Piecewise Boolean algebra

Chris Heunen
Boolean algebra: example
Boolean algebra: definition

A Boolean algebra is a set $B$ with:

- a distinguished element $1 \in B$;
- a unary operation $\neg: B \to B$;
- a binary operation $\land: B \times B \to B$;

such that for all $x, y, z \in B$:

- $x \land (y \land z) = (x \land y) \land z$;
- $x \land y = y \land x$;
- $x \land 1 = x$;
- $x \land \neg x = 0$ (0 = $\neg 1$ is the least element);

The above conditions define the operations of the Boolean algebra.

“Sets of independent postulates for the algebra of logic”
Transactions of the American Mathematical Society 5:288–309, 1904
Boolean algebra: definition

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such that for all $x, y, z \in B$:

- $x \land (y \land z) = (x \land y) \land z$;
- $x \land y = y \land x$;
- $x \land 1 = x$;
- $x \land x = x$;
- $x \land \neg x = \neg 1 = \neg 1 \land x$;  
  ($\neg x$ is a complement of $x$)
- $x \land \neg y = \neg 1 \iff x \land y = x$  
  ($0 = \neg 1$ is the least element)

“Sets of independent postulates for the algebra of logic”
Transactions of the American Mathematical Society 5:288–309, 1904
Boole’s algebra
Boolean algebra ≠ Boole’s algebra
Boolean algebra ≠ Boole’s algebra

1. The previous chapters of this work have been devoted to the investigation of the fundamental laws of the operations of the mind in reasoning; of their development in the laws of the symbols of Logic; and of the principles of expression, by which that species of propositions called primary may be represented in the language of symbols. These inquiries have been in the strictest sense preliminary. They form an indispensable introduction to one of the chief objects of this treatise—the construction of a system or method of Logic upon the basis of an exact summary of the fundamental laws of thought. There are certain considerations touching the nature of this end, and the means of its attainment, to which I deem it necessary here to direct attention.

2. I would remark in the first place that the generality of a method in Logic must very much depend upon the generality of its elementary processes and laws. We have, for instance, in the previous sections of this work investigated, among other things, the laws of that logical process of addition which is symbolized by the sign +. Now those laws have been determined from the study of instances, in all of which it has been a necessary condition, that the classes or things added together in thought should be mutually exclusive. The expression \( x + y \) seems indeed uninterpretable, unless it be assumed that the things represented by \( x \) and the things represented by \( y \) are entirely separate; that they embrace no individuals in common. And conditions analogous to this have been involved in those acts of conception from the study of which the laws of the other symbolical operations have been ascertained. The question then arises, whether
Boolean algebra = Jevon's algebra

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PURE LOGIC

OR THE

LOGIC OF QUALITY APART FROM QUANTITY:

WITH

REMARKS ON BOOLE'S SYSTEM AND

ON THE RELATION OF LOGIC AND MATHEMATICS.

BY

W. STANLEY JEVONS, M.A.

Logica est ars artium et scientiarum. — SCOTUS.

LONDON:
EDWARD STANFORD, 6 CHARING CROSS.
1864.
Boolean algebra = Jevon’s algebra

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Boole’s algebra isn’t Boolean algebra

Boole’s Algebra Isn’t Boolean Algebra

A description, using modern algebra, of what Boole really did create.

THEODORE HAILPERIN
Lehigh University
Bethlehem, PA 18015

To Boole and his mid-nineteenth century contemporaries, the title of this article would have been very puzzling. For Boole’s first work in logic, The Mathematical Analysis of Logic, appeared in 1847 and, although the beginnings of modern abstract algebra can be traced back to the early part of the nineteenth century, the subject had not fully emerged until towards the end of the century. Only then could one clearly distinguish and compare algebras. (We use the term algebra here as standing for a formal system, not a structure which realizes, or is a model for, it—for instance, the algebra of integral domains as codified by a set of axioms versus a particular structure, e.g., the integers, which satisfies these axioms.) Granted, however, that this later full degree of understanding has been attained, and that one can conceptually distinguish algebras, is it not true that Boole’s “algebra of logic” is Boolean algebra?
Piecewise Boolean algebra: definition

A piecewise Boolean algebra is a set $B$ with:

- a reflexive symmetric binary relation $\odot \subseteq B^2$;
- a (partial) binary operation $\wedge : \odot \rightarrow B$;
- a (total) function $\neg : B \rightarrow B$;
- an element $1 \in B$ with $\{1\} \times B \subseteq \odot$;

such that every $S \subseteq B$ with $S^2 \subseteq \odot$ is contained in a $T \subseteq B$ with $T^2 \subseteq \odot$ where $(T, \wedge, \neg, 1)$ is a Boolean algebra.
Piecewise Boolean algebra: example
Piecewise Boolean algebra $\preceq$ quantum logic

Subsets of a set
Subspaces of a Hilbert space

“The logic of quantum mechanics”
Piecewise Boolean algebra $\preceq$ quantum logic

Subsets of a set
Subspaces of a Hilbert space
An orthomodular lattice is:

- A partial order set $(B, \leq)$ with min 0 and max 1
- that has greatest lower bounds $x \wedge y$;
- an operation $\perp : B \rightarrow B$ such that
- $x^{\perp\perp} = x$, and $x \leq y$ implies $y^\perp \leq x^\perp$;
- $x \lor x^\perp = 1$;
- if $x \leq y$ then $y = x \lor (y \land x^\perp)$

“The logic of quantum mechanics”
Piecewise Boolean algebra \( \leq \) quantum logic

Subsets of a set
Subspaces of a Hilbert space
**An orthomodular lattice is not distributive:**

\[
(\text{tea} \text{ or } \text{coffee}) \text{ and } (\text{cookies}) \neq (\text{tea} \text{ and } \text{cookies}) \text{ or } (\text{coffee} \text{ and } \text{cookies})
\]

“The logic of quantum mechanics”
Piecewise Boolean algebra $\leq$ quantum logic

Subsets of a set
Subspaces of a Hilbert space

“The logic of quantum mechanics”
Piecewise Boolean algebra ⊆ quantum logic

Subsets of a set
Subspaces of a Hilbert space

However: fine when within orthogonal basis (Boolean subalgebra)

“The logic of quantum mechanics”
Boole’s algebra $\neq$ Boolean algebra

Quantum measurement is probabilistic
(state $\alpha|0\rangle + \beta|1\rangle$ gives outcome 0 with probability $|\alpha|^2$)
Boole’s algebra $\neq$ Boolean algebra

Quantum measurement is probabilistic
(state $\alpha|0\rangle + \beta|1\rangle$ gives outcome 0 with probability $|\alpha|^2$)

A hidden variable for a state is an assignment of a consistent outcome to any possible measurement
(homomorphism of piecewise Boolean algebras to $\{0, 1\}$)
Boole’s algebra ≠ Boolean algebra

Quantum measurement is probabilistic
(state $\alpha|0\rangle + \beta|1\rangle$ gives outcome 0 with probability $|\alpha|^2$)

A hidden variable for a state is an assignment of a consistent outcome to any possible measurement
(homomorphism of piecewise Boolean algebras to $\{0, 1\}$)

Theorem: hidden variables cannot exist
(if dimension $n \geq 3$, there is no homomorphism $\text{Sub}(\mathbb{C}^n) \rightarrow \{0, 1\}$ of piecewise Boolean algebras.)

“The problem of hidden variables in quantum mechanics”
Given a piecewise Boolean algebra $B$, its piecewise Boolean domain $\text{Sub}(B)$ is the collection of its Boolean subalgebras, partially ordered by inclusion.
Piecewise Boolean domains: example

Example: if $B$ is

then $\text{Sub}(B)$ is
Piecewise Boolean domains: theorems

Can reconstruct \( B \) from \( \text{Sub}(B) \)
\((B \cong \text{colim} \text{Sub}(B))\)
(the parts determine the whole)

“Noncommutativity as a colimit”
Piecewise Boolean domains: theorems

Can reconstruct $B$ from $\text{Sub}(B)$
($B \cong \text{colim Sub}(B)$)
(the parts determine the whole)

$\text{Sub}(B)$ determines $B$
($B \cong B' \iff \text{Sub}(B) \cong \text{Sub}(B')$)
(shape of parts determines whole)

“Noncommutativity as a colimit”

“Subalgebras of orthomodular lattices”
Order 28:549–563, 2011
Piecewise Boolean domains: as complex as graphs

State space = Hilbert space
Sharp measurements = subspaces (projections)
Jointly measurable = overlapping or orthogonal (commute)
Piecewise Boolean domains: as complex as graphs

State space = Hilbert space
Sharp measurements = subspaces (projections)
Jointly measurable = overlapping or orthogonal (commute)

(In)compatibilities form graph:

\[ p \rightarrow r \rightarrow s \rightarrow q \rightarrow t \]
Piecewise Boolean domains: as complex as graphs

State space = Hilbert space
Sharp measurements = subspaces (projections)
Jointly measurable = overlapping or orthogonal (commute)

(In)compatibilities form graph:

\[ \begin{array}{c}
\text{r} \\
\text{s} \\
\text{p} \\
\text{q} \\
\text{t} \\
\end{array} \]

**Theorem**: Any graph can be realised as sharp measurements on some Hilbert space.

“Quantum theory realises all joint measurability graphs”
Piecewise Boolean domains: as complex as graphs

State space = Hilbert space
Sharp measurements = subspaces (projections)
Jointly measurable = overlapping or orthogonal (commute)

(In)compatibilities form graph:

```
   r
  / \
 /    \ 
 p     q
  \
   \ 
    s
  / \
 q   t
```

**Theorem:** Any graph can be realised as sharp measurements on some Hilbert space.

**Corollary:** Any piecewise Boolean algebra can be realised on some Hilbert space.

“Quantum theory realises all joint measurability graphs”

“Quantum probability – quantum logic”
Springer Lecture Notes in Physics 321, 1989
Piecewise Boolean domains: as complex as hypergraphs

State space = Hilbert space

Unsharp measurements = positive operator-valued measures

Jointly measurable = marginals of larger POVM
Piecewise Boolean domains: as complex as hypergraphs

State space = Hilbert space

*Unsharp* measurements = positive operator-valued measures
Jjointly measurable = marginals of larger POVM

(In)compatibilities now form hypergraph:

\[
\begin{array}{c}
p \\
\downarrow \\
q \\
\rightarrow \\
s \\
\rightarrow \\
t \\
\end{array}
\]
Piecewise Boolean domains: as complex as hypergraphs

State space = Hilbert space

Unsharp measurements = positive operator-valued measures

Jointly measurable = marginals of larger POVM

(In)compatibilities now form abstract simplicial complex:

\[
\begin{array}{c}
p \\
q \\
t
\end{array}
\]

\[
\begin{array}{c}
r \quad s
\end{array}
\]

Theorem: Any abstract simplicial complex can be realised as POVMs on a Hilbert space.

Corollary: Any interval effect algebra can be realised on some Hilbert space.

"All joint measurability structures are quantum realizable"


"Hilbert space effect-representations of effect algebras"

Piecewise Boolean domains: as complex as hypergraphs

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Unsharp measurements = positive operator-valued measures
Jointly measurable = marginals of larger POVM

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“All joint measurability structures are quantum realizable”
Piecewise Boolean domains: as complex as hypergraphs

State space = Hilbert space

*Unsharp* measurements = positive operator-valued measures

Jointly measurable = marginals of larger POVM

(In)compatibilities now form **abstract simplicial complex**:

![Diagram of abstract simplicial complex](image)

**Theorem**: Any abstract simplicial complex can be realised as POVMs on a Hilbert space.

**Corollary**: Any interval effect algebra can be realised on some Hilbert space.

---

“All joint measurability structures are quantum realizable”

“Hilbert space effect-representations of effect algebras”
Piecewise Boolean domains: partition lattices

What does Sub($B$) look like when $B$ is an honest Boolean algebra?
Piecewise Boolean domains: partition lattices

What does $\text{Sub}(B)$ look like when $B$ is an honest Boolean algebra? Boolean algebras are dually equivalent to Stone spaces
Piecewise Boolean domains: partition lattices

What does \( \text{Sub}(B) \) look like when \( B \) is an honest Boolean algebra?

Boolean algebras are dually equivalent to Stone spaces

\( \text{Sub}(B) \) becomes a partition lattice

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“The theory of representations of Boolean algebras”
Transactions of the American Mathematical Society 40:37–111, 1936

“On the lattice of subalgebras of a Boolean algebra”
Piecewise Boolean domains: partition lattices

What does \( \text{Sub}(B) \) look like when \( B \) is an honest Boolean algebra? Boolean algebras are dually equivalent to Stone spaces \( \text{Sub}(B) \) becomes a partition lattice

Idea: every downset in \( \text{Sub}(B) \) is a partition lattice (upside-down)!

“The theory of representations of Boolean algebras”
Transactions of the American Mathematical Society 40:37–111, 1936

“On the lattice of subalgebras of a Boolean algebra”
**Lemma:** Piecewise Boolean domain $D$ gives functor $F: D \to \text{Bool}$ that preserves subobjects; “$F$ is a piecewise Boolean diagram”.

$(\text{Sub}(F(x)) \cong \downarrow x$, and $B = \text{colim } F)$
Piecewise Boolean domains: characterisation

**Lemma:** Piecewise Boolean domain $D$ gives functor $F: D \to \text{Bool}$ that preserves subobjects; “$F$ is a piecewise Boolean diagram”.

$(\text{Sub}(F(x)) \cong \downarrow x$, and $B = \text{colim} F)$

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"Piecewise Boolean algebras and their domains"
Piecewise Boolean domains: characterisation

**Lemma**: Piecewise Boolean domain $D$ gives functor $F: D \rightarrow \text{Bool}$ that preserves subobjects; “$F$ is a piecewise Boolean diagram”. 
(Sub($F(x)$) $\cong \downarrow x$, and $B = \text{colim } F$)

**Theorem**: A partial order is a piecewise Boolean domain iff:

- it has directed suprema;
- it has nonempty infima;
- each element is a supremum of compact ones;
- each downset is cogeometric with a modular atom;
- each element of height $n \leq 3$ covers $(n+1)^2$ elements.
- a set of atoms has a sup iff each finite subset does

“Piecewise Boolean algebras and their domains”
Orthoalgebras

This is almost a piecewise Boolean domain $D$:

That is of the form $D = \text{Sub}(B)$ for this $B$:

But $B$ is not a piecewise Boolean algebra: $\{a, c, e\}$ not in one block
Piecewise Boolean domains: higher order

Scott topology turns directed suprema into topological convergence (closed sets = downsets closed under directed suprema)
Lawson topology refines it from dcpos to continuous lattices (basic open sets = Scott open minus upset of finite set)
Piecewise Boolean domains: higher order

Scott topology turns directed suprema into topological convergence (closed sets = downsets closed under directed suprema)
Lawson topology refines it from dcpos to continuous lattices (basic open sets = Scott open minus upset of finite set)

If $B_0$ is piecewise Boolean algebra, Sub($B_0$) is algebraic dcpo and complete semilattice,
Piecewise Boolean domains: higher order

Scott topology turns directed suprema into topological convergence (closed sets = downsets closed under directed suprema)
Lawson topology refines it from dcpo to continuous lattices (basic open sets = Scott open minus upset of finite set)

If $B_0$ is piecewise Boolean algebra, $\text{Sub}(B_0)$ is algebraic dcpo and complete semilattice, hence a Stone space under Lawson topology!

“Continuous lattices and domains”
Cambridge University Press, 2003
Piecewise Boolean domains: higher order

Scott topology turns directed suprema into topological convergence (closed sets = downsets closed under directed suprema)  
Lawson topology refines it from dcpoś to continuous lattices (basic open sets = Scott open minus upset of finite set)

If $B_0$ is piecewise Boolean algebra, $\text{Sub}(B_0)$ is algebraic dcpo and complete semilattice, hence a Stone space under Lawson topology!

It then gives rise to a new Boolean algebra $B_1$. Repeat: $B_2, B_3, \ldots$ (Can handle domains of Boolean algebras with Boolean algebra!)

“Continuous lattices and domains”  
Cambridge University Press, 2003

“Domains of commutative C*-subalgebras”  
Piecewise Boolean diagrams: topos

- Consider “contextual sets” over piecewise Boolean algebra $B$
  assignment of set $S(C)$ to each $C \in \text{Sub}(B)$
  such that $C \subseteq D$ implies $S(C) \subseteq S(D)$
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They form a topos $\mathcal{T}(B)$!
category whose objects behave a lot like sets
in particular, it has a logic of its own!
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There is one canonical contextual set $B$
$B(C) = C$
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They form a topos $\mathcal{T}(B)$!
category whose objects behave a lot like sets
in particular, it has a logic of its own!

There is one canonical contextual set $B$
$B(C) = C$

$\mathcal{T}(B)$ believes that $B$ is an honest Boolean algebra!
Operator algebra

C*-algebras: main examples of piecewise Boolean algebras.
Operator algebra

*-algebras: main examples of piecewise Boolean algebras.
Operator algebra

-\( \star \)-algebras: main examples of piecewise Boolean algebras.

Example: \( C(X) = \{ f : X \to \mathbb{C} \text{ continuous} \} \)

Theorem: Every commutative \( \star \)-algebra is of this form.

“Normierte Ringe”
Matematicheskii Sbornik 9(51):3–24, 1941
Operator algebra

$\star$-algebras: main examples of piecewise Boolean algebras.

Example: $C(X) = \{ f : X \to \mathbb{C} \text{ continuous} \}$

**Theorem:** Every commutative $\star$-algebra is of this form.

Example: $B(H) = \{ f : H \to H \text{ continuous linear} \}$

**Theorem:** Every $\star$-algebra embeds into one of this form.

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“Normierte Ringe”
Matematicheskii Sbornik 9(51):3–24, 1941

“On the imbedding of normed rings into operators on a Hilbert space”
Matematicheskii Sbornik 12(2):197–217, 1943
Operator algebra

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piecewise Boolean algebras \( \overset{\longrightarrow}{\text{ \( \star \)-algebras}} \)

projections

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\[ \text{piecewise Boolean algebras} \quad \text{\( \perp \) \quad \text{\( \star \)-algebras}} \]

\[ \text{projections} \]

“Normierte Ringe”
Matematicheskii Sbornik 9(51):3–24, 1941

“On the imbedding of normed rings into operators on a Hilbert space”
Matematicheskii Sbornik 12(2):197–217, 1943

“Active lattices determine AW*-algebras”
Operator algebra: same trick

A (piecewise) \(\star\)-algebra \(A\) gives a dcpo \(\text{Sub}(A)\).
Operator algebra: same trick

A (piecewise) $*$-algebra $A$ gives a dcpo $\text{Sub}(A)$.

Can characterize partial orders $\text{Sub}(A)$ arising this way. Involves action of unitary group $U(A)$.

“Characterizations of categories of commutative $C^*$-subalgebras”
Communications in Mathematical Physics 331(1):215–238, 2014
Operator algebra: same trick

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If $\text{Sub}(A) \cong \text{Sub}(B)$, then $A \cong B$ as Jordan algebras. Except $\mathbb{C}^2$ and $\mathbb{M}_2$.

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“Isomorphisms of ordered structures of abelian C*-subalgebras of C*-algebras”
Operator algebra: same trick

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If $\text{Sub}(A) \cong \text{Sub}(B)$, then $A \cong B$ as Jordan algebras. Except $\mathbb{C}^2$ and $\mathbb{M}_2$.

If $\text{Sub}(A) \cong \text{Sub}(B)$ preserves $U(A) \times \text{Sub}(A) \to \text{Sub}(A)$, then $A \cong B$ as $\star$-algebras.

Needs orientation!

“Characterizations of categories of commutative C*-subalgebras”
Communications in Mathematical Physics 331(1):215–238, 2014

“Isomorphisms of ordered structures of abelian C*-subalgebras of C*-algebras”

“Active lattices determine AW*-algebras”
Scatteredness

A space is scattered if every nonempty subset has an isolated point. Precisely when each continuous $f : X \to \mathbb{R}$ has countable image. Example: $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \}$.
Scatteredness

A space is scattered if every nonempty subset has an isolated point. Precisely when each continuous \( f: X \rightarrow \mathbb{R} \) has countable image. Example: \( \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \} \).

A \( \star \)-algebra \( A \) is scattered if \( X \) is scattered for all \( C(X) \in \text{Sub}(A) \). Precisely when each self-adjoint \( a = a^* \in A \) has countable spectrum. Example: \( K(H) + 1_H \)

“Inductive Limits of Finite Dimensional C*-algebras”

“Scattered C*-algebras”
Mathematica Scandinavica 41:308–314, 1977
Scatteredness

A space is scattered if every nonempty subset has an isolated point. Precisely when each continuous \( f : X \to \mathbb{R} \) has countable image.
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A ★-algebra \( A \) is scattered if \( X \) is scattered for all \( C(X) \in \text{Sub}(A) \).
Precisely when each self-adjoint \( a = a^* \in A \) has countable spectrum.
Example: \( K(H) + 1_H \)

Nonexample: \( C(\text{Cantor}) \) is approximately finite-dimensional
Nonexample: \( C([0, 1]) \) is not even approximately finite-dimensional

“Inductive Limits of Finite Dimensional C*-algebras”

“Scattered C*-algebras”
Mathematica Scandinavica 41:308–314, 1977
Scatteredness

**Theorem:** the following are equivalent for a $*$-algebra $A$:

- Sub($A$) is algebraic
- Sub($A$) is continuous
- Sub($A$) is meet-continuous
- Sub($A$) is quasi-algebraic
- Sub($A$) is quasi-continuous
- Sub($A$) is atomistic
- $A$ is scattered

“A characterization of scattered C*-algebras and application to crossed products”

“Domains of commutative C*-subalgebras”
Back to quantum logic

For $\star$-algebra $C(X)$, projections are clopen subsets of $X$. Can characterize in order-theoretic terms: (if $|X| \geq 3$)

- closed subsets of $X = \text{ideals of } C(X) = \text{elements of } \text{Sub}(C(X))$
- clopen subsets of $X = \text{‘good’ pairs of elements of } \text{Sub}(C(X))$
Back to quantum logic

For $\star$-algebra $C(X)$, projections are clopen subsets of $X$.
Can characterize in order-theoretic terms: (if $|X| \geq 3$)
- closed subsets of $X =$ ideals of $C(X) = $ elements of $\text{Sub}(C(X))$
- clopen subsets of $X =$ ‘good’ pairs of elements of $\text{Sub}(C(X))$

Each projection of $\star$-algebra $A$ is in some maximal $C \in \text{Sub}(A)$.
Can recover poset of projections from $\text{Sub}(A)!$ (if $\text{dim}(Z(A)) \geq 3$)
Back to piecewise Boolean domains

Sub\((B)\) determines \(B\)

\((B \cong B' \iff \text{Sub}(B) \cong \text{Sub}(B'))\)

\((\text{shape of parts determines whole})\)

⚠ Caveat: not 1-1 correspondence!

“Subalgebras of orthomodular lattices”
Order 28:549–563, 2011
Back to piecewise Boolean domains

\[ \text{Sub}(B) \text{ determines } B \]
\[ (B \cong B' \iff \text{Sub}(B) \cong \text{Sub}(B')) \]
\[ (\text{shape of parts determines whole}) \]

\[ \text{Caveat: not 1-1 correspondence!} \]

If \( B \) Boolean algebra, then \( \text{Sub}(B) \) partition lattice

\[ \text{Caveat: not constructive, not categorical} \]

“Subalgebras of orthomodular lattices”
Order 28:549–563, 2011

“On the lattice of subalgebras of a Boolean algebra”
Different kinds of atoms

If \( B = \{1, 2, 3, 4\} \), then \( \text{Sub}(B) = \cdots \)
Different kinds of atoms
Principal pairs

Reconstruct pairs \((x, \neg x)\) of \(B\):

- principal ideal subalgebra of \(B\) is of the form

- they are the elements \(p\) of \(\text{Sub}(B)\) that are dual modular and

- atom or relative complement

\[
(p \lor m) \land n = p \lor (m \land n) \quad \text{for} \quad n \geq p
\]

\[
a \land m = a, \quad a \lor m = B \quad \text{for atom} \quad a
\]
Principal pairs

Reconstruct pairs \((x, \neg x)\) of \(B\):

- *principal ideal subalgebra* of \(B\) is of the form

- they are the elements \(p\) of \(\text{Sub}(B)\) that are *dual modular* and
  
  \[
  (p \lor m) \land n = p \lor (m \land n) \quad \text{for } n \geq p
  \]

  atom or *relative complement*
  
  \[
  a \land m = a, \quad a \lor m = B \quad \text{for atom } a
  \]

Reconstruct elements \(x\) of \(B\):

- *principal pairs* of \(B\) are \((p, q)\) with atomic meet
Principal pairs

Reconstruct pairs \((x, \neg x)\) of \(B\):

- **principal ideal subalgebra** of \(B\) is of the form

- they are the elements \(p\) of \(\text{Sub}(B)\) that are **dual modular** and

atom or **relative complement** \(a \land m = a, a \lor m = B\) for atom \(a\)

Reconstruct elements \(x\) of \(B\):

- **principal pairs** of \(B\) are \((p, q)\) with atomic meet

**Theorem:** \(B \simeq \text{Pp}(\text{Sub}(B))\) for Boolean algebra \(B\) of size \(\geq 4\)

\(D \simeq \text{Sub}(\text{Pp}(D))\) for Boolean domain \(D\) of size \(\geq 2\)
Directions

If $B$ is

\[
\begin{array}{cccc}
\neg v & \neg w & \neg x & \neg y & \neg z \\
v & w & x & y & z \\
0 & 1
\end{array}
\]

then $\text{Sub}(B)$ is
Directions

If $B$ is

```
 1
¬v ¬w ¬x ¬y ¬z

v w x y z
0
```

or

```
 1
¬v ¬w ¬x y z

v w x ¬y ¬z
0
```

then $\text{Sub}(B)$ is

```
bullet bullet bullet

bullet bullet bullet

bullet
```
Directions

If $B$ is

$\neg v \quad \neg w \quad \neg x \quad \neg y \quad \neg z$ or

$v \quad w \quad x \quad y \quad z$

then $\text{Sub}(B)$ is

A direction for a Boolean domain is map $d : D \to D^2$ with

$\bullet d(1) = (p, q)$ is a principal pair

$\bullet d(m) = (p \land m, q \land m)$
Directions

If $B$ is

If $B$ is

or

or

then Sub($B$) is

then Sub($B$) is

A direction for a piecewise Boolean domain is map $d: D \rightarrow D^2$ with

- if $a \leq m$ then $d(m)$ is a principal pair with meet $a$ in $m$
- $d(m) = \bigvee \{(m, m) \land f(n) \mid a \leq n\}$
- if $m, n$ cover $a$, $d(m) = (a, m)$, $d(n) = (n, a)$, then $m \lor n$ exists
Orthoalgebras

Almost theorem:

- $B \simeq \text{Dir}(\text{Sub}(B))$ for orthoalgebra $B$ of size $\geq 4$
- $D \simeq \text{Sub}(\text{Dir}(D))$ for piecewise orthodomain $D$ of size $\geq 2$

Problems:

- subalgebras of a Boolean orthoalgebra need not be Boolean
- intersection of two Boolean subalgebras need not be Boolean
- two Boolean subalgebras might have no meet
- two Boolean subalgebras might have upper bound but no join

“Boolean subalgebras of orthoalgebras”
Ongoing work
Conclusion

- Should consider piecewise Boolean algebras
- Give rise to domain of honest Boolean subalgebras
- Complicated structure, but can characterize
- Shape of parts enough to determine whole
- Same trick works for scattered operator algebras
- Direction needed for almost categorical equivalence
Conclusion

- Should consider piecewise Boolean algebras
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**Question**

**Theorem:** any Boolean algebra is isomorphic to the global sections of a sheaf on its Stone space

**Question:** is any piecewise Boolean algebra isomorphic to the global sections of a sheaf on its Stone space?

Would give logic of contextuality

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“The theory of representations of Boolean algebras”
Transactions of the American Mathematical Society 40:37–111, 1936

“Representations of algebras by continuous sections”

“The sheaf-theoretic structure of nonlocality and contextuality”

“?”