Multi-task Learning with Gaussian Processes, with Applications to Robot Inverse Dynamics

Chris Williams
with Kian Ming A. Chai, Stefan Klanke, Sethu Vijayakumar

December 2009

Motivation

- Examples of multi-task learning
  - Co-occurrence of ores (geostats)
  - Object recognition for multiple object classes
  - Personalization (personalizing spam filters, speaker adaptation in speech recognition)
  - Compiler optimization of many computer programs
  - Robot inverse dynamics (multiple loads)
- Gain strength by sharing information across tasks
- More general questions: meta-learning, learning about learning

Outline

1. Gaussian process prediction
2. Co-kriging
3. Intrinsic Correlation Model
4. Multi-task learning:
    - A. MTL as Hierarchical Modelling
    - B. MTL as Input-space Transformation
    - C. MTL as Shared Feature Extraction
5. Theory for the Intrinsic Correlation Model
6. Multi-task learning in Robot Inverse Dynamics

1. What is a Gaussian process?

- A Gaussian process (GP) is a generalization of a multivariate Gaussian distribution to infinitely many variables
- Informally: infinitely long vector $\simeq$ function
- Definition: a Gaussian process is a collection of random variables, any finite number of which have (consistent) Gaussian distributions
- A Gaussian distribution is fully specified by a mean vector $\boldsymbol{\mu}$ and covariance matrix Σ

$$\mathbf{f} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$$

- A Gaussian process is fully specified by a mean function $m(\mathbf{x})$ and a covariance function $k(\mathbf{x}, \mathbf{x}')$

$$f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x}'))$$

The Marginalization Property

- Thinking of a GP as an infinitely long vector may seem impractical. Fortunately we are saved by the marginalization property
- So generally we need only consider the n locations where data is observed, and the test point $\mathbf{x}_*$, the remainder are marginalized out

Drawing Random Functions from a Gaussian Process

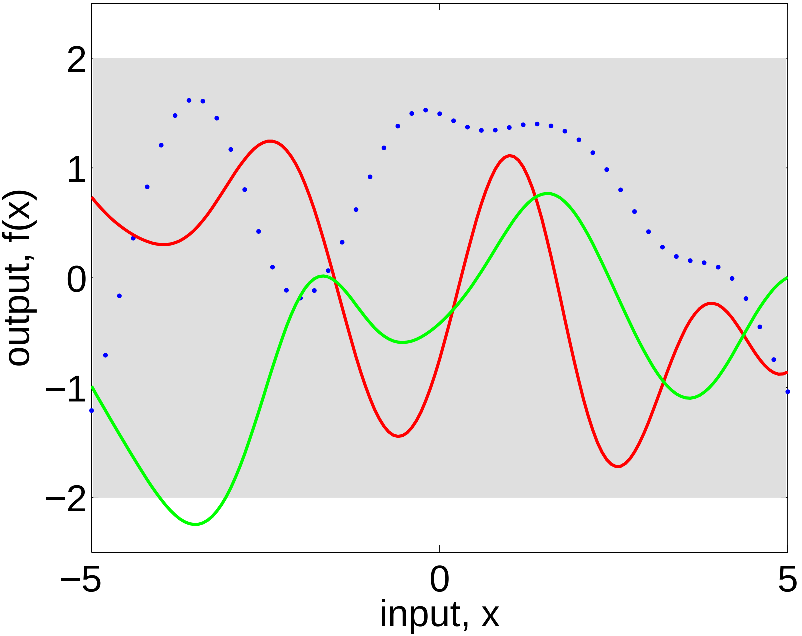
Example one-dimensional Gaussian process

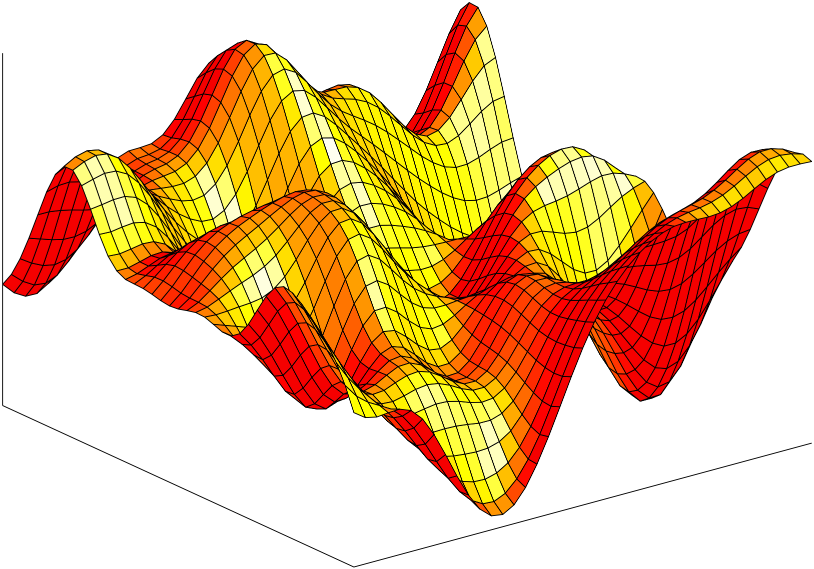
$$f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}) = 0, k(\mathbf{x}, \mathbf{x}') = \exp(-\frac{1}{2}(x - x')^2))$$

To get an indication of what this distribution over functions looks like, focus on a finite subset of $\mathbf{x}$-values, $\mathbf{f} = (f(\mathbf{x}_1), f(\mathbf{x}_2), \ldots, f(\mathbf{x}_n))^T$, for which

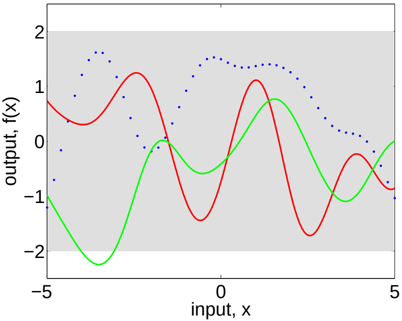
$$\mathbf{f} \sim \mathcal{N}(\mathbf{0}, \Sigma)$$

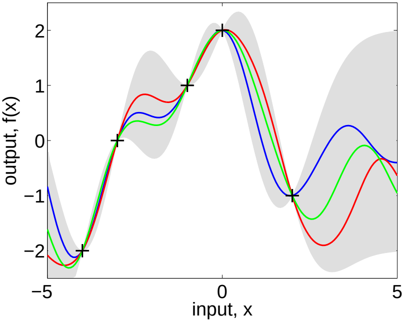
where $\Sigma_{ij} = k(\mathbf{x}_i, \mathbf{x}_j)$


2
1
0
−1
−2
output, f(x)
−5
0
5
input, x



From Prior to Posterior





Predictive distribution

$$p(y_*|\mathbf{x}_*, X, y, M) = \mathcal{N}(\mathbf{k}^T(\mathbf{x}_*, X)[K + \sigma_n^2 I]^{-1}\mathbf{y},$$
$$k(\mathbf{x}_*, \mathbf{x}_*) + \sigma_n^2 - \mathbf{k}^T(\mathbf{x}_*, X)[K + \sigma_n^2 I]^{-1}\mathbf{k}(\mathbf{x}_*, X))$$

Marginal Likelihood

$$\log p(\mathbf{y}|X, M) = -\frac{1}{2}\mathbf{y}^T K_y^{-1}\mathbf{y} - \frac{1}{2}\log|K_y| - \frac{n}{2}\log(2\pi)$$

where $K_y = K + \sigma_n^2 I$.

- This can be used to adjust the free parameters (hyperparameters) of a kernel.
- There can be multiple local optima of the marginal likelihood, corresponding to different interpretations of the data

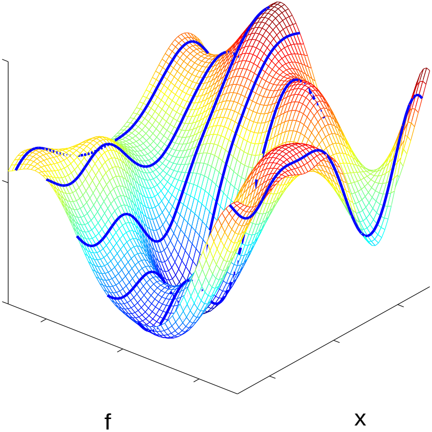
2. Co-kriging

Consider M tasks, and N distinct inputs $\mathbf{x}_1, \ldots, \mathbf{x}_N$:

- $f_{i\ell}$ is the response for the ℓ^{th} task on the i^{th} input $\mathbf{x}_i$
- Gaussian process with covariance function

$$k(\mathbf{x}, \ell; \mathbf{x}', m) = \langle f_\ell(\mathbf{x}) f_m(\mathbf{x}') \rangle$$

- Goal: Given noisy observations $\mathbf{y}$ of $\mathbf{f}$ make predictions of unobserved values $\mathbf{f}_*$ at locations X_*
- Solution Use the usual GP prediction equations


f
x

Some questions

- What kinds of (cross)-covariance structures match different ideas of multi-task learning?
- Are there multi-task relationships that don't fit well with co-kriging?

3. Intrinsic Correlation Model (ICM)

$$\langle f_\ell(\mathbf{x}) f_m(\mathbf{x}') \rangle = K^f_{\ell m} k^x(\mathbf{x}, \mathbf{x}') \qquad y_{i\ell} \sim \mathcal{N}(f_\ell(\mathbf{x}_i), \sigma^2_\ell),$$

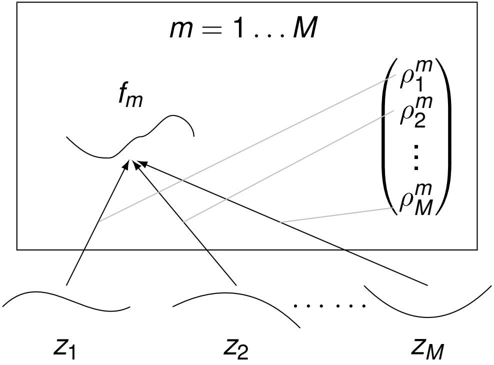
- K^f: PSD matrix that specifies the inter-task similarities (could depend parametrically on task descriptors if these are available)
- k^x: Covariance function over inputs
- σ^2_ℓ: Noise variance for the ℓ^{th} task.
- Linear Model of Coregionalization is a sum of ICMs

ICM as a linear combination of indepenent GPs

- Independent GP priors over the functions $z_j(\mathbf{x}) \Rightarrow$ multi-task GP prior over $f_m(\mathbf{x})$s

$$\langle f_\ell(\mathbf{x}) f_m(\mathbf{x}') \rangle = K^f_{\ell m} k^x(\mathbf{x}, \mathbf{x}')$$

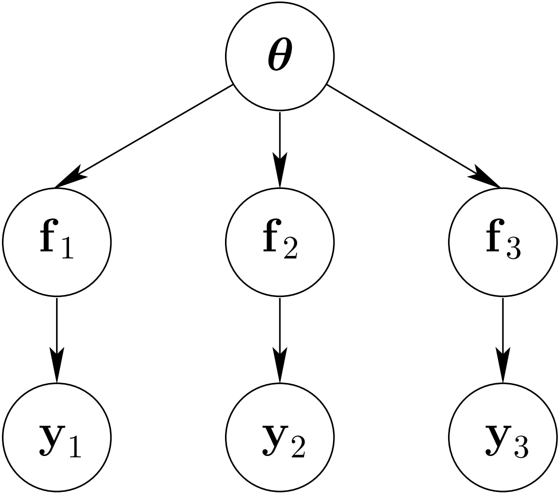
- $K^f \in \mathbb{R}^{M \times M}$ is a task (or context) similarity matrix with $K^f_{\ell m} = (\boldsymbol{\rho}^m)^T \boldsymbol{\rho}^\ell$



- Some problems conform nicely to the ICM setup, e.g. robot inverse dynamics (Chai, Williams, Klanke, Vijayakumar 2009; see later)
- Semiparametric latent factor model (SLFM) of Teh et al (2005) has P latent processes each with its own covariance function. Noiseless outputs are obtained by linear mixing of these latent functions

4 A. Multi-task Learning as Hierarchical Modelling

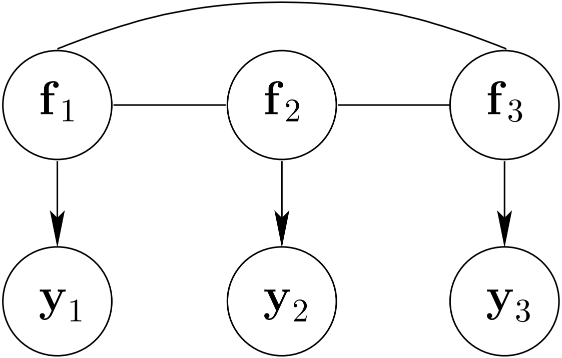
e.g. Baxter (JAIR, 2000), Goldstein (2003)



- Prior on θ may be generic (e.g. isotropic Gaussian) or more structured
- Mixture model on $\theta \rightarrow$ task clustering
- Task clustering can be implemented in the ICM model using a block diagonal K^f, where each block is a cluster
- Manifold model for θ, e.g. linear subspace $\Rightarrow$ low-rank structure of K^f (e.g. linear regression with correlated priors)
- Combination of the above ideas $\rightarrow$ a mixture of linear subspaces
- If task descriptors are available then can have $K^f_{\ell m} = k^f(\mathbf{t}_\ell, \mathbf{t}_m)$
- Regularization framework: Evgeniou et al (JMLR, 2005),

GP view

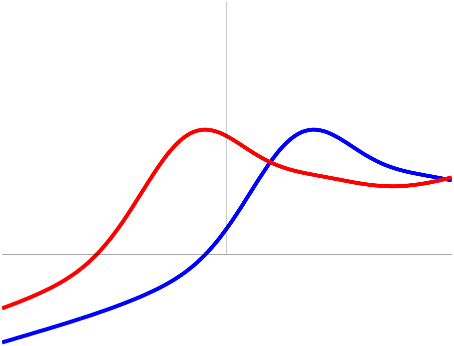
Integrate out θ



4 B. MTL as Input-space Transformation

- Ben-David and Schuller (COLT, 2003), $f_2(\mathbf{x})$ is related to $f_1(\mathbf{x})$ by a $\mathcal{X}$-space transformation $f : \mathcal{X} \rightarrow \mathcal{X}$
- Suppose $f_2(\mathbf{x})$ is related to $f_1(\mathbf{x})$ by a *shift* $\mathbf{a}$ in $\mathbf{x}$-space
- Then

$$\langle f_1(\mathbf{x}) f_2(\mathbf{x}') \rangle = \langle f_1(\mathbf{x}) f_1(\mathbf{x}' - \mathbf{a}) \rangle = k_1(\mathbf{x}, \mathbf{x}' - \mathbf{a})$$



- More generally can consider *convolutions*, e.g.

$$f_i(\mathbf{x}) = \int h_i(\mathbf{x} - \mathbf{x}')g(\mathbf{x}')d\mathbf{x}'$$

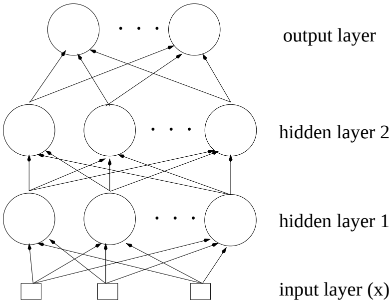
  to generate dependent f's (e.g. Ver Hoef and Barry, 1998; Higdon, 2002; Boyle and Frean, 2005). $\delta(\mathbf{x} - \mathbf{a})$ is a special case
- Alvarez and Lawrence (2009) generalize this to allow a linear combination of several latent processes

$$f_i(\mathbf{x}) = \sum_{r=1}^{R} \int h_{ir}(\mathbf{x} - \mathbf{x}')g_r(\mathbf{x}')d\mathbf{x}'$$

- ICM and SPFM are special cases using the $\delta()$ kernel

4 C. Shared Feature Extraction

- Intuition: multiple tasks can depend on the same extracted features; all tasks can be used to help learn these features
- If data is scarce for each task this should help learn the features
- Bakker and Heskes (2003) – neural network setup



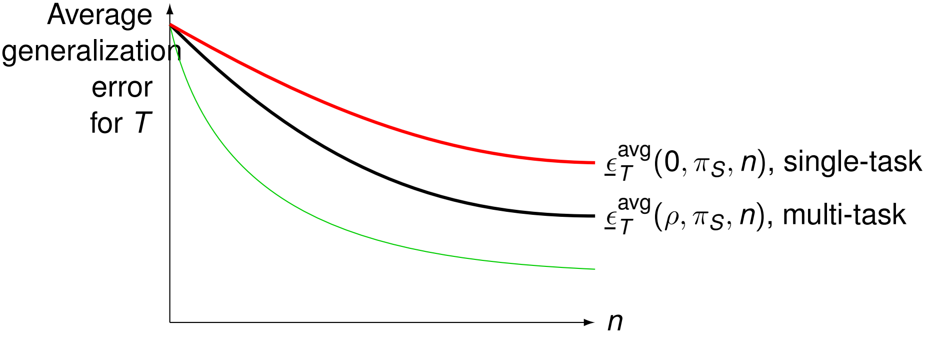
- Minka and Picard (1999): assume that the multiple tasks are independent GPs but with *shared* hyperparameters
- Yu, Tresp and Schawaighofer (2005) extend this so that all tasks share the same kernel hyperparameter, but can have different kernels
- Could also have inter-task correlations
- Interesting case if different tasks have different **x**-spaces; convert from each task-dependent **x**-space to same feature space?

5. Some Theory for the ICM

Kian Ming A. Chai, NIPS 2009

- Primary task T and secondary task S
  - Correlation ρ between the tasks
  - Proportion π_S of the total data belongs to secondary task S
- For GPs and squared loss, we can compute analytically the generalizaton error $\epsilon_T(\rho, X_T, X_S)$ given $p(\mathbf{x})$
- Average this over X_T, X_S to get $\epsilon_T^{\text{avg}}(\rho, \pi_S, n)$, the *learning curve* for primary task T given a *total* of n observations for both tasks.
- $\underline{\epsilon}_T^{\text{avg}}$ is the lower bound on ϵ_T^{avg}.
- Theory bounds benefit of multi-task learning in terms of $\underline{\epsilon}_T^{\text{avg}}$.

Theory: Result on lower bound



$$\underline{\epsilon}_T^{\text{avg}}(\rho, \pi_S, n) \geq (1 - \rho^2 \pi_S)\, \underline{\epsilon}_T^{\text{avg}}(0, \pi_S, n)$$

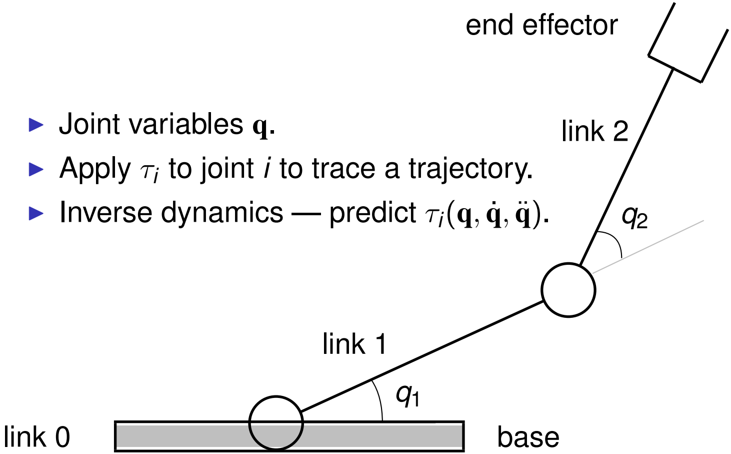
- Bound has been demonstrated on 1-d problems, and on the input distribution corresponding to the SARCOS robot arm data

Discussion

- 3 types of multi-task learning setup
- ICM and convolutional cross-covariance functions, shared feature extraction
- Are there multi-task relationships that don't fit well with a co-kriging framework?

Multi-task Learning in Robot Inverse Dynamics

- Joint variables $\mathbf{q}$.
- Apply τ_i to joint i to trace a trajectory.
- Inverse dynamics — predict $\tau_i(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$.



Inverse Dynamics
Characteristics of τ

- Torques are non-linear functions of $\mathbf{x} \stackrel{\text{def}}{=} (\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$.
- (One) idealized rigid body control:

$$\tau_i(\mathbf{x}) = \underbrace{\mathbf{b}_i^{\mathrm{T}}(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{q}}^{\mathrm{T}} H_i(\mathbf{q})\dot{\mathbf{q}}}_{\text{kinetic}} + \overbrace{g_i(\mathbf{q})}^{\text{potential}} + \underbrace{f_i^{\mathrm{v}} \dot{q}_i + f_i^{\mathrm{c}} \operatorname{sgn}(\dot{q}_i)}_{\text{viscous and Coulomb frictions}},$$

- Physics-based modelling can be hard due to factors like unknown parameters, friction and contact forces, joint elasticity, making analytical predictions unfeasible
- This is particularly true for compliant, lightweight humanoid robots

- Functions *change* with the loads handled at the end effector
- Loads have different mass, shapes, sizes.
- Bad news (1): Need a different inverse dynamics model for different loads.
- Bad news (2): Different loads may go through different trajectory in data collection phase and may explore different portions of the $\mathbf{x}$-space.

- Good news: the changes enter through changes in the dynamic parameters of the last link
- Good news: changes are linear wrt the dynamic parameters

$$\tau_i^m(\mathbf{x}) = \mathbf{y}_i^T(\mathbf{x})\boldsymbol{\pi}^m$$

where $\boldsymbol{\pi}^m \in \mathbb{R}^{11}$ (e.g. Petkos and Vijayakumar,2007)
- Reparameterization:

$$\tau_i^m(\mathbf{x}) = \mathbf{y}_i^T(\mathbf{x})\boldsymbol{\pi}^m = \mathbf{y}_i^T(\mathbf{x})A_i^{-1}A_i\boldsymbol{\pi}^m = \mathbf{z}_i^T(\mathbf{x})\boldsymbol{\rho}_i^m$$

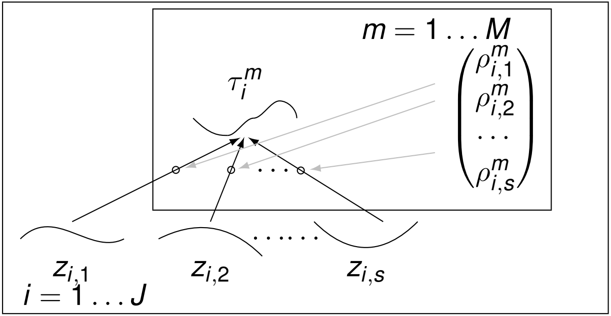
where A_i is a non-singular 11×11 matrix

GP prior for Inverse Dynamics for multiple loads

- Independent GP priors over the functions $z_{ij}(\mathbf{x}) \Rightarrow$ multi-task GP prior over τ_i^ms

$$\left\langle \tau_i^\ell(\mathbf{x}) \tau_i^m(\mathbf{x}') \right\rangle = (K_i^\rho)_{\ell m} k_i^x(\mathbf{x}, \mathbf{x}')$$

- $K_i^\rho \in \mathbb{R}^{M \times M}$ is a task (or context) similarity matrix with $(K_i^\rho)_{\ell m} = (\boldsymbol{\rho}_i^m)^T \boldsymbol{\rho}_i^\ell$



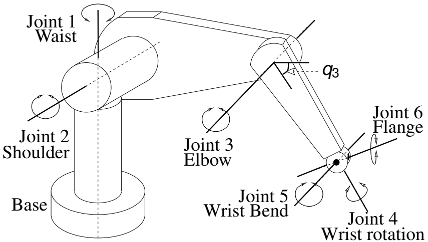
GP prior for $k(\mathbf{x}, \mathbf{x}')$

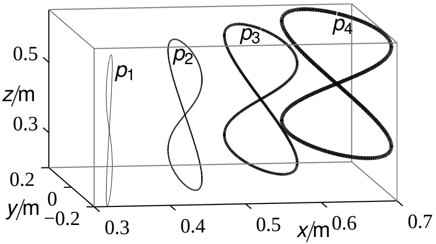
$$
\begin{aligned}
k(\mathbf{x}, \mathbf{x}') = \text{bias} + [\text{linear with ARD}](\mathbf{x}, \mathbf{x}') \\
+ [\text{squared exponential with ARD}](\mathbf{x}, \mathbf{x}') \\
+ [\text{linear (with ARD)}](\operatorname{sgn}(\dot{q}), \operatorname{sgn}(\dot{q}'))
\end{aligned}
$$

- Domain knowledge relates to last term (Coulomb friction)

Data

- Puma 560 robot arm manipulator: 6 degrees of freedom
- Realistic simulator (Corke, 1996), including viscous and asymmetric-Coulomb frictions.
- 4 paths $\times$ 4 speeds $=$ 16 different trajectories:
- Speeds: 5s, 10s, 15s and 20s completion times.
- 15 loads (contexts): 0.2kg . . . 3.0kg, various shapes and sizes.





Data

Training data

- 1 reference trajectory common to handling of all loads.
- 14 unique training trajectories, one for each context (load)
- 1 trajectory has no data for any context; thus this is always novel

Test data

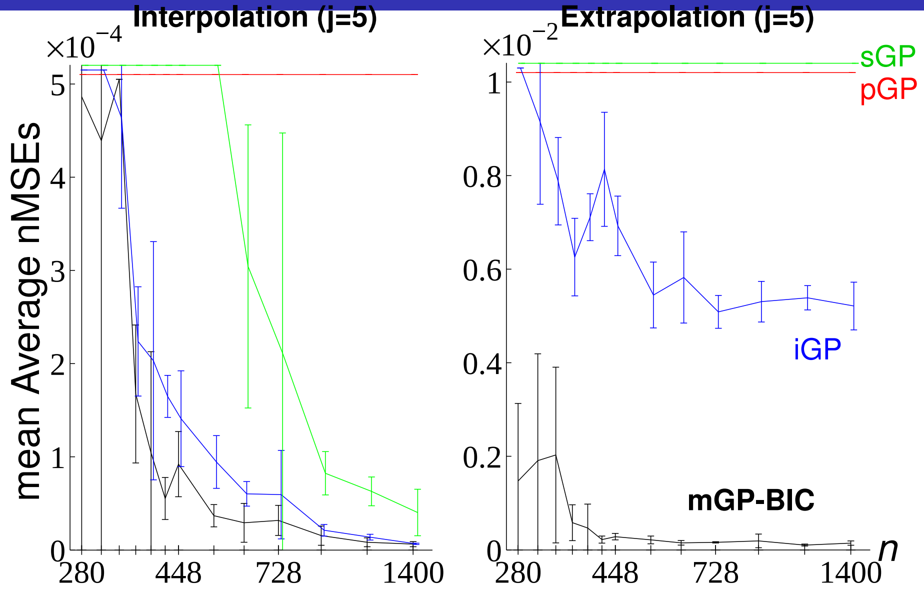
- Interpolation data sets for testing on reference trajectory and the unique trajectory for each load.
- Extrapolation data sets for testing on all trajectories.

Methods

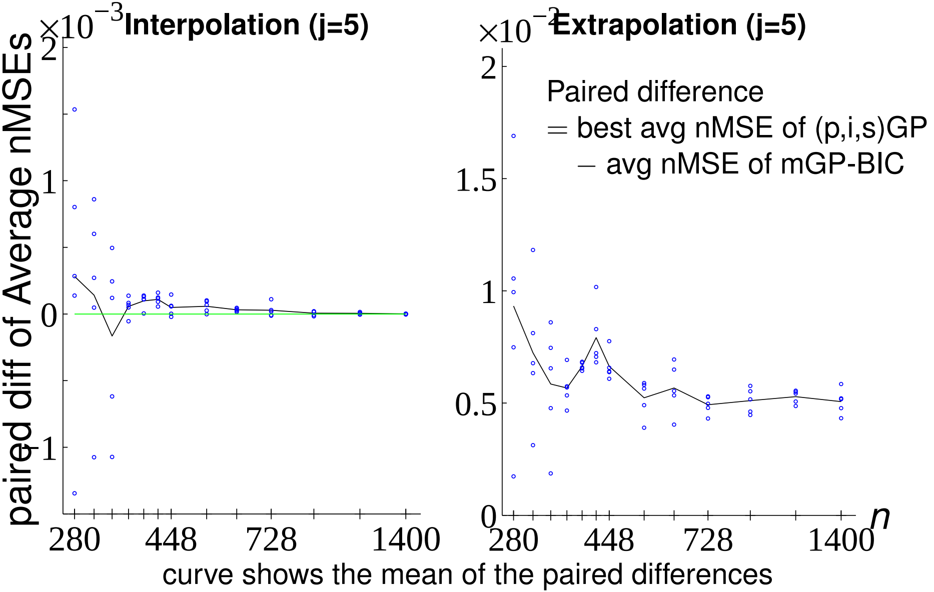
sGP	Single task GPs	GPs trained separately for each load
iGP	Independent GP	GPs trained independently for each load but tying parameters across loads
pGP	pooled GP	one single GP trained by pooling data across loads
mGP	multi-task GP with BIC	sharing latent functions across loads, selecting similarity matrix using BIC

- For mGP, the rank of K^f is determined using BIC criterion

Results



error bars: 90% confidence interval of the mean (5 replications)



Interpolation (j=5)
Extrapolation (j=5)
paired diff of Average nMSEs
$\times 10^{-3}$
$\times 10^{-2}$
Paired difference
= best avg nMSE of (p,i,s)GP
− avg nMSE of mGP-BIC
280
448
728
1400
n
curve shows the mean of the paired differences

Conclusions and Discussion

- GP formulation of MTL with factorization $k^x(\mathbf{x}, \mathbf{x}')$ and K^f, and encoding of task similarity
- This model fits exactly for multi-context inverse dynamics
- Results show that MTL can be effective
- This is one model for MTL, but what about others, e.g. cov functions that don't factorize?