How to realise arbitrary (in)compatibilities with quantum observables

Tobias Fritz   Chris Heunen   Ravi Kunjwal   Manny Reyes

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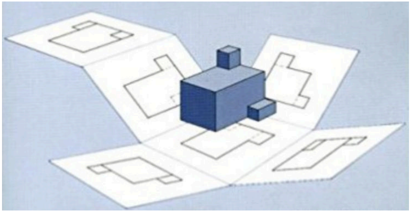
Overview:

- Yes-no questions: projections
- Sharp measurements: projection-valued measures
- Unsharp measurements: positive operator-valued measures

Yes-no questions

A projection is a bounded operator $p\colon H \to H$ with $p^2 = p = p^\dagger$.



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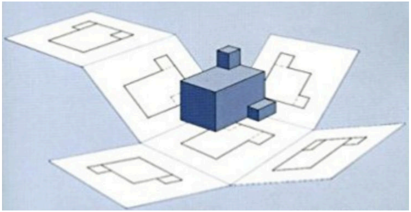

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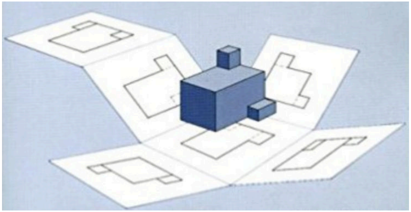

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- Vertices are observables
- Edge between vertices when jointly measurable

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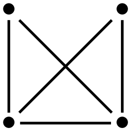
Theorem 1: Any graph can be realised

- First assume all vertices are connected except for two.



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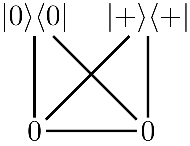

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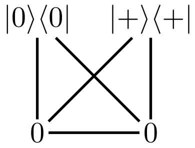


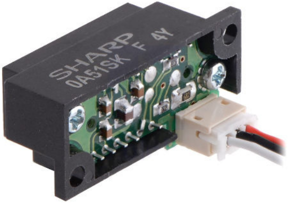
  Label vertex x with projection $p_x^{v \not\sim w}$.

- In general, take Hilbert space $\bigoplus_{v \not\sim w} \mathbb{C}^2$.
  Label vertex x with $\bigoplus_{v \not\sim w} p_x^{v \not\sim w}$

Sharp measurements

Projection-valued measure is covering set of orthogonal projections

$$i \neq j \implies p_i p_j = 0 \qquad\qquad \sum p_i = 1$$



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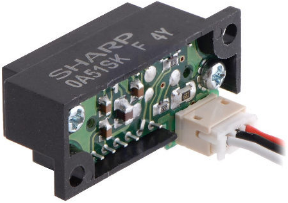

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Joint measurability is pairwise, so determines graph:

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Realising graphs with sharp measurements

Theorem 2: any graph can be realised with PVMs

- Realise with projections, and take PVM $P(x) = \{p_x, 1 - p_x\}$
- Can extend to PVMs with $n > 2$ outcomes:
  extend Hilbert space to $H \oplus \bigoplus_x \mathbb{C}^{n-2}$
  extend $P(x)$ to $\{p_x \oplus 0,\ (1 - p_x) \oplus 0,\ 0 \oplus |i\rangle\langle i|\}$

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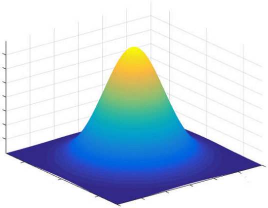
Kochen-Specker: there is 'a lot of room' in $\mathbb{C}^3$
But: no fixed dimension d can realise all graphs as PVMs

Unsharp measurements

Positive-operator valued measure is covering set of effects

$$0 \le E(i) \le 1 \qquad \sum E(i) = 1$$



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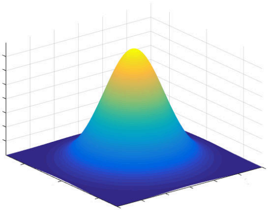

POVMs are jointly measurable when there is a *joint* POVM E of which they are *marginals*

$$E_1(i_1) = \sum_{i_2, i_3, \ldots} E(i_1, i_2, i_3)$$

$$E_2(i_2) = \sum_{i_1, i_3, \ldots} E(i_1, i_2, i_3)$$

Hypergraphs

Joint measurability of POVMs is not pairwise!

$$
\begin{aligned}
E_1(\pm) &= \tfrac{1}{2}\mathbf{1} \pm \tfrac{\eta}{2} X \cdot (1,0,0) \\
E_2(\pm) &= \tfrac{1}{2}\mathbf{1} \pm \tfrac{\eta}{2} Y \cdot (0,1,0) \\
E_3(\pm) &= \tfrac{1}{2}\mathbf{1} \pm \tfrac{\eta}{2} Z \cdot (0,0,1)
\end{aligned}
$$

- Pairwise measurable $\iff \eta \leq 1/\sqrt{2}$
- Triplewise measurable $\iff \eta \leq 1/\sqrt{3}$

Dilation

Theorem (Neumark dilation): For a POVM $\{E(i)\}$ on H, there are

- a Hilbert space K
- an isometry $V \colon H \to K$
- and a PVM $\{P(i)\}$ on K

satisfying

$$E(i) = V^\dagger \, P(i) \, V \, .$$

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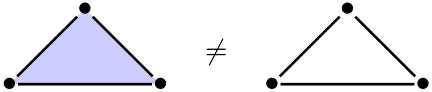
But: this need not reflect joint measurability relations!

Hypergraphs

Hypergraph:

- Vertices are observables
- Hyperedge between vertices when jointly measurable



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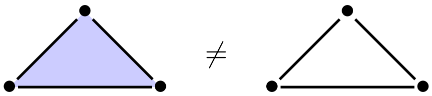

In fact: abstract simplicial complex

- If set of observables jointly measurable, so is any subset
- Hyperedges have finite number of vertices

Realising hypergraphs with unsharp measurements

Theorem 3: any hypergraph can be realised with POVMs

Proof:

1. Analyse when POVMs are(n't) jointly measurable
2. Realise simplest interesting hypergraph
3. Reduce arbitrary hypergraph to combination of simple ones

Step 1: Clifford algebra

A Clifford algebra consists of a finite set of hermitian matrices $\Gamma_1, \ldots, \Gamma_N$ satisfying

$$\Gamma_j \Gamma_k + \Gamma_k \Gamma_j = 2\delta_{jk} \mathbf{1}.$$

It follows that $\mathrm{Tr}(\Gamma_i) = 0$. They describe *spinors* and *Dirac equation*.

Construction:

- For $N = 1$: set $\Gamma_1 = \mathbf{1}$ on $H = \mathbb{C}$
- For higher N:

$$\begin{aligned} H &\rightsquigarrow H \otimes \mathbb{C}^2 \\ \Gamma_i &\rightsquigarrow \Gamma_i \otimes Z \\ \Gamma_{N+1} &= \mathbf{1} \otimes X \\ \Gamma_{N+2} &= \mathbf{1} \otimes Y \end{aligned}$$

Step 2: Minimal incompatible sets

- Set $E_i(\pm) = \frac{1}{2}(\mathbf{1} \pm \eta\Gamma_i)$.
- Then $\{E_i(\pm)\}$ jointly measurable $\iff \eta \leq 1/\sqrt{N}$
- Observe this does not rely on any ordering of the Γ_i
- So can realize hypergraph with N vertices where every $N-1$ vertex subset is jointly measurable, but not all N vertices

Step 3: Put it all together

- Identify minimal incompatible sets of vertices
- Realise each one on H_i; assign vertices outside the set $\mathbf{1}$
- Take $H = \bigoplus H_i$

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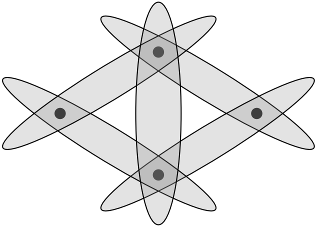
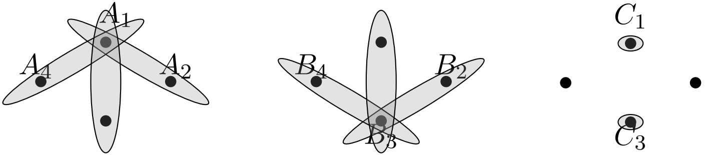
Works because:

- For edge e, compatible on each H_i, so on H
- If e' not an edge, contained in some minimal incompatible set, so incompatible on H_i, so incompatible on H

Example





$$
\begin{array}{ccc}
A_1 = \frac{1}{2}(\mathbf{1} + Z/\sqrt{2}) & B_1 = \{0, \mathbf{1}\} & C_1 = \frac{1}{2}(\mathbf{1} + Z) \\
A_2 = \frac{1}{2}(\mathbf{1} + X/\sqrt{2}) & B_2 = \frac{1}{2}(\mathbf{1} + Z/\sqrt{2}) & C_2 = \{0, \mathbf{1}\} \\
A_3 = \{0, \mathbf{1}\} & B_3 = \frac{1}{2}(\mathbf{1} + X/\sqrt{2}) & C_3 = \frac{1}{2}(\mathbf{1} + X) \\
A_4 = \frac{1}{2}(\mathbf{1} + Y/\sqrt{2}) & B_4 = \frac{1}{2}(\mathbf{1} + Y/\sqrt{2}) & C_4 = \{0, \mathbf{1}\}
\end{array}
$$

$$E_i(\pm) = A_i(\pm) \oplus B_i(\pm) \oplus C_i(\pm)$$

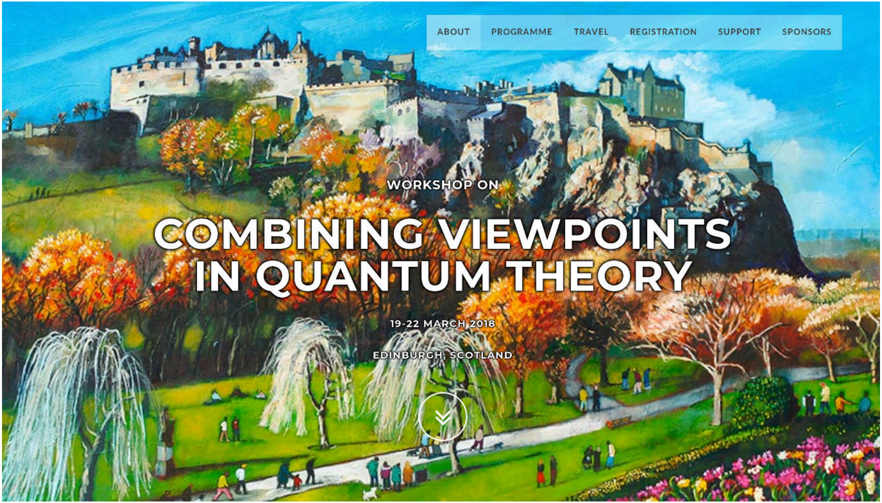
Conclusion

Can realise any:

- Graph with projections
- Graph with PVMs
- Hypergraph with POVMs

Further:

- Optimal dimension?
- Homology?
- Protocols?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