A \to B$ of $\mathbf{C}$ becomes a morphism $f^\dagger f \to ff^\dagger$. The identity on $ff^\dagger$ is $ff^\dagger$ itself, and composition is inherited from $\mathbf{C}$. Inverses are given by $f^{-1} = f^\dagger$. The order $f \leq g$ holds when $f = gf^\dagger f$; clearly two identity morphisms are comparable exactly when they endomorphisms on the same object. The restriction of $f \colon f^\dagger f \to ff^\dagger$ to $s^\dagger s = s \leq f^\dagger f$ is fs. □

Lemma 8. *If* $F \colon \mathbf{S}^{\mathrm{op}} \to \mathbf{Gpd}$ *is a semilattice of groupoids, there is a well-defined inverse category* $\mathbf{C}$ *with the same objects as* $F(\top)$ *and morphisms*

$$\mathbf{C}(A,B) = \coprod_{s \in \mathbf{S}} F(s)\big(A,B\big).$$

If (φ, θ) *is a morphism* $F \to F'$ *of semilattices of groupoids, then there is a dagger functor* $\mathbf{C} \to \mathbf{C}'$ *between their associated categories, given by* $A \mapsto \theta_\top(A)$ *on objects and* $F(s) \ni f \mapsto \theta_s(f) \in F'(\varphi(s))$ *on morphisms. This gives a functor* $\mathbf{SLat}[\mathbf{Gpd}] \to \mathbf{InvCat}$.

Proof. The composition of $f \in F(s)(A,B)$ and $g \in F(t)(A,B)$ is given by $F(s \wedge t \leq t)(g) \circ F(s \wedge t \leq s)(f) \in F(s \wedge t)(A,C)$; this is clearly associative. The identity on A is given by $\mathrm{id}_A \in F(\top)(A,A)$: if $f \in F(s)(A,B)$, then $f \circ \mathrm{id}_A = F(s \wedge \top \leq \top)(\mathrm{id}_A) \circ F(s \wedge \top \leq s)(f) = \mathrm{id} \circ F(s \leq s)(f) = f$. The dagger of $f \in F(s)(A,B)$ is given by $f^{-1} \in F(s)(B,A)$; this clearly is an inverse category. □

Combining Theorem 7 and Lemma 8, we see that a semilattice of groupoids $F \colon \mathbf{S}^{\mathrm{op}} \to \mathbf{Gpd}$ gives rise to a locally complete inductive groupoid $\mathbf{G}$ where:

- objects are $\coprod_{A \in F(\top)} \coprod_{s \in \mathbf{S}} \{f^\dagger f \mid f \in F(s)(A,A)\}$;
- there is an arrow $(f^\dagger f)_{A,s} \to (ff^\dagger)_{B,s}$ for each $f \in F(s)(A,B)$;
- the composition of $f \in F(s)(A,B)$ and $g \in F(t)(B,C)$ is computed as $F(s \wedge t \leq t)(g) \circ F(s \wedge t \leq s)(f)$.

Not every locally complete inductive groupoid comes from a semilattice of groupoids in this way. Instead, locally complete inductive groupoids correspond to certain functors $\mathbf{S}^{\mathrm{op}} \to \mathbf{Gpd}$ where $\mathbf{S}$ may be a disjoint union of several semilattices; a 'multi-semilattice' of groupoids.

Notice that the objects of $\mathbf{G}$ are doubly-indexed: once by an object of the category $F(\top)$, and once by an element of the semilattice $\mathbf{S}$. Locally complete inductive groupoids and semilattices of groupoids have different ways of bookkeeping the same data, each emphasising one of these two indices. In the remainder of the paper, we will prefer to work with semilattices of groupoids rather than the more general locally complete inductive groupoids for two reasons. First, the extra structure we will consider does not require 'multi-semilattices', but instead is uniform enough so semilattices suffice. Second, semilattices of groupoids form a purely categorical concept, whereas ordered groupoids require extra conditions on groupoids internal to the category of partially ordered sets that are somewhat ad hoc. For example, this perspective will later enable us to remove the restriction that all groupoids in a semilattice of groupoids must have the same objects; see Lemma 23 below.

4. Compact inverse categories

There is another way to categorify inverse monoids, that takes advantage of a degree of commutativity. Instead of moving from inverse monoids to inverse categories, in this section we move to *compact inverse categories*. The presence of the tensor product means that the latter specialise to commutative inverse monoids in the one-object case. By a *compact inverse category* we mean an inverse category that is also a compact dagger category under the same dagger [17]. Here, a dagger category is compact when it is symmetric monoidal, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all morphisms f and g, all coherence isomorphisms are inverted by their own daggers, and every object A allows an object A^* and a morphism $\eta_A \colon I \to A^* \otimes A$ satisfying

$$\mathrm{id}_A = \lambda_A \circ (\varepsilon \otimes \mathrm{id}_A) \circ \alpha \circ (\mathrm{id}_A \otimes \eta) \circ \rho_A^{-1} \tag{1}$$

for $\varepsilon = \sigma \circ \eta^\dagger$ where σ is the swap map. Let us first show that compact inverse categories indeed generalise commutative inverse monoids, because the property of compactness is hidden in the one-object case.

Proposition 9. *One-object compact (dagger/inverse) categories are exactly commutative (involutive/inverse) monoids.*

Proof. Let M be a commutative monoid. Regard it as a one-object monoidal category. The one object is the tensor unit, and in any monoidal category, the tensor unit I is its own dual $I^* = I$, since $\eta = \lambda_I^{-1}$ and $\varepsilon = \rho_I$ satisfy (1) by coherence [17, Lemma 3.6]. If the monoid is involutive/inverse, then the category is clearly dagger/inverse.

Conversely, a one-object (dagger) category is clearly an (involutive) monoid. If the category is monoidal, then the monoid is necessarily that of scalars $I \to I$, where tensor and composition coincide and are commutative [1]. □

We now set out to generalise Theorem 5 to compact inverse categories $\mathbf{C}$. They have the right modicum of commutativity to take advantage of Lemma 8: the monoid $\mathbf{C}(I, I)$ of *scalars* is always commutative, any morphism $f \colon A \to B$ can be multiplied with a scalar $s \colon I \to I$ to give $s \bullet f = \lambda \circ (s \otimes f) \circ \lambda^{-1}$, and any endomorphism $f \colon A \to A$ has a *trace* $\mathrm{Tr}(f) = \varepsilon \circ (f \otimes \mathrm{id}_{A^*}) \circ \sigma \circ \eta \colon I \to I$. Furthermore, any morphism $f \colon A \to B$ has a *dual* $f^* = (\mathrm{id}_{A^*} \otimes \varepsilon_B) \circ (\mathrm{id}_{A^*} \otimes f \otimes \mathrm{id}_{B^*}) \circ (\eta_A \otimes \mathrm{id}_{B^*}) \colon B^* \to A^*$, satisfying $\mathrm{Tr}(f^*) = \mathrm{Tr}(f)^*$ when $A = B$. We will write $\mathrm{tr}(f)$ instead of $\mathrm{Tr}(f)^*$. The form of the following lemma resembles the categorical no-cloning theorem [2], and is the heart of the matter.

Lemma 10. *In a compact inverse category, any endomorphism f equals* $\mathrm{tr}(f) \bullet \mathrm{id}$.

Proof. Let $f \colon A \to A$ be an endomorphism. Compactness provides $\eta \colon I \to A^* \otimes A$ and $\varepsilon \colon A \otimes A^* \to I$ satisfying the snake equations. In terms of $g = \varepsilon \otimes \mathrm{id}_A$ and

$h = \mathrm{id}_A \otimes \eta^\dagger = \mathrm{id}_A \otimes (\varepsilon \circ \sigma)$, and suppressing coherence isomorphisms, these equations read $gh^\dagger = \mathrm{id}_A = hg^\dagger$. It follows that

$$hh^\dagger = gh^\dagger hh^\dagger = gh^\dagger = \mathrm{id}_A,$$
$$g^\dagger h = g^\dagger g h^\dagger h = h^\dagger h g^\dagger g = h^\dagger g.$$

Therefore $g = hh^\dagger g = hg^\dagger h = h$, and so

$$f = g \circ (\mathrm{id}_A \otimes f^* \otimes \mathrm{id}_A) \circ h^\dagger = h \circ (\mathrm{id}_A \otimes f^* \otimes \mathrm{id}_A) \circ h^\dagger = \mathrm{Tr}(f^*) \bullet \mathrm{id}_A. \qquad \square$$

Proposition 11. *A compact dagger category is a compact inverse category if and only if every morphism f satisfies $f = \mathrm{tr}(ff^\dagger) \bullet f$.*

Proof. Suppose we're given a compact inverse category. By Lemma 10, the endomorphism $ff^\dagger$ equals $\mathrm{tr}(ff^\dagger ff^\dagger) \bullet \mathrm{id} = \mathrm{tr}(ff^\dagger) \bullet \mathrm{id}$. Hence $f = \mathrm{tr}(ff^\dagger) \bullet f$.

Conversely, suppose given a compact dagger category in which every morphism satisfies $f = \mathrm{tr}(ff^\dagger) \bullet f$. We will prove that this is a *restriction category* with $\bar{f} = \mathrm{tr}(ff^\dagger) \bullet \mathrm{id}$, by verifying the four axioms [6].

First, $f\bar{f} = \mathrm{tr}(ff^\dagger) \bullet f = f$. Second, $\bar{f}\bar{g} = \mathrm{tr}(ff^\dagger) \bullet \mathrm{tr}(gg^\dagger) \bullet \mathrm{id} = \bar{g}\bar{f}$ if $\mathrm{dom}(f) = \mathrm{dom}(g)$. Third,

$$\begin{aligned}
& \mathrm{tr}(ff^\dagger)^\dagger \circ \mathrm{tr}(ff^\dagger) \\
&= (\varepsilon \otimes \varepsilon) \circ (\sigma \otimes \mathrm{id}) \circ (ff^\dagger \otimes \mathrm{id} \otimes ff^\dagger \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma) \circ (\eta \otimes \eta) \\
&= \varepsilon \circ (ff^\dagger ff^\dagger \otimes \mathrm{id}) \circ \sigma \circ \eta \\
&= \varepsilon \circ (ff^\dagger \otimes \mathrm{id}) \circ \sigma \circ \eta \\
(*) \qquad &= \mathrm{tr}(ff^\dagger)
\end{aligned}$$

by Lemma 10. Therefore, for $\mathrm{dom}(f) = \mathrm{dom}(g)$:

$$\begin{aligned}
\overline{g\bar{f}} &= \overline{\mathrm{tr}(ff^\dagger) \bullet g} \\
&= \mathrm{tr}\left[\mathrm{tr}(ff^\dagger)^\dagger \bullet \mathrm{tr}(ff^\dagger) \bullet gg^\dagger\right] \bullet \mathrm{id} \\
&= \mathrm{tr}(ff^\dagger)^\dagger \bullet \mathrm{tr}(ff^\dagger) \bullet \mathrm{tr}(gg^\dagger) \bullet \mathrm{id} \\
&= \mathrm{tr}(ff^\dagger) \bullet \mathrm{tr}(gg^\dagger) \bullet \mathrm{id} \\
&= \bar{g}\bar{f}.
\end{aligned}$$

Fourth, $\bar{g}f = \mathrm{tr}(gg^\dagger) \bullet f = \mathrm{tr}(gg^\dagger) \bullet \mathrm{tr}(ff^\dagger) \bullet f$, and $f\overline{gf} = \mathrm{tr}(gff^\dagger g^\dagger) \bullet f$. The two are equal by a similar computation as $(*)$.

Finally, taking $g = f^\dagger$ shows that $\bar{f} = \mathrm{tr}(ff^\dagger) \bullet \mathrm{id} = gf$ by Lemma 10 and similarly $\bar{g} = fg$. Therefore the category is compact inverse [6, Theorem 2.20]. $\square$

Next we build up to generalise Theorem 5, starting with the replacement for abelian groups. A *compact groupoid* is a compact dagger category where any morphism f is inverted by $f^\dagger$.

Lemma 12. *Compact groupoids are precisely compact inverse categories with invertible scalars.*

Proof. Let $\mathbf{C}$ be a compact inverse category with invertible scalars. By Lemma 10, all endomorphisms are invertible. Let $f\colon A \to B$ be any morphism. Then $ff^\dagger$ is an isomorphism, and so f is (split) monic. Because $f = ff^\dagger f$, it follows that $ff^\dagger = \mathrm{id}_B$. Similarly $f^\dagger f$ is an isomorphism, so f is (split) epic, whence $f^\dagger f = \mathrm{id}_A$. Thus f is invertible. $\square$

We can now show that any compact inverse category is a semilattice of compact groupoids. Write **CptInvCat** for the category of compact inverse categories and (strong) monoidal dagger functors, and **CptGpd** for the full subcategory of compact groupoids and (strong) monoidal functors.

Proposition 13. *If* $\mathbf{C}$ *is a compact inverse category, then*

$$\mathbf{S} = \{s \in \mathbf{C}(I,I) \mid ss^\dagger = s\}, \qquad s \wedge t = st, \qquad \top = \mathrm{id}_I,$$

is a semilattice, and for each $s \in \mathbf{S}$*, there is a compact groupoid* $F(s)$ *with the same objects as* $\mathbf{C}$ *and morphisms*

$$F(s)(A,B) = \{f \in \mathbf{C}(A,B) \mid \mathrm{tr}(ff^\dagger) = s\},$$

giving a semilattice $F \colon \mathbf{S}^{\mathrm{op}} \to \mathbf{CptGpd}$ *of compact groupoids* $F(s \leq t)(f) \mapsto s \bullet f$*.*

The assignment $\mathbf{C} \mapsto F$ *extends to a functor* $\mathbf{CptInvCat} \to \mathbf{SLat}[\mathbf{CptGpd}]$ *by sending a morphism* $G \colon \mathbf{C} \to \mathbf{C}'$ *to*

$$\varphi(s) = \psi_0^{-1} \circ G(s) \circ \psi_0, \qquad \theta_s(A) = G(A), \qquad \theta_s(f) = G(f),$$

where $\psi_0 \colon I' \to G(I)$ *is the structure isomorphism.*

Proof. First, $\mathbf{S}$ is a commutative idempotent monoid by definition.

Next, we verify that $F(s)$ is a compact groupoid. Composition is well-defined: if $f \colon A \to B$ and $g \colon B \to C$ satisfy $\mathrm{tr}(ff^\dagger) = s = \mathrm{tr}(gg^\dagger)$, then by Lemma 10 and linearity and cyclicity of trace:

$$\begin{aligned}
\mathrm{tr}\left((gf)(gf)^\dagger\right) &= \mathrm{tr}(g^\dagger g f f^\dagger) \\
&= \mathrm{tr}\left[(\mathrm{tr}(g^\dagger g) \bullet \mathrm{id}_B) \circ (\mathrm{tr}(ff^\dagger) \bullet \mathrm{id}_B)\right] \\
&= \mathrm{tr}(g^\dagger g) \bullet \mathrm{tr}(ff^\dagger) \bullet \mathrm{tr}(\mathrm{id}_B) \\
&= \mathrm{tr}(ff^\dagger) \bullet \mathrm{tr}(\mathrm{id}_B) \\
&= \mathrm{tr}\left[\mathrm{id}_B \circ (\mathrm{tr}(ff^\dagger) \bullet \mathrm{id}_B)\right] \\
&= \mathrm{tr}(\mathrm{id}_B \circ ff^\dagger) \\
&= \mathrm{tr}(ff^\dagger) \\
&= s.
\end{aligned}$$

It is clear that $s \bullet \mathrm{id}_A$ play the role of identities in $F(s)$. The category $F(s)$ is monoidal, because if $\mathrm{tr}(ff^\dagger) = s = \mathrm{tr}(gg^\dagger)$, then $\mathrm{tr}((f \otimes g)(f \otimes g)^\dagger) = \mathrm{tr}(ff^\dagger \otimes gg^\dagger) = \mathrm{tr}(ff^\dagger)\,\mathrm{tr}(gg^\dagger) = s$. It also inherits the dagger from $\mathbf{C}$: if $\mathrm{tr}(ff^\dagger) = s$, then also $\mathrm{tr}(f^\dagger f) = \mathrm{tr}(ff^\dagger) = s$. Consequently, $F(s)$ inherits the property of being an inverse category from $\mathbf{C}$. Moreover, $F(s)$ is a compact dagger category: the units and counits are given by $s \bullet \eta_A$ and $s \bullet \varepsilon_A$. Finally, scalars $x \in F(s)(I,I)$ are those scalars $x \in \mathbf{C}(I,I)$ satisfying $x^\dagger x = s$, and form an abelian group with inverse $x^\dagger$ and unit s: for $xs = xx^\dagger x = x$; if $x^\dagger x = s = y^\dagger y$ then $(xy)^\dagger(xy) = x^\dagger x y^\dagger y = s^\dagger s = s$; and $xx^\dagger = s$. Lemma 12 therefore makes $F(s)$ a compact groupoid. Notice that F is a well-defined functor: if $s \leq t$ and $\mathrm{tr}(ff^\dagger) = t$, then $st = t$, so $\mathrm{tr}((sf)(sf)^\dagger) = ss^\dagger\,\mathrm{tr}(ff^\dagger) = st = s$.

Now consider morphisms. If $\mathbf{G} \colon \mathbf{C} \to \mathbf{C}'$ is a monoidal dagger functor, say with structure isomorphisms $\psi_0 \colon I' \to G(I)$ and $\psi_{A,B} \colon G(A) \otimes' G(B) \to G(A \otimes B)$, then it is easy to see that φ is a semilattice homomorphism, and that θ_s is a well-defined monoidal dagger functor that is moreover natural in s, because monoidal functors preserve dual objects and hence traces. Finally, it is clear that the assignment $G \mapsto (\varphi, f)$ is functorial. $\square$

Notice that $\mathbf{S}$ contains all *dimension* scalars $\dim(A) = \mathrm{tr}(\mathrm{id}_A)$.

Lemma 14. *If $F \colon \mathbf{S} \to \mathbf{CptGpd}$ is a semilattice of compact groupoids, then the category $\mathbf{C}$ of Lemma 8 is a compact inverse category, and this gives a functor $\mathbf{SLat}[\mathbf{CptGpd}] \to \mathbf{CptInvCat}$.*

Proof. Define the tensor product on objects on $\mathbf{C}$ as in $F(\top)$, and set the tensor unit I in $\mathbf{C}$ to be that of $F(\top)$. The fact that $F(s \leq \top)$ are monoidal functors gives structure isomorphisms $\psi_s \colon A \otimes_s B \to A \otimes B$, where we write $\otimes_s$ for the tensor product in $F(s)$, and $\psi \colon I_s \to I$, where we write I_s for the tensor unit in $F(s)$. Define the tensor product of $f \in F(s)(A,B)$ and $g \in F(t)(C,D)$ to be

$$\psi_{s\wedge t} \circ \big(F(s \wedge t \leq s)(f) \otimes_{s\wedge t} F(s\wedge t \leq t)(g)\big) \circ \psi_{s\wedge t}^{-1}$$

in $F(s \wedge t)\big(A \otimes C, B \otimes D\big)$. Taking coherence isomorphisms and dual objects as in $F(\top)$, a tedious but straightforward calculation proves that the triangle and pentagon axioms are satisfied, that the snake equations are satisfied, and that $\mathbf{C}$ is a compact inverse category.

An even more tedious but still straightforward calculation shows that the functor induced by a morphism of semilattices of compact groupoids is monoidal. $\square$

Theorem 15. *The functors of Proposition 13 and Lemma 14 implement an equivalence $\mathbf{CptInvCat} \simeq \mathbf{SLat}[\mathbf{CptGpd}]$.*

Proof. Starting with a compact inverse category $\mathbf{C}$, turning it into a semilattice of compact groupoids F, and turning that into compact inverse category again, results in the exact same compact inverse category $\mathbf{C}$. For example, the old homset $\mathbf{C}(A,B)$ equals the new homset $\coprod_{s\in\mathbf{C}(I,I) \mid ss^\dagger = s}\{f \in \mathbf{C}(A,B) \mid \mathrm{tr}(ff^\dagger) = s\}$ because any morphism f in $\mathbf{C}$ is of the form $s \bullet f$ for some scalar $ss^\dagger = s = \mathrm{tr}(ff^\dagger)$ by Proposition 11. Similarly, the new tensor product of $f \in F(s)(A,B)$ and $g \in F(t)(C,D)$ is

$$\begin{aligned}
&\psi_{s\wedge t} \circ \big(F(s\wedge t \leq s)(f) \otimes F(s \wedge t \leq s)(g)\big) \circ \psi_{s\wedge t}^{-1} \\
&= \psi_{s\wedge t} \circ (stf \otimes stg) \circ \psi_{s\wedge t}^{-1} \\
&= \psi_{s\wedge t} \circ (st \bullet (f \otimes g)) \circ \psi_{s\wedge t}^{-1} \\
&= (st \bullet (f \otimes g)) \circ \psi_{s\wedge t} \circ \psi_{s\wedge t}^{-1} \\
&= (s \bullet f) \otimes (t \bullet g) \\
&= f \otimes g,
\end{aligned}$$

again by Proposition 11, and because the natural isomorphism ψ cooperates with unitors and hence scalar multiplication, and so equals the old tensor product.

Now start with a semilattice of compact groupoids $F \colon \mathbf{S}^{\mathrm{op}} \to \mathbf{CptGpd}$. Lemma 14 turns it into a compact inverse category $\mathbf{C}$, which in turn becomes the following semilattice of compact groupoids $G \colon \mathbf{T}^{\mathrm{op}} \to \mathbf{CptGpd}$. The semilattice $\mathbf{T}$ is

$$\coprod_{s\in\mathbf{S}}\{t \in F(s)(I,I) \mid tt^\dagger = t\} = \coprod_{s\in\mathbf{S}}\{\mathrm{id}_I \in F(s)(I,I)\}$$

because each $F(s)$ is a groupoid, so $s \mapsto \mathrm{id}_I \in F(s)(I,I)$ is a semilattice isomorphism $\varphi \colon \mathbf{S} \to \mathbf{T}$. The construction of Proposition 13 gives $G(\varphi(s))$ the same objects as $F(\top)$. Morphisms $A \to B$ in $G(\varphi(s))$ are $f\colon A \to B$ in $F(t)(A,B)$ for some $t \in \mathbf{S}$ satisfying $\varphi(s) = \mathrm{tr}(ff^\dagger)$. Because $F(t)$ is a groupoid, s must be t, so $G(\varphi(s))$ and $F(t)$ have the exact same homsets and identities, and we may take θ to be the identity functor. Going through the construction of G shows that θ is in fact a monoidal dagger functor. □

5. EXAMPLES

This section lists examples of compact inverse categories $\mathbf{C}$. For each example we will indicate how Proposition 13 works by writing $\mathbf{C}_0$ for the semilattice $\mathbf{S}$ and $\mathbf{C}_s$ for the compact groupoid $F(s)$.

Example 16 (The fundamental compact groupoid)**.** Any topological space X with a fixed chosen point $x \in X$ gives rise to a compact groupoid $\mathbf{C}$:

- The objects of $\mathbf{C}$ are paths from x_0 to x_0, more precisely, continuous functions $f\colon [0,1] \to X$ with $f(0) = f(1) = x$.
- The arrows $f \to g$ are homotopy classes of paths, more precisely, continuous functions $h\colon [0,1]^2 \to X$ such that $h(s,0) = f(s)$, $h(s,1) = g(s)$, and $h(0,t) = h(1,t) = x_0$, where h and h' are identified when there is a continuous function $H\colon [0,1]^3 \to X$ with $H(s,t,0) = h(s,t)$, $H(s,t,1) = h'(s,t)$, $H(s,0,u) = f(s)$, $H(s,1,u) = g(s)$, and $H(0,t,u) = H(1,t,u) = x_0$.
- The tensor product of objects is composition of paths according to some fixed reparametrisation, the tensor unit is the constant path. Reparametrisation leads to associators and unitors.
- Dual objects are given by reversal of paths.
- The dagger is given by reversal of homotopies.
- The unit η_f is the "birth of a double loop", a homotopy that "grows" from the constant path to the path $f^\dagger \circ f$ by travelling progressively further along f before travelling back along $f^\dagger$.
- The counit ε_f is the "contraction of a double loop", a homotopy that "shrinks" from the path $f^\dagger \circ f$ to the constant path.

In this case $\mathbf{C}_0$ is a one-element semilattice, and $\mathbf{C}_s = \mathbf{C}$ is already a groupoid.

Example 17. Any abelian group $\mathbf{C}$, considered as a discrete monoidal category, is a compact groupoid. In this case $\mathbf{C}_0$ is a one-element semilattice, and $\mathbf{C}_s = \mathbf{C}$ is already a groupoid.

Lemma 18. *If $\mathbf{C}$ is a compact (dagger/inverse) category, and S a family of (dagger) idempotents, then $\mathrm{Split}_S(\mathbf{C})$ is again (dagger/inverse) compact.*

In terms of Theorem 15, $\mathrm{Split}_S(\mathbf{C})_0 \simeq \mathbf{C}_0$, and $\mathrm{Split}_S(\mathbf{C})_s = \mathrm{Split}_{S_s}(\mathbf{C}_s)$, where $S_s = \{p \in S \mid \mathrm{tr}(p) = s\}$.

Proof. Let $p\colon A \to A$ be in S. Define $\eta_p = (p^* \otimes p) \circ \eta_A \colon \mathrm{id}_I \to p \otimes p^*$ and $\varepsilon_p = \varepsilon_A \circ (p \otimes p^*)\colon p^* \otimes p \to \mathrm{id}_I$; these are well-defined morphisms in $\mathrm{Split}_S(\mathbf{C})$. Then indeed the snake equations hold: $p = (\varepsilon_A \otimes p) \circ (p \otimes p^* \otimes p) \circ (p \otimes \eta_A) = (\varepsilon_p \otimes p) \circ (p \otimes \eta_p)$. If $\mathbf{C}$ has a dagger, then so does $\mathrm{Split}_S(\mathbf{C})$, and $\eta_p = (\varepsilon_p \circ \sigma)^\dagger$. □

Example 19. If $\mathbf{C}$ and $\mathbf{D}$ are compact inverse categories, then so is $\mathbf{C} \times \mathbf{D}$. In this case $(\mathbf{C} \times \mathbf{D})_0 \simeq \mathbf{C}_0 \times \mathbf{D}_0$, and $(\mathbf{C} \times \mathbf{D})_{(s,t)} = \mathbf{C}_s \times \mathbf{D}_t$. If $\mathbf{C}$ and $\mathbf{D}$ are compact groupoids, then so is $\mathbf{C} \times \mathbf{D}$.

Example 20. If $\mathbf{C}$ is a compact inverse category, and $\mathbf{G}$ is a groupoid, then $[\mathbf{G},\mathbf{C}]_\dagger$, the category of functors $F\colon \mathbf{G}\to\mathbf{C}$ satisfying $F(f^{-1}) = F(f)^\dagger$ and natural transformations, is again a compact inverse category.

In this case $([\mathbf{G},\mathbf{C}]_\dagger)_0 \simeq \mathbf{C}_0$, and $([\mathbf{G},\mathbf{C}]_\dagger)_s$ has as morphisms natural transformations whose every component is in $\mathbf{C}_s$.

Proof. If $\alpha\colon F \Rightarrow G$ is a natural transformation, its dagger is given by $(\alpha^\dagger)_A = (\alpha_A)^\dagger\colon G(A)\to F(A)$; naturality of $\alpha^\dagger$ follows from naturality of α together with the conditions $F(f)^\dagger = F(f^{-1})$ and $G(f)^\dagger = G(f^{-1})$. This makes $[\mathbf{G},\mathbf{C}]_\dagger$ into a dagger category. It inherits the property $\alpha = \alpha\alpha^\dagger\alpha$ componentwise from $\mathbf{C}$, and is therefore an inverse category.

The tensor product of objects is given by $(F\otimes G)(A) = F(A)\otimes G(A)$, and on morphisms by $(F\otimes G)(f) = F(f)\otimes G(f)$. The tensor unit is the functor that is constantly I. Because the coherence isomorphisms in $\mathbf{C}$ are unitary, this makes $[\mathbf{G},\mathbf{C}]_\dagger$ into a well-defined dagger symmetric monoidal category.

Finally, the dual object of $F\colon \mathbf{G}\to\mathbf{C}$ is given by $F^*(A) = F(A)^*$ and $F^*(f) = F(f)_*$. The unit $\eta_F\colon I \Rightarrow F^*\otimes F$ is given by $(\eta_F)_A = \eta_{F(A)}$, and the counit by $(\varepsilon_F)_A = \varepsilon_{F(A)}$. These are natural because any morphism $f\colon A\to B$ in $\mathbf{G}$ satisfies $ff^\dagger = \mathrm{id}_A$, whence $(F(f)_*\otimes F(f))\circ\eta_{F(A)} = (\mathrm{id}_{B^*}\otimes f)\circ(\mathrm{id}_{B^*}\otimes f^\dagger)\circ\eta_{F(B)} = \eta_{F(B)}$. This makes $[\mathbf{G},\mathbf{C}]_\dagger$ a compact inverse category. □

6. Compact groupoids

This section moves to a 2-categorical perspective, to connect to a characterisation of compact groupoids. A compact groupoid is the same thing as a *coherent 2-group* [5]. It is also known as a *crossed module*. Compact groupoids are classified by two abelian groups G and H and an element of the third cohomology group of G with coefficients in H, as worked out by Baez and Lauda [5]. The following proposition makes this more precise. In the nonsymmetric case, G need not be abelian, and there is an additional action of G on H.

Proposition 21. *A compact groupoid G is, up to equivalence, defined by the following data:*

- *the (abelian) group G of isomorphism classes of objects of $\mathbf{C}$, under $\otimes$, with unit I, and inverse given by dual objects;*
- *the abelian group H of scalars $\mathbf{C}(I,I)$ under composition with unit id_I and inverse $\dagger$;*
- *the conjugation action $G\times H\to H$ that takes (A,s) to $\mathrm{tr}(A\otimes s) = s$;*
- *the 3-cocycle $G\times G\times G\to H$ that takes (A,B,C) to $\mathrm{Tr}(\alpha_{A,B,C})$.*

The above data form the objects of a (weak) 2-category **Cocycle**, with 1- and 2-cells as in [5, Theorem 43].

Proof sketch. See [5, Section 8]. The trick is the following. First, we may assume that $\mathbf{C}$ is skeletal. Then, we may adjust the tensor product such that all unitors and units and counits (but not the associators!) are identities. The pentagon equation ensures that the trace of the associator is in fact a 3-cocycle. □

The proof of Theorem 15 is the only place where we have used that in a semilattice F of categories all $F(s)$ must have the same objects. It was needed because if the functor θ_s is to be an isomorphism, it must give a bijection between the objects of $F(s)$ and $F(\top)$. We now move to a (weak) 2-categorical perspective to remove this restriction.

Definition 22. Redefine the category $\mathbf{SLat}[\mathbf{V}]$ of Definition 4 to become a (weak) 2-category as follows:

- 0-cells are functors $F\colon \mathbf{S}^{\mathrm{op}} \to \mathbf{Cat}$ for some semilattice $\mathbf{S}$;
- 1-cells $F \to F'$ consist of a morphism $\varphi\colon \mathbf{S} \to \mathbf{S}'$ of semilattices and a natural transformation $\theta\colon F \Rightarrow F' \circ \varphi$;
- 2-cells $(\varphi, \theta) \to (\varphi', \theta')$ exist when $\varphi \leq \varphi'$ and then are natural transformations $\gamma\colon \theta \Rrightarrow \theta' \circ (\mathrm{id} * (\varphi \leq \varphi'))$.

Composition is by pasting.

$$\begin{tikzcd}
\mathbf{S}^{\mathrm{op}} \arrow[d, bend right, "\varphi"'] \arrow[d, bend left, "\varphi'"] \arrow[d, phantom, "\leq"] \arrow[rr, bend left, "F"] & & \\
 & \theta \overset{\gamma}{\Rrightarrow} \theta' & \mathbf{V} \\
\mathbf{S}'^{\mathrm{op}} \arrow[rr, bend right, "F'"'] & &
\end{tikzcd}$$

Write $\mathbf{SLat}_{=}[\mathbf{CptGpd}]$ for the full sub-2-category where all categories $F(s)$ have the same objects.

To be precise, in $\mathbf{SLat}[\mathbf{CptGpd}]$, 2-cells γ are modifications: for each $s \in \mathbf{S}$ and $A \in F'(\varphi(s))$, there is a morphism $\gamma_{s,A}\colon \theta_s(A) \to \theta'_s\big(F'\big(\varphi(s) \leq \varphi'(s)\big)(A)\big)$ that is natural in s as well as A.

Lemma 23. *There is a (weak) 2-equivalence* $\mathbf{SLat}[\mathbf{CptGpd}] \simeq \mathbf{SLat}_{=}[\mathbf{CptGpd}]$.

Proof. First, observe that two 0-cells $F, G\colon \mathbf{S}^{\mathrm{op}} \to \mathbf{CptGpd}$ are equivalent in $\mathbf{SLat}[\mathbf{CptGpd}]$ exactly when there is a natural monoidal equivalence $F(s) \simeq G(s)$. Therefore, it suffices to construct, for each F, such a G such that each $G(s)$ has the same objects. Let κ_s be the cardinality of the objects of $F(s)$, and let κ be the maximum of all κ_s. Define $G(s)$ to be equal to $F(s)$, except that we add κ isomorphic copies of the tensor unit I. There is an obvious monoidal structure on $G(s)$, and by construction there is a monoidal equivalence $F(s) \simeq G(s)$, so that $G(s)$ is automatically a compact groupoid. We may furthermore relabel the objects of $G(s)$ to be ordinal numbers, so that all $G(j)$ have the same objects. □

Theorem 24. *There is a (weak) 2-equivalence* $\mathbf{CptInvCat} \simeq \mathbf{SLat}[\mathbf{Cocycle}]$, *where* $\mathbf{CptInvCat}$ *has natural transformations as 2-cells.*

Proof. The (weak) 2-equivalence $\mathbf{CptGpd} \simeq \mathbf{Cocycle}$ of [5, Theorem 43] induces a (weak) 2-equivalence $\mathbf{SLat}[\mathbf{CptGpd}] \simeq \mathbf{SLat}[\mathbf{Cocycle}]$ by postcomposition. Combine this with the equivalence $\mathbf{SLat}_{=}[\mathbf{CptGpd}] \simeq \mathbf{SLat}[\mathbf{CptGpd}]$ of Lemma 23 and the equivalence $\mathbf{CptInvCat} \simeq \mathbf{SLat}_{=}[\mathbf{CptGpd}]$ of Theorem 15; the latter still holds after the change of Definition 22. □

7. Concluding remarks

We conclude by discussing the many questions left open and raised in this paper. First, one could investigate generalising the results in this paper from categories to semicategories. Second, one could investigate generalising the results in this paper from compact categories to monoidal categories where every object has a dual.

7.1. **Traced inverse categories.** Inverse categories provide semantics for reversible programs, but higher-order aspects of reversible programming remain unclear. Compact categories are closed and hence provide semantics for higher-order programming. Theorem 15 shows that compact inverse categories are, in a sense, degenerate. But one of the most interesting aspects of higher-order programming, tail recursion, doesn't need compact categories for semantics, and can already be modeled in traced monoidal categories. (But see also [21].) Now every traced monoidal category can be monoidally embedded in a compact category [20]. One can prove that there exists a left dagger biadjoint to the forgetful functor from dagger compact categories to dagger traced categories. There is also a left adjoint to the forgetful functor from compact inverse categories to compact dagger categories, but the latter is not faithful. Hence there is a left dagger biadjoint to the forgetful functor from compact inverse categories to traced inverse categories, but its unit does not embed any traced inverse category into a compact inverse category. Therefore Theorem 15 does not show that all traced inverse categories degenerate. Indeed, the category **PInj** of sets and injections is the universal inverse category [22], and is also traced [18, 14], but it fails Lemma 10, irrespective of which tensor product it carries, as the swap map on the two element set is not a scalar multiple of the identity. That leaves a valid question: what do traced inverse categories look like?

7.2. **Idempotents.** A *subunit* in a monoidal category $\mathbf{C}$ is a subobject $r\colon R \rightarrowtail I$ for which $r \otimes \mathrm{id}_R$ is invertible [11]; they form a semilattice $\mathrm{ISub}(\mathbf{C})$. The following lemma shows that in compact inverse categories, up to splitting idempotents, the semilattice $\mathbf{C}_0$ is precisely that of subunits. See also [28] for structure theorems of inverse categories in which all idempotents split.

Lemma 25. *Let $\mathbf{C}$ be a compact inverse category.*

(a) *A map $r\colon R \to I$ is a subunit if and only if $r^\dagger r = \mathrm{id}$.*
(b) *Any subunit r induces an element $rr^\dagger$ of $\mathbf{C}_0$.*
(c) *If idempotents split in $\mathbf{C}$, any element $\mathbf{C}_0$ is $rr^\dagger$ for a unique subunit r; this gives a isomorphism between the semilattices $\mathbf{C}_0$ and $\mathrm{ISub}(\mathbf{C})$.*

Proof. For (a), first notice that if $r\colon R \to I$ is monic, then because $r = rr^\dagger r$ in fact r is an isometry, that is, $r^\dagger r = \mathrm{id}$. We will show that for isometries r, the condition that $r \otimes \mathrm{id}_R$ is invertible holds automatically, with the inverse being $r^\dagger \otimes \mathrm{id}_R$. It suffices to show that $(r \otimes \mathrm{id}_R)(r^\dagger \otimes \mathrm{id}_R) = \mathrm{id}_{I\otimes R}$. But

$$\mathrm{id}_{I\otimes R} = \mathrm{id}_I \otimes (r^\dagger r) = \mathrm{id}_I(r^\dagger(rr^\dagger)r) = (rr^\dagger) \otimes (r^\dagger r) = (rr^\dagger) \otimes \mathrm{id}_R.$$

Thus the subunits are precisely the (subobjects represented by) isometries.

Part (b) is obvious: if r is an isometry, then $s = rr^\dagger\colon I \to I$ satisfies $s = ss^\dagger$.

Part (c) follows from [6, Lemma 2.25], as does the fact that the maps of (b) and (c) are each other's inverses. It is easy to see that both maps preserve the order structure using [11, Proposition 2.8]. □

Now there are two ways to 'localise' $\mathbf{C}$ to $r \in \mathrm{ISub}(\mathbf{C})$. The localisation $\mathbf{C}\big|_r$ according to [11] has objects A such that $r \otimes \mathrm{id}_A$ is invertible, and all morphisms between those objects. The localisation $\mathbf{C}_{rr^\dagger}$ above has all objects, but only those morphisms f satisfying $\mathrm{tr}(ff^\dagger) = rr^\dagger$. These two localisations are different. The former localises with respect to the tensor product, whereas the latter localises with respect to composition.

Generally, taking semilattices of categories is a completion procedure. Does it generalise to (weak) 2-categories? If so, the above may be the special cases of a single object and of unique 2-cells, and could form a higher-categorical analogue of the Eckmann-Hilton argument in the Baez-Dolan stabilisation hypothesis [4]. Is there a relationship with [15]?

7.3. **Internal descriptions.** Groupoids are precisely special dagger Frobenius algebras in the category **Rel** of sets and relations [16]. Compact groupoids are precisely special dagger Frobenius algebras in the category **Rel**(**Gp**) of relations over the regular category of groups, see [13]. Can inverse categories similarly be described as certain monoids in a category of relations?

7.4. **Bratteli diagrams and C*-algebras.** Describing compact inverse categories through a diagram of groupoids resembles describing an AF C*-algebra as a diagram of finite-dimensional C*-algebras [3]. It is very fruitful to work with this so-called Bratteli diagram directly rather than with the C*-algebra itself. More generally, inverse semigroups are a popular way to generate C*-algebras [8], as it is easier to work with the inverse semigroup directly, and moreover this captures many important classes of C*-algebras (see *e.g.* [30]): AF C*-algebras, graph C*-algebras, tiling C*-algebras, self-similar group C*-algebras, subshift C*-algebras, C*-algebras of ample étale groupoids, and C*-algebras of Boolean dynamical systems. There is also a multiply-typed version building a C*-algebra from a so-called higher rank graph [23]. Can one similarly generate a C*-algebra from a compact inverse category, and is there a relationship to these other constructions? A first step might be to extend [25] to possibly infinite categories by adding a norm.

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