# The category of Hilbert modules

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informatics

Hilbert modules form a monoidal category

- categorical quantum mechanics can take place
- geometry: continuous fields of Hilbert spaces
- Frobenius structures: algebraic quantum field theory
- restriction: recovering open subsets of base space
- spacetime structure: restriction becomes propagation
- quantum teleportation: proof of concept

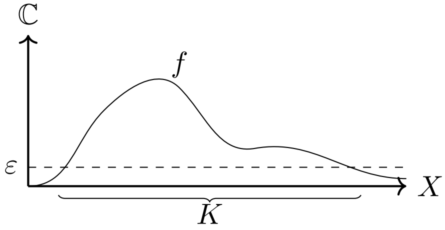
# Hilbert spaces

$\mathbb{C}$-module $H$ with complete inner product valued in $\mathbb{C}$

| | |
|---|---|
| tensor product over $\mathbb{C}$ | monoidal category |
| tensor unit $\mathbb{C}$ | tensor unit $I$ |
| complex numbers $\mathbb{C}$ | scalars $I \to I$ |
| finite dimension | dual objects |
| adjoints | dagger |
| orthonormal basis | commutative dagger Frobenius structure |
| C*-algebra | dagger Frobenius structure |

# Base space

Let $X$ be locally compact Hausdorff space.
$C_0(X) = \{f\colon X \to \mathbb{C} \text{ cts} \mid \forall\varepsilon > 0\, \exists K \subseteq X \text{ cpt}\colon f(X \setminus K) < \varepsilon\}$




$C_b(X) = \{f\colon X \to \mathbb{C} \text{ cts} \mid \exists 0 < \|f\| < \infty\, \forall t \in X\colon |f(t)| \leq \|f\|\}$

# Hilbert modules

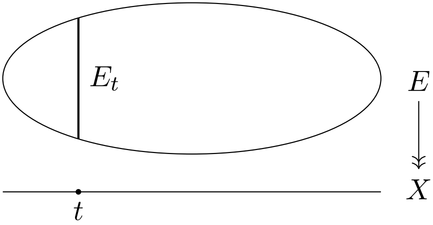
$C_0(X)$-module with complete inner product valued in $C_0(X)$

| | |
|---|---|
| tensor product over $C_0(X)$ | monoidal category |
| tensor unit $C_0(X)$ | tensor unit $I$ |
| $C_b(X)$ | scalars $I \to I$ |
| ? | dual objects |
| adjointable morphisms | dagger |
| ? | dagger Frobenius structure |

"Scalars are not numbers"

# Bundles of Hilbert spaces

Bundle $E \twoheadrightarrow X$, each fibre Hilbert space, operations continuous



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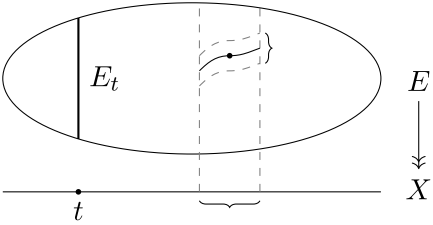

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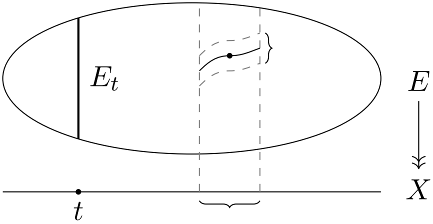


| | | |
|---|---|---|
| Hilbert $C_0(X)$-modules | $\simeq$ | bundles of Hilbert spaces over $X$ |
| sections vanishing at infinity | $\leftarrow\!\!\dashv$ | $E \twoheadrightarrow X$ |
| $E$ | $\mapsto$ | localisation |

# Dual objects

$E$ has dual object when $\cup\colon I \to E^* \otimes E$ and $\cap\colon E \otimes E^* \to I$ satisfy  $= |$ and  $= |$

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finite Hilbert bundle:
$\sup_{t \in X} \dim(E_t) < \infty$

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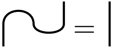
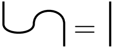
$E$ has dual object when $\cup \colon I \to E^* \otimes E$ and $\cap \colon E \otimes E^* \to I$ satisfy  and 

$$\iff$$

finite Hilbert bundle:

$\sup_{t \in X} \dim(E_t) < \infty$

$$\iff$$

finitely presented projective Hilbert module:

$$\mathrm{id} \circlearrowright E \underset{i^\dagger}{\overset{i}{\rightleftarrows}} C_0(X)^n$$

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$E$ has dual object when $\cup\colon I \to E^* \otimes E$ and $\cap\colon E \otimes E^* \to I$ satisfy  $= |$ and  $= |$

$$\iff$$

finite Hilbert bundle:
$\forall t \in X\colon\ \dim(E_t) < \infty$

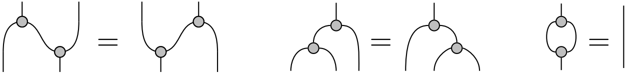
$$\iff$$

finitely generated projective Hilbert module:

$$\mathrm{span}_{C_0(X)}(x_1, \ldots, x_n) = E \dashrightarrow F \twoheadrightarrow G, \quad E \to G$$

# Frobenius structures

$E$ has special dagger Frobenius structure : $E \otimes E \to E$:

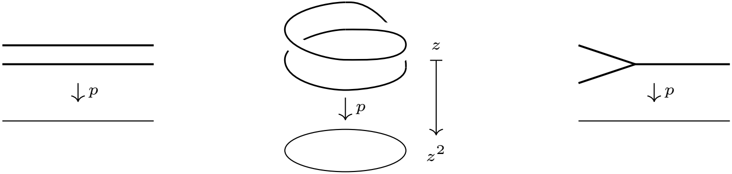


$$\iff$$

$E$ is a finite bundle of C*-algebras:
each fibre is C*-algebra, operations continuous, $\sup \dim(E_t) < \infty$

# Commutative Frobenius structures

$p\colon Y \twoheadrightarrow X$ branched covering: continuous, open, $\sup_{t\in X} |p^{-1}(t)| < \infty$



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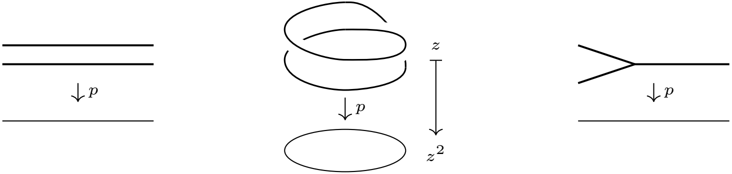


$C_0(Y)$ special dagger Frobenius: $\langle f \mid g \rangle(t) = \sum_{p(y)=t} f(y)^* g(y)$

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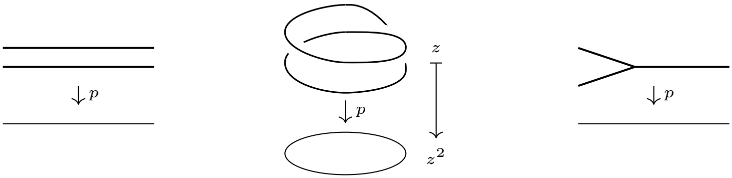


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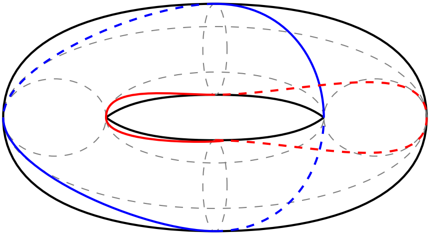
comultiplication comes from $Y \times_X Y = \{(a,b) \in S^1 \times S^1 \mid a^2 = b^2\}$



# Nontrivial central Frobenius structure

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$$
$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$
$$X = S^2 = \{t \in \mathbb{R}^3 \mid \|t\| = 1\}$$

$$\left\{ x \in C_0(\mathbb{D}, \mathbb{M}_n) \colon\; x(z) = \begin{pmatrix} \overline{z} & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} x(1) \begin{pmatrix} z & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \text{ if } z \in S^1 \right\}$$

is special dagger Frobenius structure: $\langle x \mid y \rangle(t) = \mathrm{tr}(x(t)^* y(t))$

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central: $Z(E) = C_0(X)$

# Transitivity

$E$ is special dagger Frobenius structure in $\mathbf{Hilb}_{C_0(X)}$

$$\iff$$

$E$ is special dagger Frobenius structure in $\mathbf{Hilb}_{Z(E)}$

and

$Z(E)$ is specialisable dagger Frobenius structure in $\mathbf{Hilb}_{C_0(X)}$

# Idempotent subunits

Subobject $s\colon S \rightarrowtail E$ is idempotent
if $s \otimes \mathrm{id}_S \colon S \otimes S \to E \otimes S$ isomorphic

Subunit $s\colon S \rightarrowtail C_0(X)$ idempotent

$$\iff$$

$$S = C_0(U) \simeq \{f \in C_0(X) \mid f(X \setminus U) = 0\}$$
for open $U \subseteq X$

# Order

Category is firm when $s \otimes \mathrm{id}_T$ monic for subunits $s, t$

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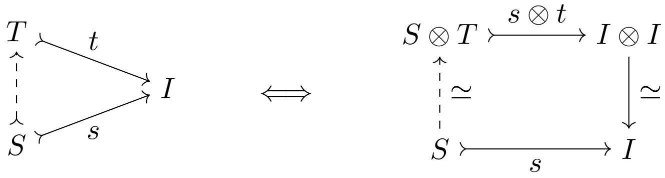
Idempotent subunits in a firm braided monoidal category
form a meet-semilattice



# Restriction

For $s$ idempotent subunit in monoidal category $\mathbf{C}$,
write $\mathbf{C}|_s$ for full subcategory of $E$ with $\mathrm{id}_E \otimes s$ iso

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Coreflective monoidal subcategory:

$$\mathbf{C} \underset{\hookleftarrow}{\overset{(-)\otimes S}{\underset{\top}{\rightleftarrows}}} \mathbf{C}|_s$$

coreflector $(-) \otimes S \colon \mathbf{C} \to \mathbf{C}|_s$ is restriction to $s$

# Conditional expectation

Conditional expectation is

$$X \xrightarrow{g} U \xrightarrow{f} \mathrm{Radon}(X) \text{ with } \mathrm{supp}(f(t)) \subseteq g^{-1}(t)$$

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Induces monoidal functor $\mathbf{Hilb}_{C_0(X)} \to \mathbf{Hilb}_{C_0(U)}$

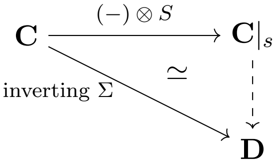
whereas in general

$$\mathbf{Hilb}_{C_0(X)}|_{C_0(U)} \simeq \{E \mid \forall x \in E \colon \|x\|(X \setminus U) = 0\}$$

# Localisation

Localisation formally inverts given class of morphisms

Restriction to $s$ is localisation at $\Sigma = \{\mathrm{id}_E \otimes s \mid E \in \mathbf{C}\}$



# Graded monad

Let $(\mathbf{I}, \otimes, 1)$ be monoidal category

Graded monad is strong monoidal functor $T\colon \mathbf{I} \to [\mathbf{C}, \mathbf{C}]$

- functor $T\colon \mathbf{I} \to [\mathbf{C}, \mathbf{C}]$
- natural iso $\eta\colon \mathrm{id}_{\mathbf{C}} \Rightarrow T(1)$
- natural isos $\mu_{s,t}\colon T(s) \circ T(t) \Rightarrow T(s \otimes t)$
- associative and unital

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Restriction forms graded monad $(\mathrm{ISub}(\mathbf{C}), \otimes, \mathrm{I}) \to [\mathbf{C}, \mathbf{C}]$

$$T(s) = (-) \otimes S$$
$$\eta = \lambda$$
$$\mu = \alpha$$

# Spatial structure

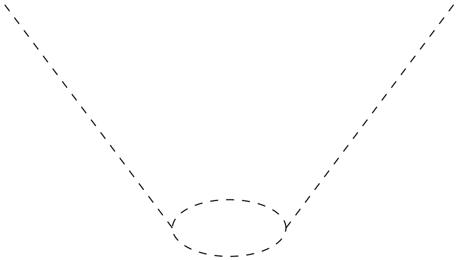
$f\colon E \to F$ restricts to $s$ when it factors through $F \otimes S$

if $f$ restricts to $s$ and $g$ restricts to $t$,
then $g \circ f$ and $g \otimes f$ restrict to $t \otimes s$

# Causal structure

What if $X$ is spacetime?

On open $U \subseteq X$, causal/chronological cones $I^{\pm}(U) = J^{\pm}(U)$ same



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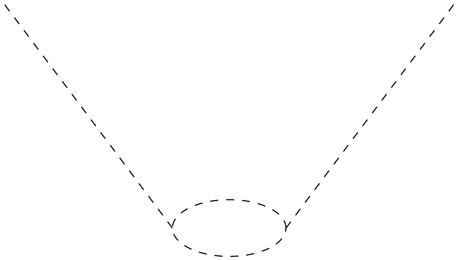

Closure operator is $I^{+}\colon \mathrm{ISub}(\mathbf{C}) \to \mathrm{ISub}(\mathbf{C})$

satisfying $s \leq I^{+}(s) \geq I^{+}(I^{+}(s))$ and monotone

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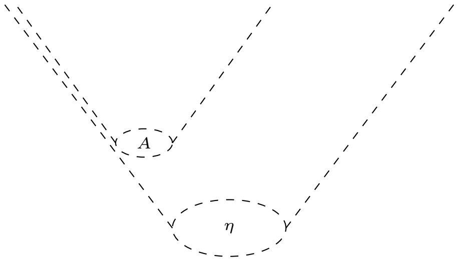


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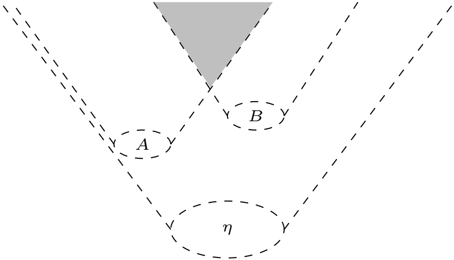


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Restriction = propagation

Teleportation only successful on intersection of Alice and Bob's cones

# This is just the beginning

- Continuous extension of higher quantum theory
- Infinite dimension with standard methods
- Deformation quantization?
- Relativistic quantum theory: summoning?
- Graphical calculus?
- Logic?