Sparse representations and compressed sensing

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with thanks to

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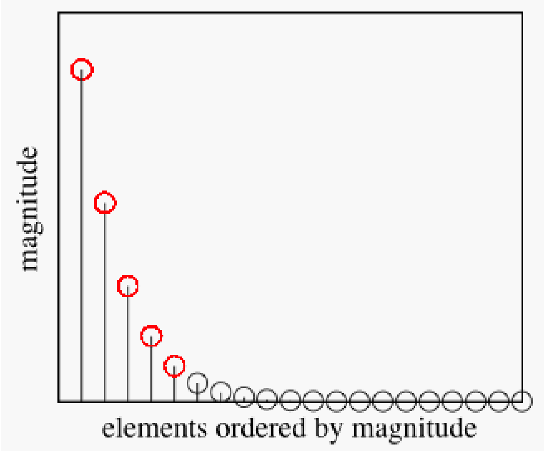
Sparsity: formal definition

A vector x is K-sparse, if only K of its elements are non-zero.

$$[0 \; 0.5 \; 0 \; 0 \; 0.1 \; 0 \; -0.2 \; 0 \; 0 \; 0 \; 0]^T$$

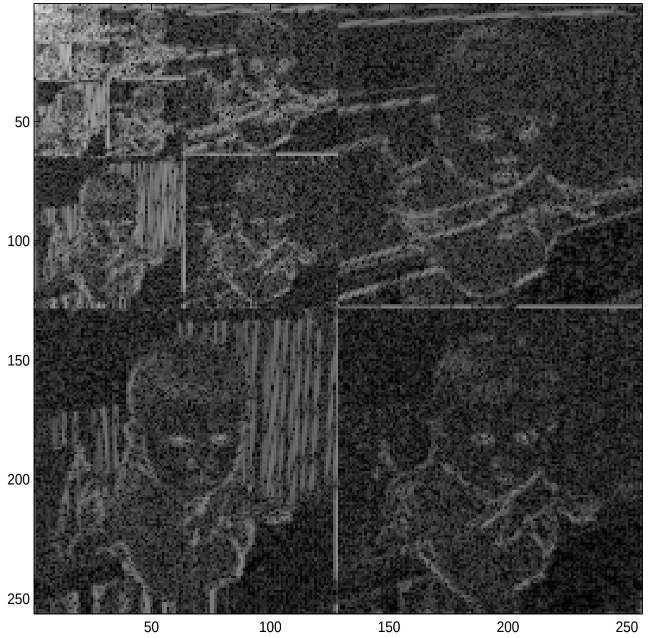
In the real world “exact” sparseness is uncommon, however, many signals are “approximately” K-sparse. That is, there is a K-sparse vector $\mathbf{x}_K$ such that the error $\|\mathbf{x} - \mathbf{x}_K\|_2$ is small.



Why Sparsity?

Why does sparsity make for a good transform?

“TOM” image

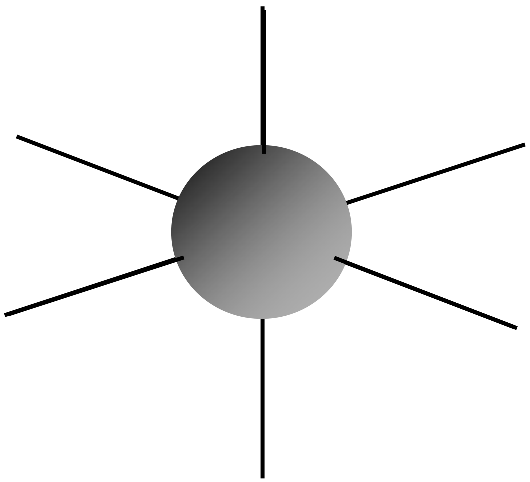


Wavelet Domain

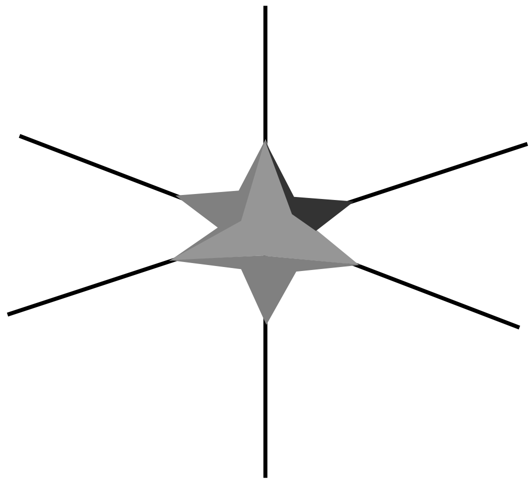
Good representations are efficient - Sparse!

Signals of interest

Efficient transform domain representations imply that our signals of interest live in a very small set.



L_2 ball (not sparse)



Sparse signal model

Sparsity and ill-posed inverse problems

Sparse signal models can be used to help solve various ill-posed linear inverse problems:

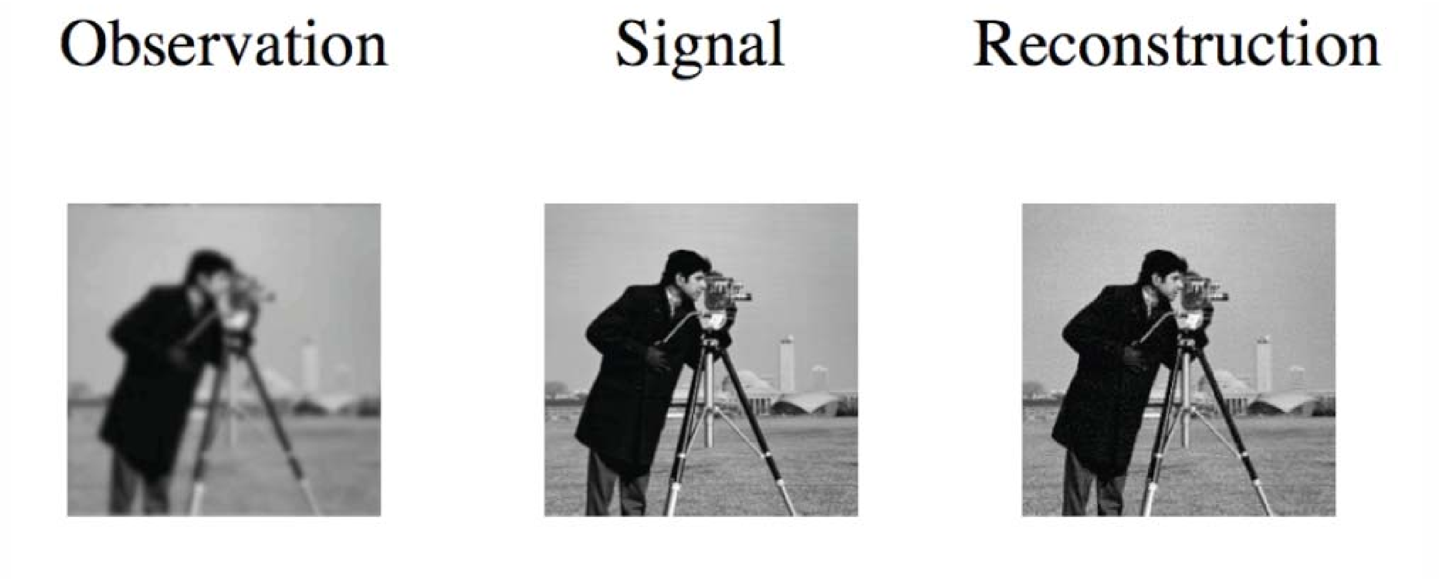


Image De-blurring

Sparse Representations and Generalized Sampling: compressed sensing

Compressed sensing

Traditionally when compressing a signal we take lots of samples move to a transform domain and then throw most of the coefficients away!

Why can't we just sample signals at the "Information Rate"?

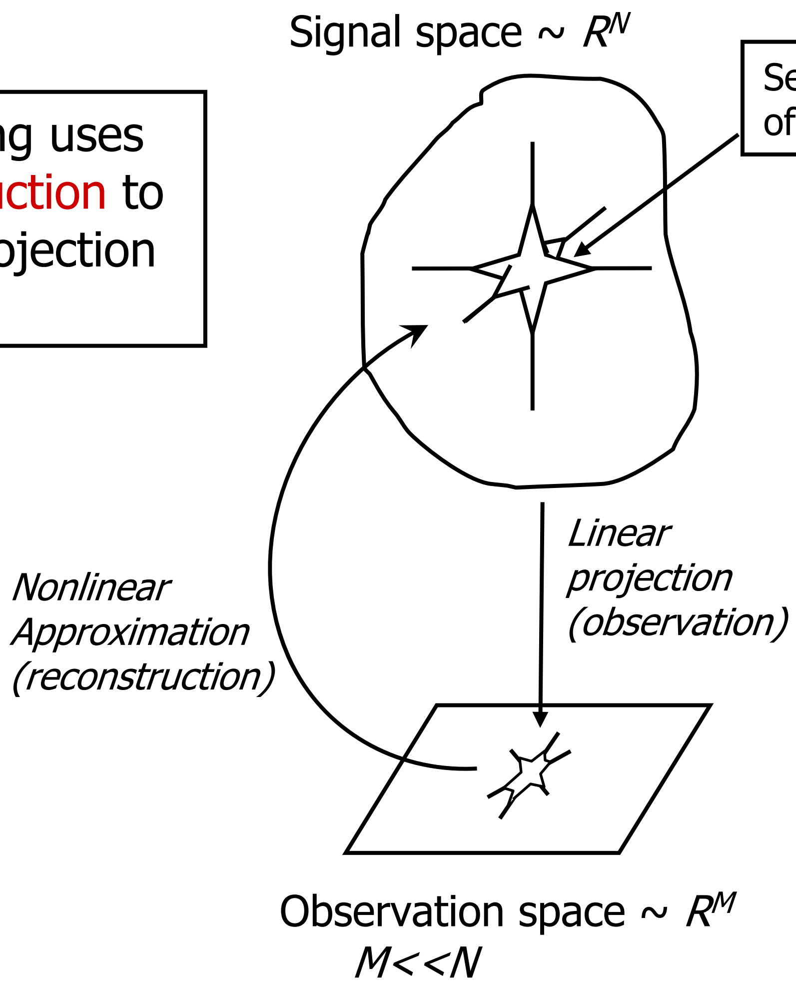
This is the philosophy of Compressed Sensing…

E. Candès, J. Romberg, and T. Tao, **"Robust Uncertainty principles: Exact signal reconstruction from highly incomplete frequency information,"** IEEE Trans. Information Theory, 2006

D. Donoho, **"Compressed sensing,"** IEEE Trans. Information Theory, 2006

Compressed sensing

Compressed Sensing uses nonlinear reconstruction to invert the linear projection operator



Compressed sensing

Compressed Sensing Challenges:

- Question 1: In which domain is the signal sparse (if at all)?

  Many natural signals are sparse in some time-frequency or space scale representation

- Question 2: How should we take good measurements?

  Current theory suggests that a random element in the sampling process is important

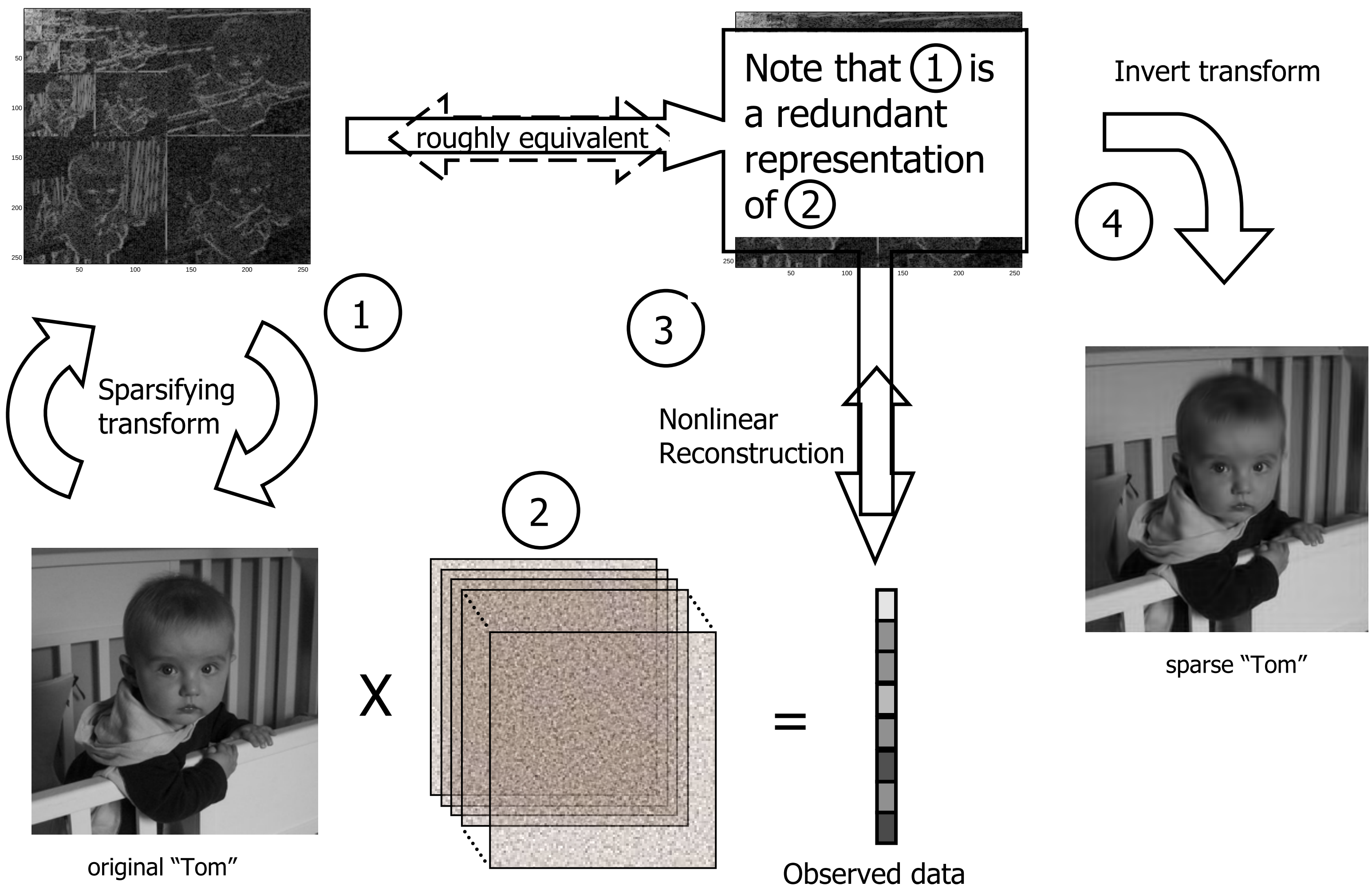
- Question 3: How many measurements do we need?

  Compressed sensing theory provides strong bounds on this as a function of the sparsity

- Question 4: How can we reconstruct the original signal from the measurements?

  Compressed sensing concentrates of algorithmic solutions with provable performance and provably good complexity

Compressed sensing principle

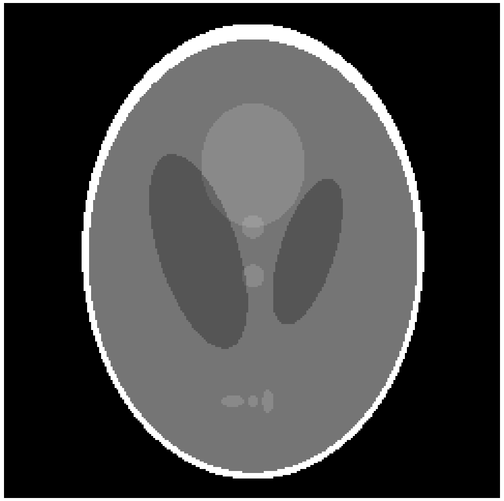


MRI acquisition

Compressed Sensing ideas can be applied to reduced sampling in Magnetic Resonance Imaging:

- MRI samples lines of spatial frequency
- Each line takes time & heats up the patient!

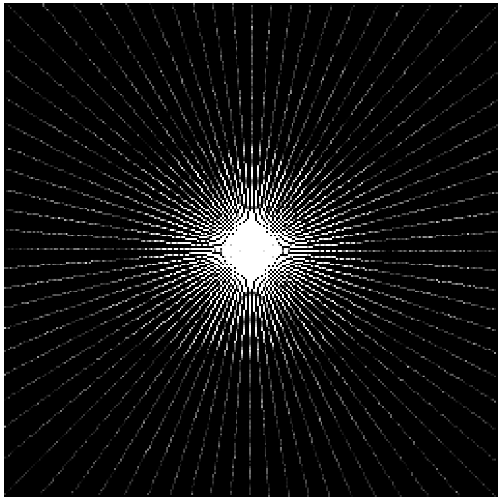
The Logan-Shepp phantom image illustrates this:



Logan-Shepp phantom

Sub-sampled Fourier Transform

≈ 7 x down sampled (no longer invertible)



…but we wish to sample here

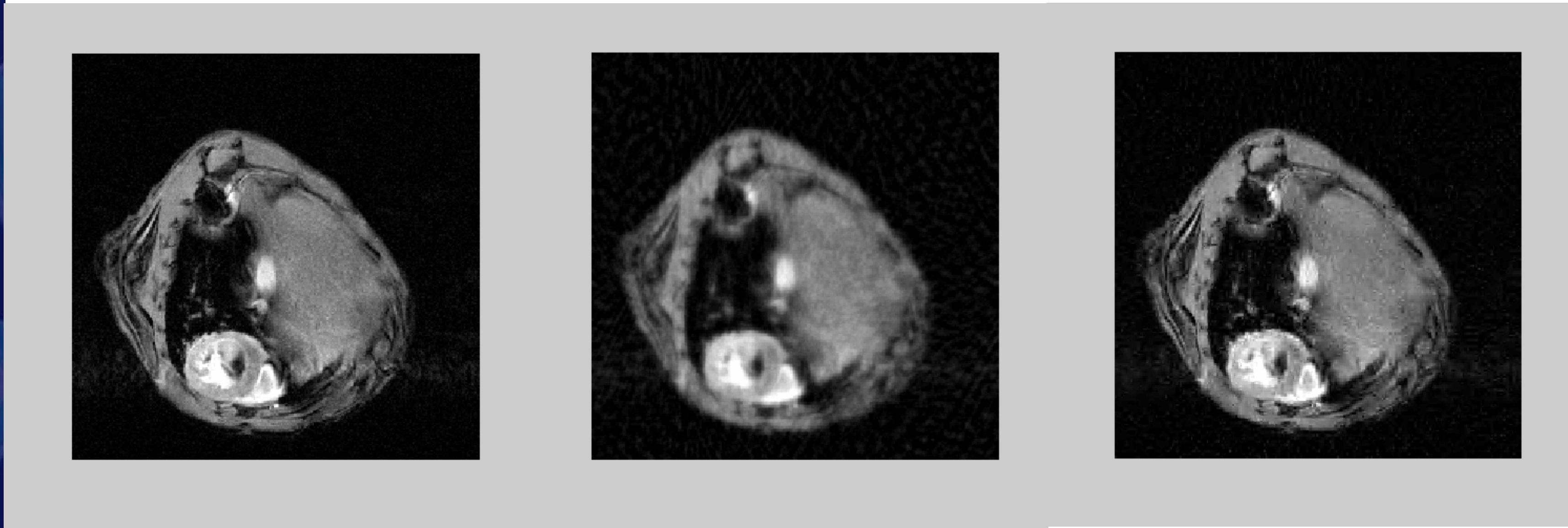
Rapid dynamic MRI acquisition in practice

However what we really want to tackle problems like this…

original

Linear reconstruction
(5x under-sampled)

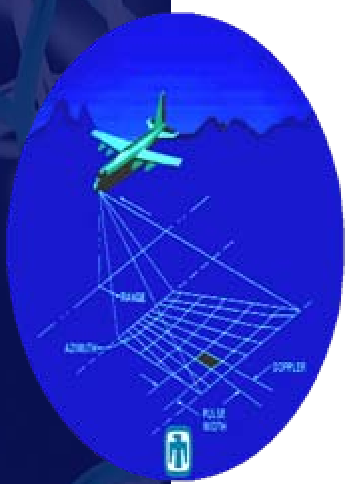
Nonlinear reconstruction
(5x under-sampled)

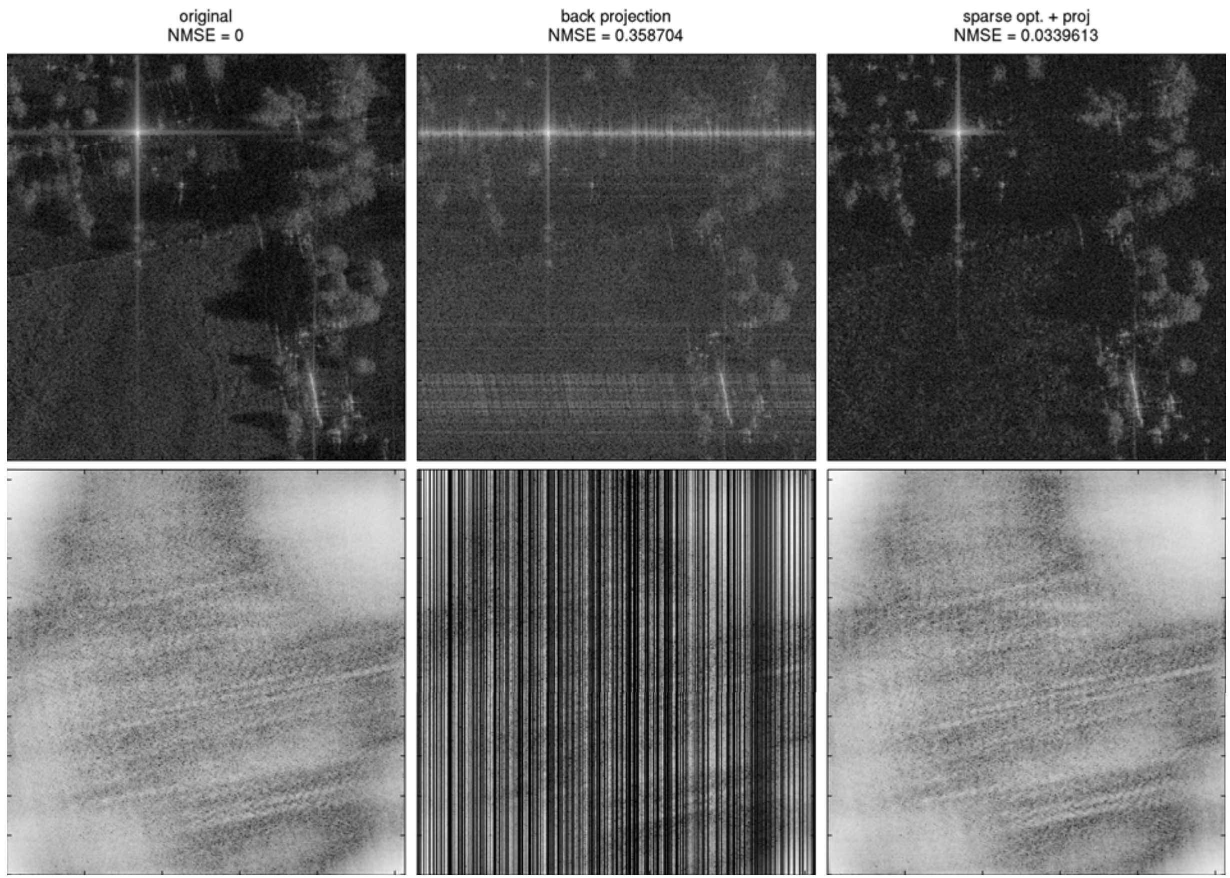


(data courtesy of Ian Marshall & Terry Tao, SFC Brain Imaging Centre)

Synthetic Aperture Radar

Compressed Sensing ideas can also be applied to reduced sampling in Synthetic Aperture Radar: Samples lines from spatial Fourier transform. Sub-sampling lines allows adaptive antennas to multi-task (interrupted SAR)





Potential applications…

Compressed Sensing provides a new way of thinking about signal acquisition. Potential applications areas include:

- Medical imaging
- Distributed sensing
- Seismic imaging
- Remote sensing (e.g. Synthetic Aperture Radar)
- Acoustic array processing (source separation)
- High rate analogue-to-digital conversion
  (DARPA A2I research program)
- The single pixel camera (novel Terahertz imaging)