Functorial spectra and discretization of C*-algebras

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Introduction

$$\text{Equivalence} \quad \mathbf{cCstar} \underset{\text{Hom}(-,\mathbb{C})}{\overset{\text{Hom}(-,\mathbb{C})}{\rightleftarrows}} \mathbf{KHaus}^{\text{op}}$$

1. Many attempts at noncommutative version, none functorial
2. Idea: noncommutative space = set of commutative subspaces
3. Active lattices: 'functions' on noncommutative space
4. Discretization: 'continuous' functions on noncommutative space

Obstruction

Theorem: If $\mathbf{C}$ has strict initial object $\emptyset$ and I continuous,

$$\begin{array}{ccc} \mathbf{cCstar} & \xrightarrow{\text{Spec}} & \mathbf{KHaus}^{\text{op}} \\ \big\downarrow & & \big\downarrow I \\ \mathbf{Cstar} & \xrightarrow[F]{} & \mathbf{C}^{\text{op}} \end{array}$$

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Proof:

1. define $K\colon \mathbf{cCstar} \to \mathbf{C}^{\text{op}}$ by $A \mapsto \lim_{C \subseteq A} I(\text{Spec}(C))$
2. then $K(C) = I(\text{Spec}(C))$ for commutative C
3. K is final with this property
4. $I \circ \text{Spec}$ preserves limits, so $K(A) = I(\text{Spec}(\text{colim}_{C \subseteq A} C))$
5. Kochen-Specker: $\text{colim}_{C \subseteq M_n(\mathbb{C})} \text{Proj}(C)$ is Boolean algebra 1
6. so $F(M_n(\mathbb{C})) \to K(M_n(\mathbb{C})) = \emptyset$

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Remarks:

- Rules out sets, schemes, locales, quantales, ringed toposes, ...
- Not just $M_n(\mathbb{C})$: W*-algebras without summands $\mathbb{C}$ or $M_2(\mathbb{C})$
- Not just Gelfand duality: also Stone, Zariski, Pierce
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Lesson: Set of commutative subalgebras important

Commutative subalgebras

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- Type and dimension: [Lindenhovius 15]
  $\mathcal{C}(A) \simeq \mathcal{C}(B)$ and A is W*/AW* $\implies$ so is B
  $\mathcal{C}(A) \simeq \mathcal{C}(B)$ and $\dim(A) < \infty \implies A \simeq B$

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  projection valued measures compatible when commute
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Lesson: $\mathcal{C}(A)$ has lots of structure, interesting to study

Characterization

When is a partially ordered set of the form $\mathcal{C}(A)$?

If A has weakly terminal abelian subalgebra $C(X)$: [H 14]

1. $\mathcal{C}(A) \simeq \mathcal{C}(C(X))$
2. $\mathcal{C}(C(X)) \simeq P(X) \rtimes S(X)$
3. Axiomatization known for partition lattice $P(X)$ [Firby 73]
4. Axiomatize monoid $S(X)$ of epimorphisms $X \twoheadrightarrow X$
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Lesson: Not just partial order $\mathcal{C}(A)$ important, also action

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$$\mathbf{Poset} \xleftarrow{\text{Proj}} \mathbf{AWstar} \xrightarrow{U} \mathbf{Group}$$

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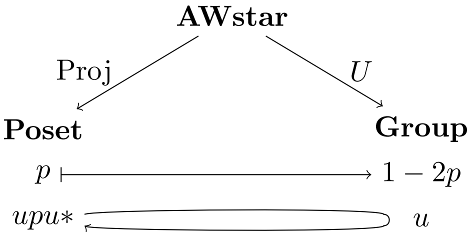

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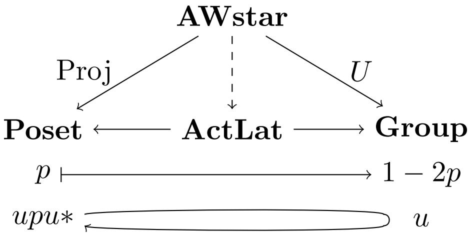

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- Symmetry group $\mathrm{Sym}(A) \subseteq U(A)$ generated by $\{1 - 2p\}$
  - if A commutative, then $\mathrm{Sym}(A)$ is Boolean ring $\mathrm{Proj}(A)$
  - if $A = M_n(C)$ type $\mathrm{I}_{\geq 2}$, then $\mathrm{Sym}(A) = \det^{-1}(\mathrm{Sym}(C))$
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Lesson: 'Noncommutative sets' have hidden actionn

Discretization

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Definition: a discretization of a C*-algebra A is a morphism

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- $A \mapsto \lim_I A/I$ faithful functorial into **Wstar** or **proCstar** for residually finite-dimensional subhomogeneous A

Discretization: another obstruction

Definition: State $\int - \mathrm{d}\mu \colon C(X) \to \mathbb{C}$ *diffuse* when μ has no atoms.
Pair $C, D \in \mathcal{C}(A)$ is *relatively diffuse* when every extension of pure state of D to A restricts to diffuse state on C.

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Lesson: Discretization needs other global coherence structure of projections than that of AW*-algebras.

Conclusion

- It pays to take commutative subalgebras seriously
- Functoriality crucial to ensure they fit together
- Leads to active lattices as ‘noncommutative sets’
- But not good enough for ‘noncommutative topology’

References

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Topos trick

- Consider 'contextual sets' over C*-algebra A:
  assignment of set $S(C)$ to each $C \in \mathcal{C}(A)$
  such that $C \subseteq D$ implies $S(C) \hookrightarrow S(D)$
- They form a *topos* $T(A)$:
  category whose objects behave a lot like sets
  in particular, it has a logic of its own!
- There is a canonical contextual set $\underline{A}$ given by $C \mapsto C$
- $T(A)$ believes that $\underline{A}$ is a *commutative* C*-algebra
- $\underline{A}$ has spectrum within $T(A)$
  corresponds externally to map into $\mathcal{C}(A)$