The category of Hilbert modules

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Hilbert modules form a monoidal category

- **categorical quantum mechanics** can take place
- **geometry**: continuous fields of Hilbert spaces
- **Frobenius structures**: algebraic quantum field theory
- **restriction**: recovering open subsets of base space
- **spacetime structure**: restriction becomes propagation
- **quantum teleportation**: proof of concept
Hilbert spaces

\[ \mathbb{C} \text{-module } H \text{ with complete inner product valued in } \mathbb{C} \]

| tensor product over \( \mathbb{C} \) | monoidal category |
| tensor unit \( \mathbb{C} \) | tensor unit \( I \) |
| complex numbers \( \mathbb{C} \) | scalars \( I \rightarrow I \) |
| finite dimension | dual objects |
| adjoints | dagger |
| orthonormal basis | commutative dagger Frobenius structure |
| \( \mathbb{C}^\ast \)-algebra | dagger Frobenius structure |
Let $X$ be locally compact Hausdorff space.

$C_0(X) = \{ f : X \to \mathbb{C} \text{ cts} \mid \forall \varepsilon > 0 \exists K \subseteq X \text{ cpt: } f(X \setminus K) < \varepsilon \}$

$C_b(X) = \{ f : X \to \mathbb{C} \text{ cts} \mid \exists 0 < \| f \| < \infty \forall t \in X : |f(t)| \leq \| f \| \}$
Hilbert modules

$C_0(X)$-module with complete inner product valued in $C_0(X)$

| tensor product over $C_0(X)$ | monoidal category tensor unit $I$ |
| tensor unit $C_0(X)$          | scalars $I \to I$ |
| $C_b(X)$                      | dual objects |
| ?                              | dagger |
| adjointable morphisms          | dagger Frobenius structure |
| ?                              | “Scalars are not numbers” |
Bundles of Hilbert spaces

Bundle $E \rightarrow X$, each fibre Hilbert space, operations continuous
Bundles of Hilbert spaces

Bundle $E \to X$, each fibre Hilbert space, operations continuous, with
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Hilbert $C_0(X)$-modules $\simeq$ bundles of Hilbert spaces over $X$
sections vanishing at infinity $\leftrightarrow E \to X$

$E$ $\mapsto$ localisation
Dual objects

\( E \) has dual object when \( \odot : I \rightarrow E^* \otimes E \) and
\( \odot : E \otimes E^* \rightarrow I \) satisfy \( \odot = \) and \( \odot = \)
Dual objects

if $X$ paracompact

$E$ has dual object when $\bigodot : I \to E^* \otimes E$ and $\bigodot : E \otimes E^* \to I$ satisfy $\bigodot = \bigodot$ and $\bigodot = \bigodot$\

$\iff$

finite Hilbert bundle:
$\sup_{t \in X} \dim(E_t) < \infty$
Dual objects

if \( X \) paracompact

\[ E \text{ has dual object when } \bigcap: I \to E^* \otimes E \text{ and } \bigcup: E \otimes E^* \to I \text{ satisfy } \bigcap = \bigcup \text{ and } \bigcup = \bigcap \]

\[ \iff \]

finite Hilbert bundle:

\( \sup_{t \in X} \dim(E_t) < \infty \)

\[ \iff \]

finitely presented projective Hilbert module:

\[ \text{id } \bigcap E \xrightleftharpoons[i]{} i \bigcup C_0(X)^n \]
Dual objects

if $X$ compact

$E$ has dual object when $\cup : I \to E^* \otimes E$ and $\cap : E \otimes E^* \to I$ satisfy $\cup = \cap$ and $\cap = \cup$

$\iff$

finite Hilbert bundle:
$\forall t \in X : \dim(E_t) < \infty$

$\iff$

finitely generated projective Hilbert module:
$\text{span}_{C_0(X)}(x_1, \ldots, x_n) = E$
Frobenius structures

\( E \) has special dagger Frobenius structure \( \triangleright: E \otimes E \to E: \)

\( \begin{align*}
\text{= } & \\
\text{=} & \\
\text{=} & \\
\text{=} & \end{align*} \)

\( \iff \)

\( E \) is a finite bundle of \( \mathrm{C}^* \)-algebras:
each fibre is \( \mathrm{C}^* \)-algebra, operations continuous, sup dim(\( E_t \)) < \( \infty \)
Commutative Frobenius structures

$p: Y \rightarrow X$ branched covering: continuous, open, $\sup_{t \in X} |p^{-1}(t)| < \infty$

\[\begin{align*}
\text{Diagram 1} & \\
\text{Diagram 2} & \\
\text{Diagram 3} & \\
\end{align*}\]
Commutative Frobenius structures

$p: Y \to X$ branched covering: continuous, open, \( \sup_{t \in X} |p^{-1}(t)| < \infty \)

\[ C_0(Y) \text{ special dagger Frobenius: } \langle f | g \rangle(t) = \sum_{p(y) = t} f(y)^* g(y) \]
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comultiplication comes from \( Y \times_X Y = \{(a, b) \in S^1 \times S^1 \mid a^2 = b^2\} \)
Nontrivial central Frobenius structure

\[ \mathbb{D} = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \]

\[ S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \]

\[ X = S^2 = \{ t \in \mathbb{R}^3 \mid \|t\| = 1 \} \]

\[ \left\{ x \in C_0(\mathbb{D}, \mathbb{M}_n) : x(z) = \begin{pmatrix} \bar{z} & 1 \\ 1 & 1 \end{pmatrix} x(1) \begin{pmatrix} \bar{z} & 1 \\ 1 & 1 \end{pmatrix} \right\} \text{ if } z \in S^1 \]

is special dagger Frobenius structure: \( \langle x | y \rangle(t) = \text{tr}(x(t)^*y(t)) \)
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is special dagger Frobenius structure: \[ \langle x \mid y \rangle(t) = \text{tr}(x(t)^* y(t)) \]

central: \[ Z(E) = C_0(X) \]
Transitivity

\[ E \text{ is special dagger Frobenius structure in } \text{Hilb}_{C_0(X)} \]

\[ \iff \]

\[ E \text{ is special dagger Frobenius structure in } \text{Hilb}_{Z(E)} \]

and

\[ Z(E) \text{ is specialisable dagger Frobenius structure in } \text{Hilb}_{C_0(X)} \]
Idempotent subunits

Subobject $s: S \rightarrow E$ is idempotent if $s \otimes \text{id}_S: S \otimes S \rightarrow E \otimes S$ is isomorphic.

Subunit $s: S \rightarrow C_0(X)$ idempotent

$\iff$

$S = C_0(U) \simeq \{ f \in C_0(X) \mid f(X \setminus U) = 0 \}$

for open $U \subseteq X$
Order

Category is **firm** when $s \otimes \text{id}_T$ monic for subunits $s, t$
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Idempotent subunits in a firm braided monoidal category form a meet-semilattice

\[
\begin{align*}
T & \xrightarrow{t} I \\
S & \xrightarrow{s} I
\end{align*}
\]

\[
\begin{align*}
S \otimes T & \xrightarrow{s \otimes t} I \otimes I \\
S & \xrightarrow{s} I
\end{align*}
\]
Restriction

For $s$ idempotent subunit in monoidal category $C$, write $C|_s$ for full subcategory of $E$ with $\text{id}_E \otimes s$ iso
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Coreflective monoidal subcategory:

\[
\begin{array}{ccc}
C & \overset{T}{\leftrightarrow} & C|_s \\
\downarrow & & \downarrow \\
(\cdot) \otimes S & & (\cdot) \otimes S
\end{array}
\]

coreflector $(-) \otimes S: C \to C|_s$ is restriction to $s$
Conditional expectation

Conditional expectation is

\[ X \xrightarrow{g} U \xrightarrow{f} \text{Radon}(X) \text{ with } \text{supp}(f(t)) \subseteq g^{-1}(t) \]
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Induces monoidal functor \( \text{Hilb}_{C_0(X)} \to \text{Hilb}_{C_0(U)} \)

whereas in general
\[ \text{Hilb}_{C_0(X)}|_{C_0(U)} \simeq \{ E \mid \forall x \in E : \|x\|(X \setminus U) = 0 \} \]
Localisation formally inverts given class of morphisms
Restriction to $s$ is localisation at $\Sigma = \{\text{id}_E \otimes s \mid E \in C\}$

\[
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{(-) \otimes S} & \mathbf{C}|_s \\
\downarrow \sim & & \downarrow \sim \\
\mathbf{D} & \sim & \mathbf{D}
\end{array}
\]
Graded monad

Let \((I, \otimes, 1)\) be monoidal category. 

**Graded monad** is strong monoidal functor \(T: I \to [C, C]\)

- functor \(T: I \to [C, C]\)
- natural iso \(\eta: id_C \Rightarrow T(1)\)
- natural isos \(\mu_{s,t}: T(s) \circ T(t) \Rightarrow T(s \otimes t)\)
- associative and unital
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Restriction forms graded monad \((\text{ISub}(C), \otimes, I) \to [C, C]\)

\[T(s) = (-) \otimes S\]

\[\eta = \lambda\]

\[\mu = \alpha\]
Spatial structure

\[ f : E \to F \text{ restricts to } s \text{ when it factors through } F \otimes S \]

if \( f \) restricts to \( s \) and \( g \) restricts to \( t \),
then \( g \circ f \) and \( g \otimes f \) restrict to \( t \otimes s \)
Causal structure

What if $X$ is spacetime?

On open $U \subseteq X$, causal/chronological cones $I^{\pm}(U) = J^{\pm}(U)$ same.
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Closure operator is $I^+: \text{ISub}(C) \to \text{ISub}(C)$
satisfying $s \leq I^+(s) \geq I^+(I^+(s))$ and monotone
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Restriction = propagation
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Restriction = propagation

Teleportation only successful on intersection of Alice and Bob’s cones
This is just the beginning

- Continuous extension of higher quantum theory
- Infinite dimension with standard methods
- Deformation quantization?
- Relativistic quantum theory: summoning?
- Graphical calculus?
- Logic?