Sparse representations and compressed sensing

Mike Davies
Institute for Digital Communications (IDCOM) &
Joint Research Institute for Signal and Image Processing
University of Edinburgh

with thanks to
Thomas Blumensath, Gabriel Riling
Sparsity: formal definition

A vector $x$ is $K$-sparse, if only $K$ of its elements are non-zero.

$$[0 \ 0.5 \ 0 \ 0 \ 0.1 \ 0 \ -0.2 \ 0 \ 0 \ 0 \ 0]^T$$

In the real world “exact” sparseness is uncommon, however, many signals are “approximately” $K$-sparse. That is, there is a $K$-sparse vector $x_K$ such that the error $\|x - x_K\|_2$ is small.
Why Sparsity?

Why does sparsity make for a good transform?

“TOM” image

Wavelet Domain

Good representations are efficient - Sparse!
Signals of interest

Efficient transform domain representations imply that our signals of interest live in a very small set.

\[ L_2 \text{ ball (not sparse)} \quad \text{Sparse signal model} \]
Sparsity and ill-posed inverse problems

Sparse signal models can be used to help solve various ill-posed linear inverse problems:

Observation  Signal  Reconstruction

Image De-blurring
Sparse Representations and Generalized Sampling: compressed sensing
Compressed sensing

Traditionally when compressing a signal we take lots of samples move to a transform domain and then throw most of the coefficients away!

Why can’t we just sample signals at the “Information Rate”?

This is the philosophy of Compressed Sensing...


Compressed Sensing uses **nonlinear reconstruction** to invert the linear projection operator.
Compressed Sensing Challenges:

• Question 1: In which domain is the signal sparse (if at all)?
  Many natural signals are sparse in some time-frequency or space scale representation

• Question 2: How should we take good measurements?
  Current theory suggests that a random element in the sampling process is important

• Question 3: How many measurements do we need?
  Compressed sensing theory provides strong bounds on this as a function of the sparsity

• Question 4: How can we reconstruct the original signal from the measurements?
  Compressed sensing concentrates on algorithmic solutions with provable performance and provably good complexity
Compressed sensing principle

1. Sparsifying transform
2. Reconstruction
3. Nonlinear Reconstruction
4. Invert transform

Note that 1 is a redundant representation of 2.

Original “Tom”

Observed data

sparse “Tom”
Compressed Sensing ideas can be applied to reduced sampling in Magnetic Resonance Imaging:

- MRI samples lines of spatial frequency
- Each line takes time & heats up the patient!

The Logan-Shepp phantom image illustrates this:

Sub-sampled Fourier Transform

\[ \approx 7 \times \text{down sampled (no longer invertible)} \]

Logan-Shepp phantom...but we wish to sample here
Rapid dynamic MRI acquisition in practice

However what we really want to tackle problems like this...

original  Linear reconstruction (5x under-sampled)  Nonlinear reconstruction (5x under-sampled)

(data courtesy of Ian Marshall & Terry Tao, SFC Brain Imaging Centre)
Compressed Sensing ideas can also be applied to reduced sampling in Synthetic Aperture Radar: Samples lines from spatial Fourier transform. Sub-sampling lines allows adaptive antennas to multi-task (interrupted SAR)
Compressed Sensing provides a new way of thinking about signal acquisition. Potential applications areas include:

- Medical imaging
- Distributed sensing
- Seismic imaging
- Remote sensing (e.g. Synthetic Aperture Radar)
- Acoustic array processing (source separation)
- High rate analogue-to-digital conversion (DARPA A2I research program)
- The single pixel camera (novel Terahertz imaging)