The geometry of Boolean algebra

Chris Heunen

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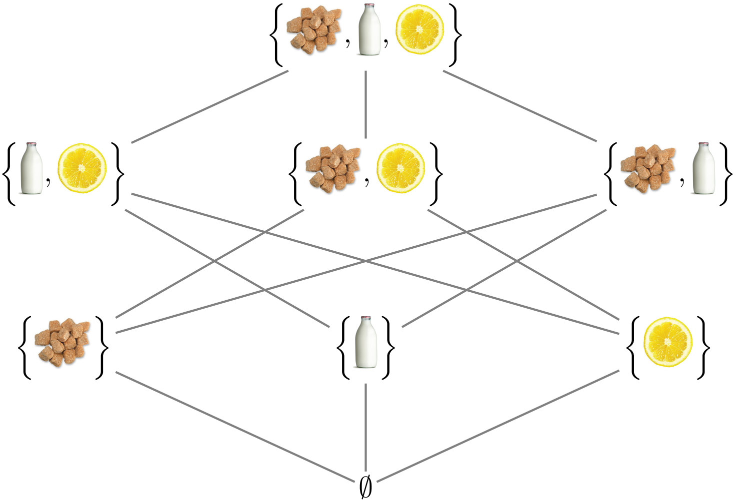
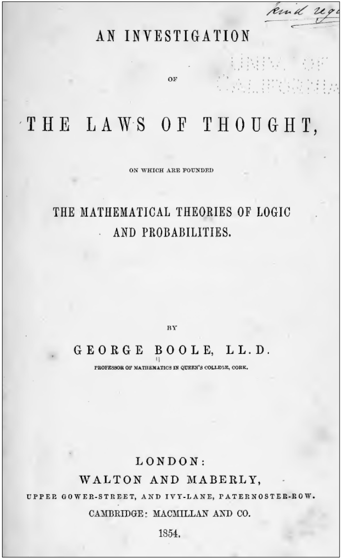
informatics

Boolean algebra: example



Boole's algebra

Boolean algebra ≠ Boole's algebra



AN INVESTIGATION

OF

THE LAWS OF THOUGHT,

ON WHICH ARE FOUNDED

THE MATHEMATICAL THEORIES OF LOGIC AND PROBABILITIES.

BY

GEORGE BOOLE, LL.D.

PROFESSOR OF MATHEMATICS IN QUEEN'S COLLEGE, CORK.

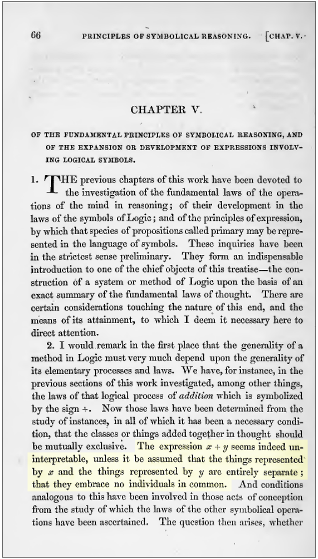
LONDON:

WALTON AND MABERLY,

UPPER GOWER-STREET, AND IVY-LANE, PATERNOSTER-ROW.

CAMBRIDGE: MACMILLAN AND CO.

1854.



66 PRINCIPLES OF SYMBOLICAL REASONING. [CHAP. V.

CHAPTER V.

OF THE FUNDAMENTAL PRINCIPLES OF SYMBOLICAL REASONING, AND OF THE EXPANSION OR DEVELOPMENT OF EXPRESSIONS INVOLVING LOGICAL SYMBOLS.

1. THE previous chapters of this work have been devoted to the investigation of the fundamental laws of the operations of the mind in reasoning; of their development in the laws of the symbols of Logic; and of the principles of expression, by which that species of propositions called primary may be represented in the language of symbols. These inquiries have been in the strictest sense preliminary. They form an indispensable introduction to one of the chief objects of this treatise—the construction of a system or method of Logic upon the basis of an exact summary of the fundamental laws of thought. There are certain considerations touching the nature of this end, and the means of its attainment, to which I deem it necessary here to direct attention.

2. I would remark in the first place that the generality of a method in Logic must very much depend upon the generality of its elementary processes and laws. We have, for instance, in the previous sections of this work investigated, among other things, the laws of that logical process of *addition* which is symbolized by the sign +. Now those laws have been determined from the study of instances, in all of which it has been a necessary condition, that the classes or things added together in thought should be mutually exclusive. The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common. And conditions analogous to this have been involved in those acts of conception from the study of which the laws of the other symbolical operations have been ascertained. The question then arises, whether

Boolean algebra = Jevon's algebra

PURE LOGIC

OR THE

LOGIC OF QUALITY APART FROM QUANTITY:

WITH

REMARKS ON BOOLE'S SYSTEM AND

ON THE RELATION OF LOGIC AND MATHEMATICS.

BY

William

W. STANLEY JEVONS, M.A.

Logica est ars artium et scientia scientiarum.—SCOTUS.

LONDON:

EDWARD STANFORD, 6 CHARING CROSS.

1864.

is AB; if it is C, is AC, and it is therefore either AB or AC.

Use of brackets.

67. Let a plural term enclosed in brackets (. . . . . .), and placed beside another term, mean that it is combined with it, as one single term is with another:

Thus A (B+C) = AB+AC.

Combination of plural terms.

68. One plural term is combined with another by combining each alternative of the one separately with each of the other. Each combined alternative may then be combined with each alternative of a third plural term, and so on:

Thus (D+E) (B+C)=B (D+E) +C (D+E)
=BD+BE+CD+CE.

Law of unity.

69. It is in the nature of thought and things that *same alternatives are together same in meaning, as any one taken singly.*

Thus, what is the same as A *or* A is the same as A, a self-evident truth.

A+A=A A+A+A=A A+A+B=A+B

This law is correlative to the Law of Simplicity, (§ 39), and is perhaps of equal importance and frequent use. It was not recognised by Professor Boole, when laying down the principles of his system.

Superfluous terms.

70. In a plural term, any alternative may be removed, of which a part forms another alternative. Thus the term *either B or BC* is the same in meaning with B alone, or B+BC=B. For it is a self-evident truth (§ 99) that B standing alone is either the same as BC, or as B *not*-C. Thus
B+BC=B *not*-C+BC+BC
=B *not*-C+BC=B.

Boole's algebra isn't Boolean algebra

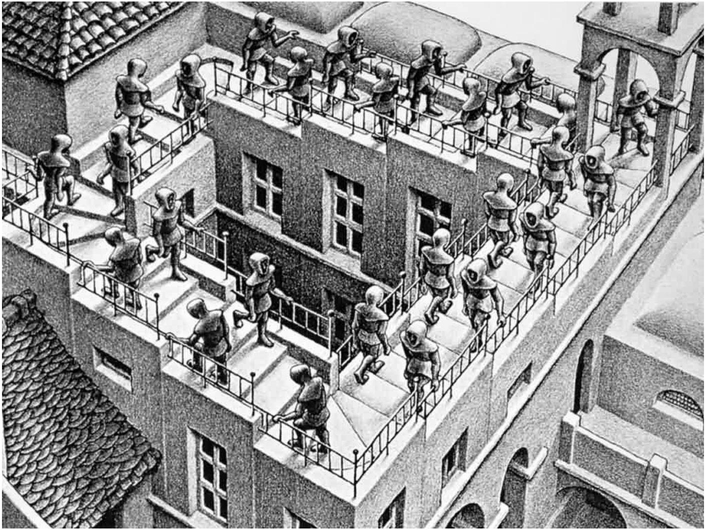
Boole's Algebra Isn't Boolean Algebra

A description, using modern algebra, of what Boole really did create.

THEODORE HAILPERIN
Lehigh University
Bethlehem, PA 18015

To Boole and his mid-nineteenth century contemporaries, the title of this article would have been very puzzling. For Boole's first work in logic, *The Mathematical Analysis of Logic*, appeared in 1847 and, although the beginnings of modern abstract algebra can be traced back to the early part of the nineteenth century, the subject had not fully emerged until towards the end of the century. Only then could one clearly distinguish and compare algebras. (We use the term **algebra** here as standing for a formal system, not a structure which realizes, or is a model for, it—for instance, the algebra of integral domains as codified by a set of axioms *versus* a particular structure, e.g., the integers, which satisfies these axioms.) Granted, however, that this later full degree of understanding has been attained, and that one can conceptually distinguish algebras, is it not true that Boole's "algebra of logic" is Boolean algebra?

Contextuality



Orthoalgebra: definition

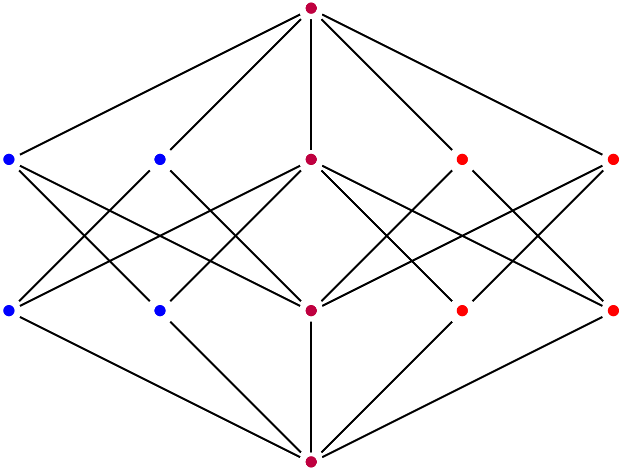
An orthoalgebra is a set A with

- a *partial* binary operation $\oplus\colon A \times A \to A$
- a unary operation $\neg\colon A \to A$
- distinguished elements $0, 1 \in A$

such that

- $\oplus$ is commutative and associative
- $\neg a$ is the unique element with $a \oplus \neg a = 1$
- $a \oplus a$ is defined if and only if $a = 0$

Orthoalgebra: example

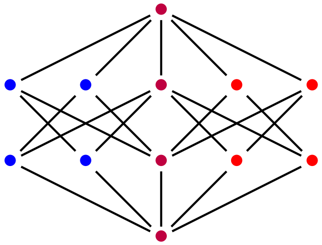


Orthodomain: definition

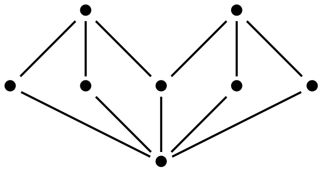
Given a piecewise Boolean algebra A,
its orthodomain $\mathrm{BSub}(A)$
is the collection of its Boolean subalgebras,
partially ordered by inclusion.

Orthodomain: example

Example: if A is



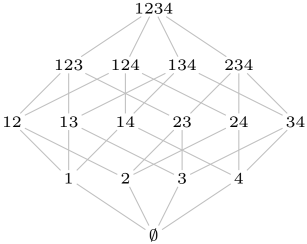
then $\mathrm{BSub}(A)$ is



Orthoalgebra: pitfalls

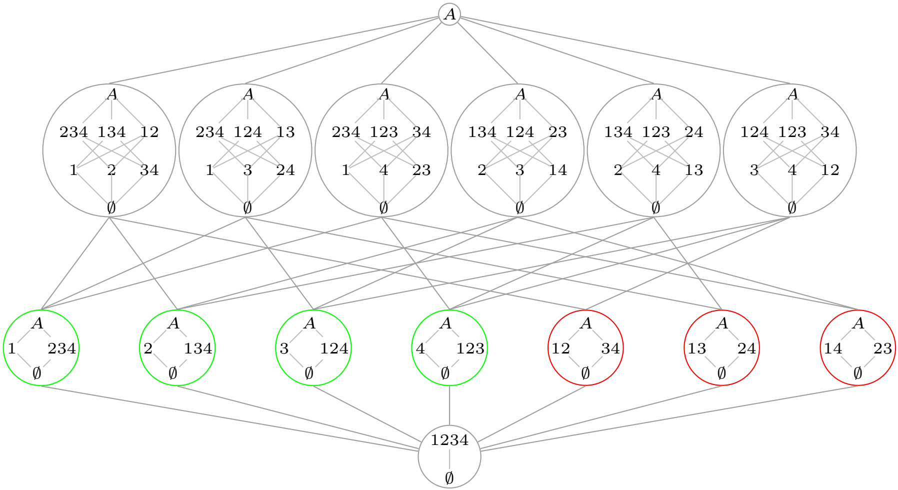
- subalgebras of a Boolean orthoalgebra need not be Boolean
- intersection of two Boolean subalgebras need not be Boolean
- two Boolean subalgebras might have no meet
- two Boolean subalgebras might have upper bound but no join

Different kinds of atoms

If $A =$ 

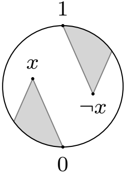
, then $\mathrm{BSub}(A) = \cdots$

Different kinds of atoms



Principal pairs

Reconstruct pairs $(x, \neg x)$ of A:



- *principal ideal subalgebra* of A is of the form
- they are the elements p of $\mathrm{BSub}(A)$ that are
  dual modular and $(p \vee m) \wedge n = p \vee (m \wedge n)$ for $n \geq p$
  atom or *relative complement* $a \wedge m = a$, $a \vee m = A$ for atom a

Principal pairs

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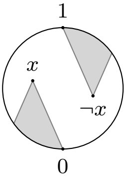

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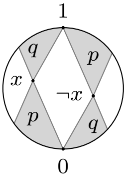
Reconstruct elements x of A:

- *principal pairs* of A are (p, q) with atomic meet



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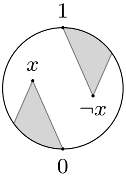

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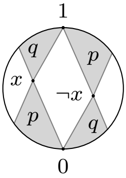
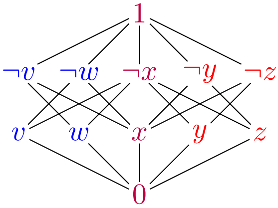
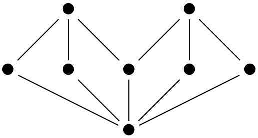


Theorem: $A \simeq \mathrm{Pp}(\mathrm{BSub}(A))$ for Boolean algebra A of size ≥ 4
$D \simeq \mathrm{BSub}(\mathrm{Pp}(D))$ for Boolean domain D of size ≥ 2

Directions

If A is 

then BSub(A) is 

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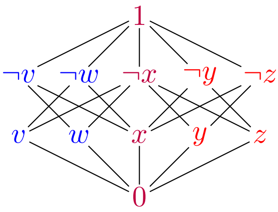
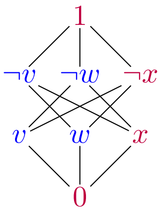
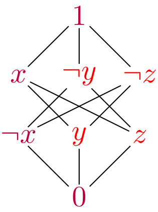
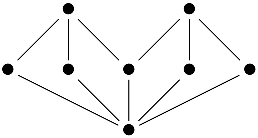
If A is 

or 



then $\mathrm{BSub}(A)$ is 

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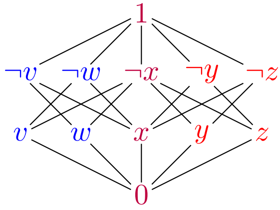
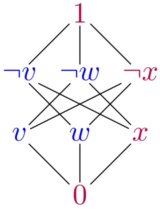
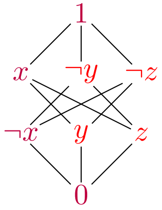
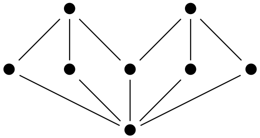
If A is 

or 



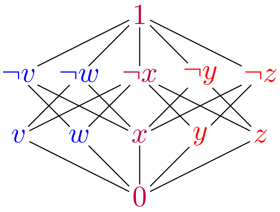
then $\mathrm{BSub}(A)$ is 

A direction for a Boolean domain is a map $d\colon D \to D^2$ with

- $d(1) = (p, q)$ is a principal pair
- $d(m) = (p \wedge m, q \wedge m)$

Directions

If A is



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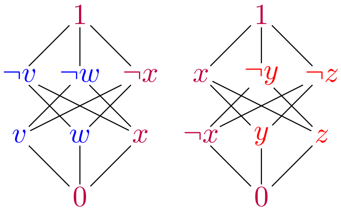

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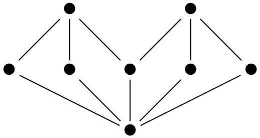


A direction for a orthodomain is a map $d\colon D \to D^2$ with

- if $a \leq m$ then $d(m)$ is a principal pair with meet a in m
- $d(m) = \bigvee\{(m,m) \wedge f(n) \mid a \leq n\}$
- if m, n cover a, $d(m) = (a, m)$, $d(n) = (n, a)$, then $m \vee n$ exists

Orthoalgebras and orthodomains

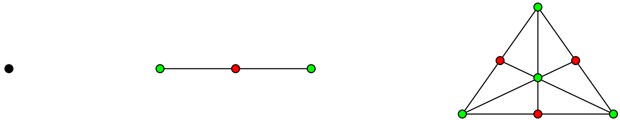
Lemma: If an atom in an orthodomain has a direction, then it has exactly two directions

Theorem:

- $A \simeq \mathrm{Dir}(\mathrm{BSub}(A))$ for orthoalgebra A
  whose blocks have > 4 elements
- $D \simeq \mathrm{BSub}(\mathrm{Dir}(D))$ for orthodomain D
  that has enough directions and is tall

Orthohypergraphs

An orthohypergraph is consists of a set of points, a set of lines, and a set of planes. A line is a set of 3 points, and a plane is a set of 7 points where the restriction of the lines to these 7 points is as:



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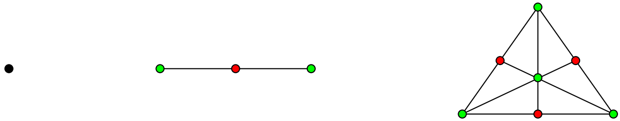
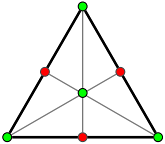


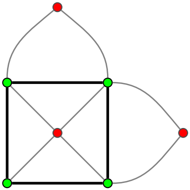
Every orthoalgebra/orthodomain gives rise to an orthohypergraph:

- points are Boolean subalgebras of size 4
- lines are Boolean subalgebras of size 8
- planes are Boolean subalgebras of size 16

Projective geometry

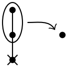
- Any two lines intersect in at most one point.
- Any two points lie on a line or plane.
- For orthomodular posets: if it looks like a plane, it is a plane.





Orthohypergraph morphisms

Morphism of orthohypergraphs is partial function such that:

- 
  none defined
  
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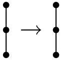
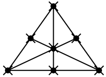
  
  isomorphism
- 
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  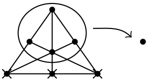
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  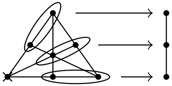
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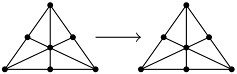
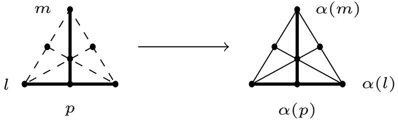
  
  isomorphism
- If lines l, m intersect in point p, and lines $\alpha(l) \neq \alpha(m)$ in plane t' intersect in edge point $\alpha(p)$, then l, m lie in plane t that is mapped isomorphically to t':



Orthodomains and orthohypergraphs

Theorem: functor that sends orthoalgebra to its orthohypergraph:

- is essentially surjective on objects
- is injective on objects except on 1- and 2-element orthoalgebras
- is full on *proper* morphisms
- is faithful on *proper* morphisms

So for all intents and purposes is equivalence

Conclusion

- Orthoalgebra: Boolean algebra as Boole intended
- Orthodomain: shape of parts enough to determine whole
- Orthohypergraph: (projective) geometry of contextuality

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