Frobenius structures over Hilbert C*-modules

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Naive quantum field theory

“Categorical quantum mechanics needs to come to terms with infinite-dimensionality. This paper is just dipping its toe into those cold waters, working with situations where all infinite-dimensionality is concentrated in the commutative part. But the theorems are sufficiently simple and beautiful that they will stand the test of time.”

– Anonymous referee
Classical limit
Theories varying over time

big bang

theory

$\rightarrow t$

big bang
Continuing higher categories

Colours = classical outcomes

finite: 

infinite: 

continuous: 

...
Let $X$ be locally compact Hausdorff space.

$C_0(X) = \{ f : X \to \mathbb{C} \text{ cts} \mid \forall \varepsilon > 0 \exists K \subseteq X \text{ cpt: } f(X \setminus K) < \varepsilon \}$

$C_b(X) = \{ f : X \to \mathbb{C} \text{ cts} \mid \exists \| f \| < \infty \forall t \in X : |f(t)| \leq \| f \| \}$
Hilbert spaces

$\mathbb{C}$-module $H$ with complete inner product valued in $\mathbb{C}$

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Hilbert modules

$C_0(X)$-module with complete inner product valued in $C_0(X)$

tensor product over $C_0(X)$
tensor unit $C_0(X)$
$C_b(X)$
? adjointable morphisms
? monoidal category

tensor unit $I$
scalars $I \rightarrow I$
dual objects
dagger
dagger Frobenius structure

“ Scalars are not numbers”
Beware

- Bounded map may not be adjointable
- Closed subspace may not be complemented
- Subobject of dual object may not be dual
Bundles of Hilbert spaces

Bundle $E \rightarrow X$, each fibre Hilbert space, operations continuous

\[ E_t \]

\[ E \rightarrow X \]
Bundles of Hilbert spaces

Bundle $E \to X$, each fibre Hilbert space, operations continuous, with
Bundles of Hilbert spaces

Bundle $E \rightarrow X$, each fibre Hilbert space, operations continuous, with

Hilbert $C_0(X)$-modules $\simeq$ bundles of Hilbert spaces over $X$
sections vanishing at infinity $\leftrightarrow E \rightarrow X$
$E \mapsto$ localisation
Dual objects

$E$ has dual object when $\bigcirc: I \to E^* \otimes E$ and $\bigcirc: E \otimes E^* \to I$ satisfy $\bigcirc \bigcirc = 1$ and $\bigcirc \bigcirc = 1$
Dual objects

if $X$ paracompact

$E$ has dual object when $\bigcirc: I \rightarrow E^* \otimes E$ and $\bigodot: E \otimes E^* \rightarrow I$ satisfy $\bigcirc = \bigodot$ and $\bigotimes = \bigcirc$

$\iff$

finite Hilbert bundle:

$\sup_{t \in X} \dim(E_t) < \infty$
Dual objects

if $X$ paracompact

$E$ has dual object when $\bigodot : I \to E^* \otimes E$ and $\bigodot : E \otimes E^* \to I$ satisfy $\bigodot = \bigodot$ and $\bigodot = \bigodot$

$\iff$

finite Hilbert bundle:

$\sup_{t \in X} \dim(E_t) < \infty$

$\iff$

finitely presented projective Hilbert module:

$id \bigodot E \xrightarrow{i} C_0(X)^n$
Dual objects

if $X$ compact

$E$ has dual object when $\cap : I \to E^* \otimes E$ and $\cup : E \otimes E^* \to I$ satisfy $\bigcap = |$ and $\bigcup = |$

$\iff$

finite Hilbert bundle:
$\forall t \in X : \dim(E_t) < \infty$

$\iff$

finitely generated projective Hilbert module:
$\text{span}_{C_0(X)}(x_1, \ldots, x_n) = E$
Frobenius structures

$E$ has special dagger Frobenius structure $\otimes: E \otimes E \to E:\n$ 

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$\iff$

$E$ is a finite bundle of C*-algebras:

each fibre is C*-algebra, operations continuous, sup dim$(E_t) < \infty$
Commutative Frobenius structures

\( p: Y \rightarrow X \) finite covering: continuous, each \( t \in X \) has open neighbourhood whose preimage is union of disjoint open sets homeomorphic to it, \( \sup_{t \in X} |p^{-1}(t)| < \infty \)
Commutative Frobenius structures

\[ p: Y \to X \text{ finite covering: continuous, each } t \in X \text{ has open neighbourhood whose preimage is union of disjoint open sets homeomorphic to it, } \sup_{t \in X} |p^{-1}(t)| < \infty \]

\[ C_0(Y) \text{ special dagger Frobenius: } \langle f | g \rangle(t) = \sum_{p(y) = t} f(y)^* g(y) \]
Commutative Frobenius structures

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\[ C_0(Y) \text{ special dagger Frobenius: } \langle f | g \rangle(t) = \sum_{p(y) = t} f(y)^* g(y) \]

comultiplication comes from \( Y \times_X Y = \{(a, b) \in S^1 \times S^1 \mid a^2 = b^2\} \)
Caveats

- not determined by orthonormal basis of sections
  might be no copyable states at all!
  (even though category is well-pointed)
Caveats

- not determined by orthonormal basis of sections
  might be no copyable states at all!
  (even though category is well-pointed)

- unital Frobenius algebra \(\iff\) finite Hilbert bundle
Nontrivial central Frobenius structure

\[ \mathbb{D} = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \]
\[ S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \]
\[ X = S^2 = \{ t \in \mathbb{R}^3 \mid \|t\| = 1 \} \]

\[ \begin{cases} x \in C_0(\mathbb{D}, \mathbb{M}_n): & x(z) = \begin{pmatrix} \bar{z} & 1 \\ 1 & 1 \end{pmatrix} x(1) \begin{pmatrix} \bar{z} & 1 \\ 1 & 1 \end{pmatrix} \text{ if } z \in S^1 \end{cases} \]

is special dagger Frobenius structure: \( \langle x \mid y \rangle(t) = \text{tr}(x(t)^* y(t)) \)
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is special dagger Frobenius structure: \[ \langle x \mid y \rangle(t) = \text{tr}(x(t)^* y(t)) \]

central: \[ Z(E) = C_0(X) \]
Transitivity

\[ E \text{ is special dagger Frobenius structure in } \text{Hilb}_{C_0(X)} \]

\[ \iff \]

\[ E \text{ is special dagger Frobenius structure in } \text{Hilb}_{Z(E)} \]

and

\[ Z(E) \text{ is specialisable dagger Frobenius structure in } \text{Hilb}_{C_0(X)} \]
Additive structure

With adjointable morphisms:

- Direct sums provide dagger biproducts
- $\text{FHilb}_{C_0}(X)$ has dagger kernels
- $\text{Hilb}_{C_0}(X)$ has dagger kernels $\implies X$ totally disconnected
- clopen subsets of $X \leftrightarrow$ dagger subobjects of tensor unit

Conjecture: if compact dagger category has dagger biproducts, equalisers, and isometries are kernels, it embeds into $\text{FHilb}_{C_0}(X)$
This is just the beginning

Message:

- New category for arsenal
- Scalars are not numbers
- Frobenius structures may have no copyable states

Next talk:

- Abstract spatial structure to any monoidal category