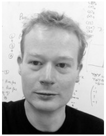
Can a quantum state over time resemble a quantum state at a single time?

Chris Heunen

THE UNIVERSITY of EDINBURGH
informatics

History



Barbados, 2011

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Information processing in generalized probabilistic theories,
Physical Review A 75(3):032304, 2007

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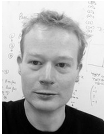

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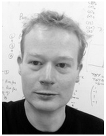
Prehistory



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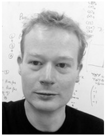

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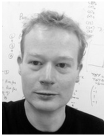

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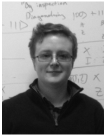

~~Oxford, 2012~~ Durham, 2015

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Idea

- ‘problem of time’ in quantum gravity
  asymmetry between space and time in quantum theory
- one solution: remove asymmetry
  quantum states across time as well as space
- many approaches to states over time
  path integrals, consistent histories, multi-time states
- but: all depend on spatio-temporal relationships
  classical probability theory does not
- can it be done?

Outline

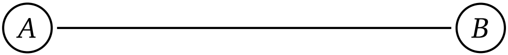
- Three proposals
- Four axioms
- The theorem
- Its proof
- What does it mean?

States across space

first region	second region
A	B
$\mathcal{H}_A$	$\mathcal{H}_B$
ρ_A	ρ_B

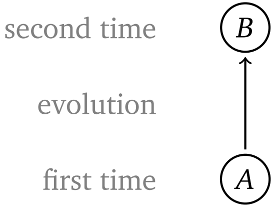
States across space

first region | composite system | second region



$\mathcal{H}_A$	$\mathcal{H}_A \otimes \mathcal{H}_B$	$\mathcal{H}_B$
ρ_A	$\rho_A \otimes \rho_B$	ρ_B

States across time



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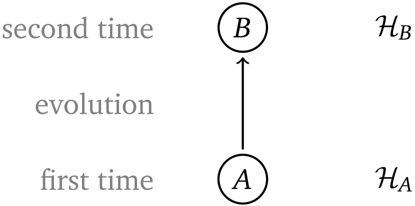
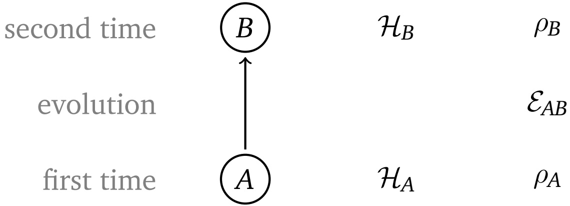


States across time

second time	$\mathcal{H}_B$	ρ_B
evolution		$\mathcal{E}_{AB}$
first time	$\mathcal{H}_A$	ρ_A

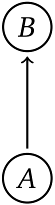
States across time



Composite system AB, operator ρ_{AB} on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$
No restrictions on ρ_{AB}, but fully defined by ρ_A, ρ_B, and $\mathcal{E}_{AB}$

$$\rho_{AB} = f(\rho_B, \mathcal{E}_{AB}, \rho_A)$$

States across time

second time		$\mathcal{H}_B$	ρ_B
evolution			$\mathcal{E}_{AB}$
first time		$\mathcal{H}_A$	ρ_A

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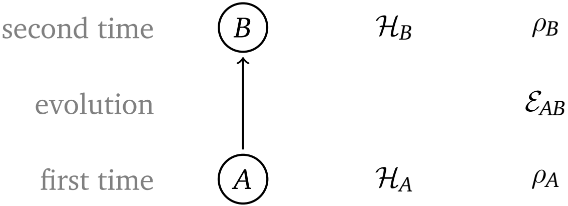
$$\rho_{AB} = f(\mathcal{E}_{AB}, \rho_A)$$

What form could f take?

States across time



Composite system AB, operator ρ_{AB} on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$
No restrictions on ρ_{AB}, but fully defined by ρ_A and $\mathcal{E}_{AB}$

$$\rho_{AB} = E_{AB} \star \rho_A$$

What form could star product $\star$: $\mathcal{B}(\mathcal{H}_{AB}) \times \mathcal{B}(\mathcal{H}_{AB}) \to \mathcal{B}(\mathcal{H}_{AB})$ take?

channel state $E_{AB} = \sum_{ij} \mathcal{E}_{AB}|i\rangle\langle j|_A \otimes |j\rangle\langle i|_B$
satisfying $\rho_B = \mathrm{Tr}_A(E_{AB}\rho_A)$

Leifer-Spekkens

“Quantum theory is like Bayesian inference”

Towards a formulation of quantum theory as a causality neutral theory of Bayesian inference
Physical Review A 88(5):052130, 2013

channel state	conditional probabilities
E_{AB}	$\mathrm{P}(B \mid A)$
$\mathrm{Tr}_B(E_{AB}) = 1$	$\sum_B \mathrm{P}(B \mid A) = 1$
$\rho_B = \mathrm{Tr}_A(E_{AB}\,\rho_A)$	$\sum_A \mathrm{P}(B \mid A)\,\mathrm{P}(A)$
?	$\mathrm{P}(AB) = \mathrm{P}(B \mid A)\,P(A)$

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?	$\mathrm{P}(AB) = \mathrm{P}(B \mid A)\,P(A)$

$$? = \rho_{AB}^{(\mathrm{LS})} = E_{AB} \star_{\mathrm{LS}} \rho_A = \sqrt{\rho_A}\,E_{AB}\,\sqrt{\rho_A}$$

Fitzsimons-Jones-Vedral

"Use pseudo-density matrices"

Quantum correlations which imply causation
Scientific Reports 5:18281, 2015

For (multi-)qubit systems A, B:

1. Measure $\sigma_i \in \{1, \sigma_x, \sigma_y, \sigma_z\}$ on A
2. Evolve according to channel $\mathcal{E}_{AB}$
3. Measure σ_j on B

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$$\rho_{AB}^{(\mathrm{FJV})} = \frac{1}{4}\left(\sum_{i=1}^{3} \langle \sigma_i \otimes \sigma_j \rangle \sigma_i \otimes \sigma_j\right)$$

Fitzsimons-Jones-Vedral

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$$\rho_{AB}^{(\mathrm{FJV})} = \frac{1}{4}\left(\sum_{i=1}^{3} \langle \sigma_i \otimes \sigma_j \rangle \sigma_i \otimes \sigma_j\right) = E_{AB} \star_{\mathrm{FJV}} \rho_A = \frac{1}{2}(\rho_A E_{AB} + E_{AB} \rho_A)$$

Discrete Wigner functions

"Use quasi-probabilities on discrete representation"

On the quantum correction for thermodynamic equilibrium
Physical Review 40:749-759, 1932

1. Pick *phase-point operator* basis $\{K_i^A\}$ for $\mathcal{H}_A$
   $\mathrm{Tr}(K_i^A K_j^A) = \delta_{ij} \dim(\mathcal{H}_A)$ and $\sum_i K_i^A = \dim(\mathcal{H}_A)$
2. Write ρ_A as *quasi-probability function* $r_A \colon \{i\} \to [-1, 1]$
3. Write E_{AB} as *conditional quasi-probability* $r_{B|A} \colon \{(i,j)\} \to [-1, 1]$
4. Define ρ_{AB} by $r_{AB}(ij) = r_{B|A}(j \mid i) r_A(i)$

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$$\rho_{AB}^{(\mathrm{W})} = \sum_{ij} r_{B|A}(j \mid i) r_A(i) K_i^A \otimes K_j^B$$

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$$\rho_{AB}^{(\mathrm{W})} = \sum_{ij} r_{B|A}(j \mid i) r_A(i) K_i^A \otimes K_j^B$$

$$= E_{AB} \star_{\mathrm{W}} \rho_A = \sum_{ij} \mathrm{Tr}_{AB}(E_{AB} K_i^A \otimes K_j^B)\, \mathrm{Tr}_A(\rho_A K_i^A) K_i^A \otimes K_j^B$$

Axiom 1: preservation of probabilistic mixtures

If A conditioned on fair classical coin
and channel is identity
then should have composite state be mixture

$\rho_{A,x=h} = |0\rangle\langle 0|$, $\rho_{A,x=t} = |1\rangle\langle 1|$
$E_{AB} = |\phi^+\rangle\langle\phi^+|^{T_B}$

$$E_{AB} \star \left(\frac{1}{2}\rho_{A,x=h} + \frac{1}{2}\rho_{A,x=t}\right) = \frac{1}{2}\left(E_{AB} \star \rho_{A,x=h}\right) + \frac{1}{2}\left(E_{AB} \star \rho_{A,x=t}\right)$$

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Axiom 1: convex-bilinearity

$$\big(px + (1-p)y\big) \star z = p(x \star z) + (1-p)(y \star z)$$
$$x \star \big(py + (1-p)z\big) = p(x \star y) + (1-p)(x \star z)$$

for all operators x, y, z and probabilities $p \in [0, 1]$

Axiom 2: preservation of classical limit

If channel completely dephasing $\quad E_{AB} = \sum_i \langle i|\rho|i\rangle \sum_j p(j \mid i)\, |j\rangle\langle j|$
and input state diagonal $\quad \rho_A = \sum_i p(i)\, |i\rangle\langle i|$
then should reproduce joint classical probabilities

$$E_{AB} \star \rho_A = \sum_i p(i)\, |i\rangle\langle i| \otimes \sum_i p(j \mid i) |j\rangle\langle j| = E_{AB}\rho_A$$

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Axiom 2: product on commuting pairs

$$[x,y] = 0 \implies x \star y = xy$$

for all operators x, y

Axiom 3: preservation of marginals

Joint state ρ_{AB} should reproduce marginal states ρ_A and ρ_B

$$\mathrm{Tr}_B\, \rho_{AB} = \rho_A \qquad \mathrm{Tr}_A\, \rho_{AB} = \rho_B$$

$$\mathrm{Tr}_B(E_{AB} \star \rho_A) = \mathrm{Tr}_B(E_{AB}\rho_A) \qquad \mathrm{Tr}_A(E_{AB} \star \rho_A) = \mathrm{Tr}_A(E_{AB}\rho_A)$$

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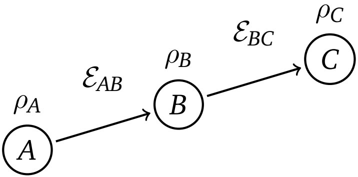
Axiom 3: product when traced

$$\mathrm{Tr}(x \star y) = \mathrm{Tr}(xy)$$

for all operators x, y

Axiom 4: preservation of compositionality

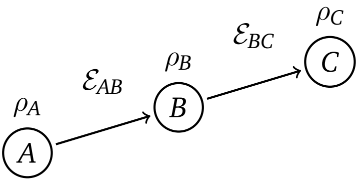


form $\mathcal{E}_{AB,C}$, get $\rho_{ABC} = E_{AB,C} \star \rho_{AB}$

or

form $\mathcal{E}_{A,BC}$, get $\rho_{ABC} = E_{A,BC} \star \rho_A$

Axiom 4: preservation of compositionality



form $\mathcal{E}_{BC}$, get $\rho_{ABC} = E_{BC} \star (E_{AB} \star \rho_A)$

or

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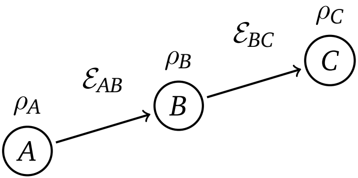
Axiom 4: preservation of compositionality



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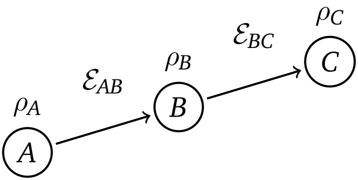

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Axiom 4: associativity

$$(x \star y) \star z = x \star (y \star z)$$

for all operators x, y, z

Three proposals, four axioms

	convex-bilinear	product on commuting pairs	product when traced	associative
LS	$\times$	$\checkmark$	$\checkmark$	$\times$
FJV				
W				

Recall $x \star_{\text{LS}} y = y^{1/2} x y^{1/2}$

if $x = |\phi^+\rangle\langle\phi^+|^{T_B}$ and $y = |0\rangle\langle 0|$, $z = |1\rangle\langle 1|$, and $p = 1/2$,
then $x \star (py + (1-p)z) \neq p(x \star y) + (1-p)(x \star z)$

Three proposals, four axioms

	convex-bilinear	product on commuting pairs	product when traced	associative
LS	$\times$	$\checkmark$	$\checkmark$	$\times$
FJV	$\checkmark$	$\checkmark$	$\checkmark$	$\times$
W				

Recall $x \star_{\text{FJV}} y = \frac{1}{2}(xy + yx)$

in general $(x \star y) \star z \neq x \star (y \star z)$

Three proposals, four axioms

	convex-bilinear	product on commuting pairs	product when traced	associative
LS	×	✓	✓	×
FJV	✓	✓	✓	×*
W				

Recall $x \star_{\text{FJV}} y = \frac{1}{2}(xy + yx)$

in general $(x \star y) \star z \neq x \star (y \star z)$

But for single qubits:

$$E_{BC} \star (E_{AB} \star \rho_A) = (E_{BC} \star E_{AB}) \star \rho_A$$

because ρ_A commutes with E_{BC}

Three proposals, four axioms

	convex-bilinear	product on commuting pairs	product when traced	associative
LS	×	✓	✓	×
FJV	✓	✓	✓	×*
W	✓	×	✓	✓

Recall $x \star_{\mathrm{W}} y = \sum_{ij} \mathrm{Tr}_{AB}(E_{AB} K_i^A \otimes K_j^B)\, \mathrm{Tr}_A(\rho_A k_i^A) K_i^A \otimes K_j^B$

Unless quasi-probabilities are non-negative,

$$[x, y] = 0 \not\Rightarrow x \star y = xy$$

because joint state not diagonal in eigenbasis
of input state and channel state even if they commute

Main result

Star theorem:
Let $\mathbb{M}_n$ be the complex vector space of n-by-n matrices.
The only functions $\star\colon \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{M}_n$ satisfying
the four axioms are $x \star y = xy$ and $x \star y = yx$.

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An operator $\rho\colon \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$ is *locally positive* when it gives nonnegative probabilities on local measurements: $\langle ab|\rho|ab\rangle \geq 0$.

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Reduction lemma:
Let $\mathbb{LP}_n$ be the set of locally positive n-by-n matrices.
Any function $\star\colon \mathbb{LP}_n \times \mathbb{LP}_n \to \mathbb{LP}_n$ satisfying the axioms
extends to a function $\star\colon \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{M}_n$ satisfying the axioms.

The proof: reduction lemma

First extend from $\mathbb{LP}_n$ to set $\mathbb{H}_n$ of hermitian matrices:

- *conic combination* of $x_i \in \mathbb{H}_n$ is $\sum_i r_i x_i$ for $r_i \in [0, \infty)$
- define $\star$ on *conic hull* of $\mathbb{LP}_n$ by

$$(\sum_i r_i x_i) \star (\sum_j s_j y_j) = \sum_{ij} r_i s_j (x_i \star y_j)$$

- unique convex-bilinear extension $\star \colon \mathbb{H}_n \times \mathbb{H}_n \to \mathbb{H}_n$
  (because $x \star 0 = 0 = 0 \star x$) in fact $\mathbb{R}$-bilinear
- other three axioms preserved by extension

The proof: reduction lemma

Next extend from $\mathbb{H}_n$ to $\mathbb{M}_n$:

- *polarize* $x \in \mathbb{M}_n$ as $x = x_h + x_a$
  for $x_h = \frac{1}{2}(x + x^\dagger) \in \mathbb{H}_n$ and $x_a = \frac{1}{2}(x - x^\dagger)$
- define $\star$ on $\mathbb{M}_n$ by

$$x \star y = x_h \star y_h - i(x_h \star iy_a) - i(ix_a \star y_h) - (ix_a \star iy_a)$$

- unique $\mathbb{R}$-bilinear extension $\star \colon \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{M}_n$
  in fact $\mathbb{C}$-bilinear
- other three axioms preserved by extension
  (product on commuting pairs uses Fuglede's theorem:
  $xy = yx$ and $yy^\dagger = y^\dagger y$ imply $xy^\dagger = y^\dagger x$)

The proof

- *Product on commuting pairs* implies:
  $x \star y = 0$ if $xy = yx = 0$, $xx = x$, $yy = y$, for rank one $x, y \in \mathbb{M}_n$

The proof

   

On bilinear maps determined by rank one idempotents
Linear Algebra and its Applications 432:738743, 2010

- *Product on commuting pairs* implies:
  $x \star y = 0$ if $xy = yx = 0$, $xx = x$, $yy = y$, for rank one $x, y \in \mathbb{M}_n$
- Additionally $\mathbb{C}$-*bilinearity* implies:

$$x \star y = xy + g(xy - yx)$$

for some linear map $g \colon \mathbb{M}_n \to \mathbb{M}_n$ and all $x, y \in \mathbb{M}_n$

The proof




On bilinear maps determined by rank one idempotents
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  for some linear map $g \colon \mathbb{M}_n \to \mathbb{M}_n$ and all $x, y \in \mathbb{M}_n$
- Additionally *product when traced* implies:
  g preserves traceless matrices

The proof: linear algebra

Finally, *associativity* means for standard matrix units e_{ij}

$$
\begin{aligned}
&-\delta_{de}\delta_{af}g(e_{cb}) + \delta_{de}e_{ab}\star g(e_{cf}) - \delta_{cf}e_{ab}\star g(e_{ed}) \\
=&-\delta_{af}\delta_{bc}g(e_{ed}) + \delta_{bc}g(e_{ad})\star e_{ef} - \delta_{ad}g(e_{cb})\star e_{ef}
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Write $g(e_{ij}) = \sum_{k,l=1}^{n} G_{kl,ij}\, e_{lk}$ for entries $G_{kl,ij} \in \mathbb{C}$.
Write $g(ij)_{kl}$ for $G_{kl,ij}$, and $g(ii-jj)_{kl}$ for $G_{kl,ii} - G_{kl,jj}$.

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\begin{array}{rrrr}
g(ij)_{kl} & g(ii-jj)_{kl} & g(ij)_{jk} & g(ij)_{kj} \\
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g(ii-jj)_{ik} & g(ii-jj)_{ki} & g(ii-jj)_{kj} & g(ii-jj)_{jk} \\
g(ij)_{jj} & g(ij)_{ii} & g(ij)_{ij} & g(ii-jj)_{ij}
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More fiddling: there is $z \in \mathbb{M}_n$ with $g(ii) = \lambda e_{ii} + z$,
and $\lambda \in \{0, -1\}$ with $g(ii-jj)_{ii} = -g(ii-jj)_{jj} = g(ij)_{ji} = \lambda$ for $i \neq j$

The proof: linear algebra

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Write $g(e_{ij}) = \sum_{k,l=1}^{n} G_{kl,ij}\, e_{lk}$ for entries $G_{kl,ij} \in \mathbb{C}$.
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Conclusion: $x \star y = xy + \lambda(xy - yx)$

Conclusion

- Quantum probabilistic reasoning cannot be Bayesian
  quantum probabilities compose differently over space and time
- States over time not attractive for quantum gravity
  need quantum way to discern space and time not in relativity
- Larger state spaces don't help
  $\star\colon X \times X' \to X''$ for $X, X' \subseteq \mathbb{M}_m$ and $X'' \subseteq \mathbb{M}_n$
- Star theorem holds without *product when traced* for qubits
  proof by brute force, might hold in any dimension
- Dropping *associativity* most likely way out
  joint FJV state is matrix product without imaginary eigenvalues
- Other possibility: operational point of view
  define joint state via experimental outcomes at different times

FJV as a star product

Recall

$$\rho_{AB}^{(\mathrm{FJV})} = \frac{1}{4}\left(\sum_{i=1}^{3}\langle\sigma_i \otimes \sigma_j\rangle \sigma_i \otimes \sigma_j\right)$$

Compute

$$\begin{aligned}
\langle\sigma_i \otimes \sigma_j\rangle &= \mathrm{Tr}(E_{AB}((\rho_A^{i+} - \rho_A^{i-}) \otimes \sigma_j) \\
&= \mathrm{Tr}(E_{AB}(\frac{1}{2}(\sigma_i\rho_A + \rho_A\sigma_i) \otimes \sigma_j) \\
&= \frac{1}{2}\,\mathrm{Tr}(\rho_A E_{AB}(\sigma_i \otimes \sigma_j) + E_{AB}\rho_A(\sigma_i \otimes \sigma_j)) \\
&= \mathrm{Tr}(\rho_{AB}^{(\mathrm{FJV})}(\sigma_i \otimes \sigma_j))
\end{aligned}$$

So

$$\rho_{AB}^{(\mathrm{FJV})} = \frac{1}{2}(\rho_A\, E_{AB} + E_{AB}\, \rho_A)$$