Frobenius structures over Hilbert C*-modules

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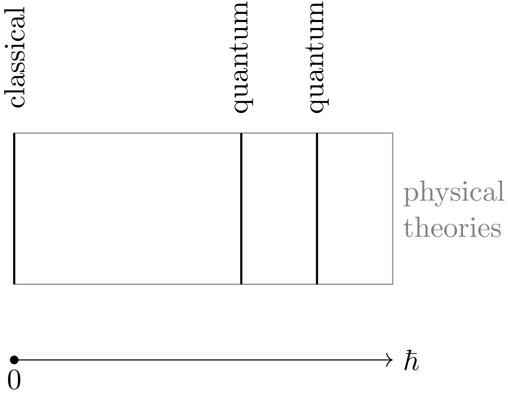
Naive quantum field theory

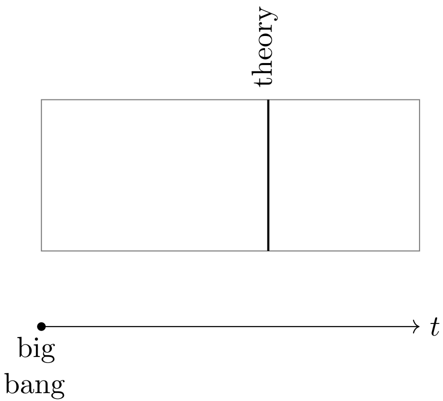
*“**Categorical quantum mechanics needs to come to terms with infinite-dimensionality.** This paper is just dipping its toe into those cold waters, working with situations where all infinite-dimensionality is concentrated in the commutative part. But the theorems are sufficiently simple and beautiful that they will stand the test of time.”*

– *Anonymous referee*

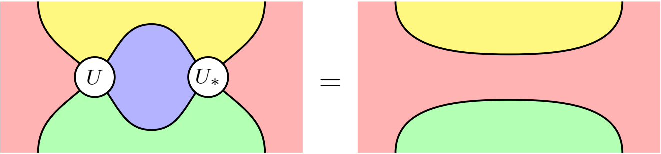
Classical limit



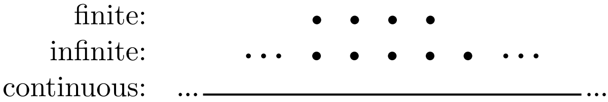
Theories varying over time



Continuing higher categories

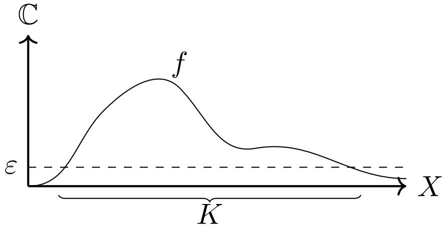


Colours = classical outcomes



Base space

Let X be locally compact Hausdorff space.
$C_0(X) = \{f\colon X \to \mathbb{C} \text{ cts} \mid \forall \varepsilon > 0\, \exists K \subseteq X \text{ cpt}\colon f(X \setminus K) < \varepsilon\}$



$C_b(X) = \{f\colon X \to \mathbb{C} \text{ cts} \mid \exists \|f\| < \infty\, \forall t \in X\colon |f(t)| \leq \|f\|\}$

Hilbert spaces

$\mathbb{C}$-module H with complete inner product valued in $\mathbb{C}$

tensor product over $\mathbb{C}$	monoidal category
tensor unit $\mathbb{C}$	tensor unit I
complex numbers $\mathbb{C}$	scalars $I \to I$
finite dimension	dual objects
adjoints	dagger
orthonormal basis	commutative dagger Frobenius structure
C*-algebra	dagger Frobenius structure

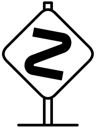
Hilbert modules

$C_0(X)$-module with complete inner product valued in $C_0(X)$

tensor product over $C_0(X)$	monoidal category
tensor unit $C_0(X)$	tensor unit I
$C_b(X)$	scalars $I \to I$
?	dual objects
adjointable morphisms	dagger
?	dagger Frobenius structure

"Scalars are not numbers"

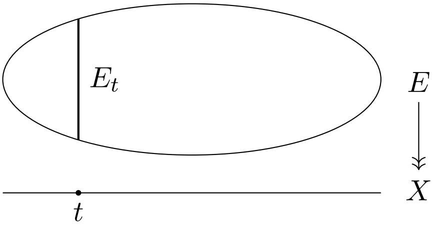
Beware



- Bounded map may not be adjointable
- Closed subspace may not be complemented
- Subobject of dual object may not be dual

Bundles of Hilbert spaces

Bundle $E \twoheadrightarrow X$, each fibre Hilbert space, operations continuous



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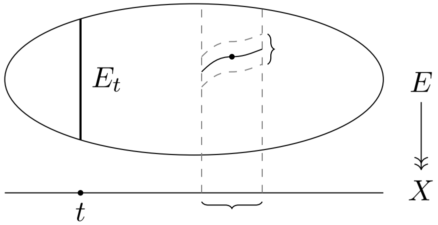

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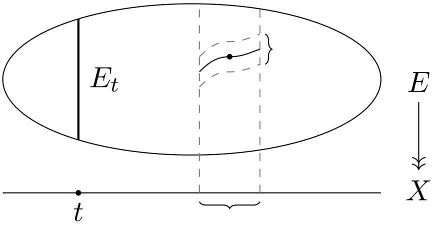
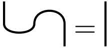


$$\begin{array}{rcl} \text{Hilbert } C_0(X)\text{-modules} & \simeq & \text{bundles of Hilbert spaces over } X \\ \text{sections vanishing at infinity} & \leftarrow\!\shortmid & E \twoheadrightarrow X \\ E & \mapsto & \text{localisation} \end{array}$$

Dual objects

E has dual object when $\cup\colon I \to E^* \otimes E$ and $\cap\colon E \otimes E^* \to I$ satisfy  and 

Dual objects

if X paracompact

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$$\iff$$

finite Hilbert bundle:
$\sup_{t \in X} \dim(E_t) < \infty$

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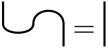
E has dual object when $\cup \colon I \to E^* \otimes E$ and $\cap \colon E \otimes E^* \to I$ satisfy  and 

$$\iff$$

finite Hilbert bundle:
$\sup_{t \in X} \dim(E_t) < \infty$

$$\iff$$

finitely presented projective Hilbert module:

$$\text{id} \circlearrowright E \underset{i^\dagger}{\overset{i}{\rightleftarrows}} C_0(X)^n$$

Dual objects

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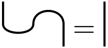
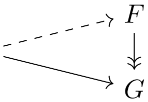
E has dual object when $\cup\colon I \to E^* \otimes E$ and $\cap\colon E \otimes E^* \to I$ satisfy  and 

$\iff$

finite Hilbert bundle:
$\forall t \in X\colon\ \dim(E_t) < \infty$

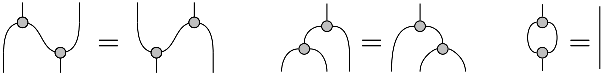
$\iff$

finitely generated projective Hilbert module:

$\operatorname{span}_{C_0(X)}(x_1, \ldots, x_n) = E$ 

Frobenius structures

E has special dagger Frobenius structure  : $E \otimes E \to E$:

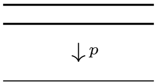


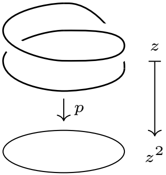
$$\iff$$

E is a finite bundle of C*-algebras:
each fibre is C*-algebra, operations continuous, $\sup \dim(E_t) < \infty$

Commutative Frobenius structures

$p\colon Y \twoheadrightarrow X$ finite covering: continuous, each $t \in X$ has open neighbourhood whose preimage is union of disjoint open sets homeomorphic to it, $\sup_{t \in X} |p^{-1}(t)| < \infty$





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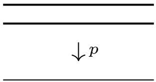

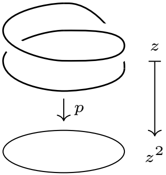

$C_0(Y)$ special dagger Frobenius: $\langle f \mid g\rangle(t) = \sum_{p(y)=t} f(y)^* g(y)$

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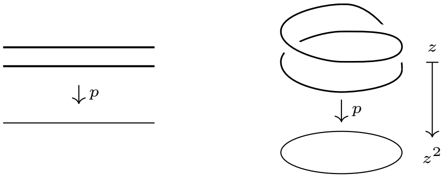

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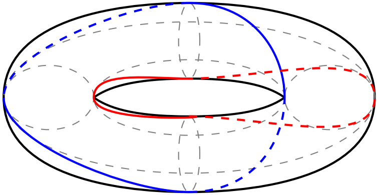
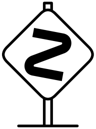
comultiplication comes from $Y \times_X Y = \{(a,b) \in S^1 \times S^1 \mid a^2 = b^2\}$



Caveats



- not determined by orthonormal basis of sections
  might be no copyable states at all!
  (even though category is well-pointed)

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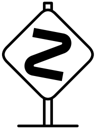

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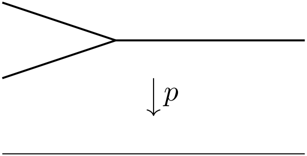
- unital Frobenius algebra $\iff$ finite Hilbert bundle



Nontrivial central Frobenius structure

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$$
$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$
$$X = S^2 = \{t \in \mathbb{R}^3 \mid \|t\| = 1\}$$

$$\left\{ x \in C_0(\mathbb{D}, \mathbb{M}_n) \colon\ x(z) = \begin{pmatrix} \overline{z} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} x(1) \begin{pmatrix} z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \text{ if } z \in S^1 \right\}$$

is special dagger Frobenius structure: $\langle x \mid y \rangle(t) = \mathrm{tr}(x(t)^* y(t))$

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is special dagger Frobenius structure: $\langle x \mid y \rangle(t) = \mathrm{tr}(x(t)^* y(t))$

central: $Z(E) = C_0(X)$

Transitivity

E is special dagger Frobenius structure in $\mathbf{Hilb}_{C_0(X)}$

$$\Longleftrightarrow$$

E is special dagger Frobenius structure in $\mathbf{Hilb}_{Z(E)}$

and

$Z(E)$ is specialisable dagger Frobenius structure in $\mathbf{Hilb}_{C_0(X)}$

Additive structure

With adjointable morphisms:

- Direct sums provide dagger biproducts
- $\mathbf{FHilb}_{C_0(X)}$ has dagger kernels
- $\mathbf{Hilb}_{C_0(X)}$ has dagger kernels $\implies X$ totally disconnected
- clopen subsets of $X \quad \leftrightarrow \quad$ dagger subobjects of tensor unit

Conjecture: if compact dagger category has dagger biproducts, equalisers, and isometries are kernels, it embeds into $\mathbf{FHilb}_{C_0(X)}$

This is just the beginning

Message:

- New category for arsenal
- Scalars are not numbers
- Frobenius structures may have no copyable states

Next talk:

- Abstract spatial structure to any monoidal category