Compact inverse categories

Robin Cockett  Chris Heunen
Inverse monoids

Every $x$ has $x^\dagger$ with $x = xx^\dagger x$, and $x^\dagger xy^\dagger y = y^\dagger yx^\dagger x$

- any group
- any semilattice
- untyped reversible computation
- partial injections on fixed set
Theorem (Ehresmann-Schein-Nambooripad):
\{\text{inverse monoids}\} \simeq \{\text{inductive groupoids}\}

(groupoid in category of posets, étale for Alexandrov topology, objects are semilattice)
(Commutative) inverse monoids

**Theorem (Ehresmann-Schein-Nambooripad):**
\{inverse monoids\} \simeq \{inductive groupoids\}
(groupoid in category of posets, étale for Alexandrov topology, objects are semilattice)

**Theorem (Jarek):**
\{commutative inverse monoids\} \simeq \{semilattices of abelian groups\}
(functor from a semilattice to category of abelian groups)
Inverse categories

Every $f$ has $f^\dagger$ with $f = ff^\dagger f$, and $f^\dagger fg^\dagger g = g^\dagger gf^\dagger f$

- fundamental groupoid of pointed topological space
- sets and partial injections
- typed reversible computation
Inverse categories

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**Theorem (DeWolf-Pronk):**

\[
\{\text{inverse categories}\} \simeq \{\text{locally complete inductive groupoids}\}
\]

(groupoid in category of posets, étale for Alexandrov topology, objects are coproduct of semilattices)
Structure theorems

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Semilattices of categories

Semilattice is partial order with greatest lower bounds $s \land t$ and $\top$.

Semilattice over a subcategory $\mathbf{V} \subseteq \mathbf{Cat}$ is functor $F : \mathbf{S}^{\text{op}} \to \mathbf{V}$ where $\mathbf{S}$ is semilattice, all categories $F(s)$ have the same objects.
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\[
\begin{array}{ccc}
S^{\text{op}} & \xrightarrow{F} & \mathbf{V} \\
\downarrow & & \downarrow \\
S'^{\text{op}} & \xrightarrow{F'} & \mathbf{V}
\end{array}
\]

Theorem (Jarek): \text{cInvMon} \simeq \text{SLat}\,[\text{Ab}]

\[
M \mapsto S = \{ s \in M \mid ss^\dagger = s \} \\
F(s) = \{ x \in M \mid xx^\dagger = s \}
\]

\[\bigsqcup_s F(s) \leftrightarrow F\]
The one-object case

\{\text{commutative inverse monoids}\} \simeq \{\text{one-object compact inverse cats}\}
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Symmetric monoidal, every object has \text{dual}

\eta: I \to A^* \otimes A \text{ with } (\varepsilon \otimes 1) \circ (1 \otimes \eta) = 1 \text{ for } \varepsilon = \sigma \circ \eta^\dagger

- A and $A^*$ adjoint in one-object 2-category
- any abelian group as discrete monoidal category
- fundamental groupoid of pointed topological space

\[ \begin{align*}
\text{In any monoidal category:} \\
\text{scalars } I \to I \text{ form commutative monoid} \\
\text{I dual to itself}
\end{align*} \]
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\vcenter{\hbox{\includegraphics[width=0.6\textwidth]{diagram.png}}}

In any monoidal category:

- scalars \( I \to I \) form commutative monoid
- \( I \) dual to itself
Compact categories

- scalar multiplication of $f : A \rightarrow B$ with $s : I \rightarrow I$

\[
\begin{array}{c}
A \xrightarrow{s \cdot f} B \\
\sim \downarrow \quad \quad \quad \quad s \otimes f \quad \quad \uparrow \sim \\
I \otimes A \xrightarrow{s \otimes f} I \otimes B
\end{array}
\]
Compact categories

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- dual morphism of $f : A \to B$

\[
f^* = (1 \otimes \varepsilon) \circ (1 \otimes f \otimes 1) \circ (\eta \otimes 1) : B^* \to A^*
\]
Compact categories

- **scalar multiplication of** $f: A \to B$ with $s: I \to I$

  $$A \xrightarrow{s \cdot f} B$$
  $$I \otimes A \xrightarrow{s \otimes f} I \otimes B$$

- **dual morphism of** $f: A \to B$

  $$f^* = (1 \otimes \varepsilon) \circ (1 \otimes f \otimes 1) \circ (\eta \otimes 1): B^* \to A^*$$

- **trace of** $f: A \to A$

  $$\text{Tr}(f) = \varepsilon \circ (f \otimes 1) \circ \eta: I \to I$$
  $$\text{tr}(f) = \text{Tr}(f)^*$$
**Endomorphisms**

**Lemma**: endomorphism $f$ in compact inverse category is $\text{tr}(f) \cdot 1$
Endomorphisms

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**Proof:**

1. because $h = hh^\dagger h$:  

$$
\begin{array}{c}
\circ = \bigcap \bigcap = \bigcap = \\
\end{array}
$$
**Endomorphisms**

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1. because $h = hh^\dagger h$:

2. $gg^\dagger$ and $hh^\dagger$ commute:

3. by 1 and 2:

4. therefore:
Arbitrary morphisms

Corollary: compact dagger category is compact inverse category

\[ \iff \]

every morphism \( f \) satisfies \( f = \text{tr}(ff^\dagger) \bullet f \)

Proof: \( \implies \): \( ff^\dagger = \text{tr}(ff^\dagger ff^\dagger) \bullet 1 = \text{tr}(ff^\dagger) \bullet 1 \)

\( \iff \): restriction category with \( \bar{f} = \text{tr}(ff^\dagger) \bullet 1 \)

every map is restriction isomorphism
Semilattices of groupoids

**Theorem:** If $C$ is compact inverse category

- $S = \{s : I \to I \mid ss^\dagger = s\}$ is semilattice
- $s \in S$ induces compact groupoid $F(s)$ with same objects, and morphisms $F(s)(A, B) = \{f : A \to B \mid \text{tr}(ff^\dagger) = s\}$
- semilattice $F : S^{\text{op}} \to \text{CptGpd}$ of compact groupoids
Semilattices of groupoids

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If $F : S^{\text{op}} \to \text{CptGpd}$ is semilattice of compact groupoids

- inverse category $\mathbf{C}$ with same objects as $F(\top)$, and morphisms $\mathbf{C}(A, B) = \coprod_{s \in S} F(s)(A, B)$
Semilattices of groupoids

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- inverse category $\mathbf{C}$ with same objects as $F(\top)$, and morphisms $\mathbf{C}(A, B) = \bigsqcup_{s \in S} F(s)(A, B)$

Equivalence $\text{CptInvCat} \simeq \text{SLat}[\text{CptGpd}]$
2-categories

Redefinition of $\text{SLat}[\mathcal{V}]$ as 2-category:

Write $\text{SLat}_{=}[\mathcal{V}]$ for full subcategory where all $F(s)$ same objects
2-categories

Redefinition of \( \text{SLat}[V] \) as 2-category:

\[
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S^{\text{op}} & \xrightarrow{F} & \mathcal{V} \\
\varphi \leq & & \\
S'^{\text{op}} & \xleftarrow{F'} & \\
\end{array}
\]

Write \( \text{SLat}_{=}[V] \) for full subcategory where all \( F(s) \) same objects

Lemma: \( \text{SLat}[\text{CptGpd}] \simeq \text{SLat}_{=}[\text{CptGpd}] \)  
(Compare inductive groupoids)
Compact groupoids

**Proposition [Baez-Lauda]:** compact groupoids $\mathbf{C}$ are, up to $\simeq$:

- abelian group $G$ of isomorphism classes of $\mathbf{C}$ under $\otimes$, $I$, $A^*$
- abelian group $H$ of scalars $\mathbf{C}(I, I)$ under $\circ$, 1, $f^\dagger$
- conjugation action $G \times H \to H$ given by $(A, s) \mapsto \text{tr}(A \otimes s)$
- 3-cocycle $G \times G \times G \to H$ given by $(A, B, C) \mapsto \text{Tr}(\alpha_{A,B,C})$

**Proof:** make $\mathbf{C}$ skeletal, strictify everything but associators
Compact groupoids

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**Proof:** make $\mathbf{C}$ skeletal, strictify everything but associators

**Theorem:** $\text{CptInvCat} \simeq \text{SLat}[\text{Cocycle}]$
Traced inverse categories

What do traced inverse categories look like?

\[
\text{TrDagCat} \quad \perp \quad \text{CptDagCat} \quad \perp \quad \text{CptInvCat}
\]

\[
\text{TrInvCat}
\]
Open ends

- SLat[V] as completion procedure?
- Bratelli diagrams?
- description internal to Rel?