An introduction to deciding higher-order matching

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- Equivalence checking: is state s equivalent to t ?
- Mostly computing dyadic fixed points e.g. bisimulations to solve it. May need algebraic/combinatorial properties of reachability sets/generators of graph

Active research goal: transfer these techniques to

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4. Application of tree automata and games to deciding higher-order matching
   [Comon, Jurski 1997; Stirling 2005-2009, work described here]

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- order of $\mathbf{0}$ is 1;
- order of $(A_1, \dots, A_n, \mathbf{0})$ is $k + 1$ where k is maximum of orders of A_is

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1. if x (f) has type A then $x : A \in T$ $(f : A \in T)$
2. if $t : B \in T$ and $x : A \in T$ then $\lambda x.t : A \rightarrow B \in T$
3. if $t : A \rightarrow B \in T$ and $u : A \in T$ then $(tu) : B \in T$

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- order of $t : A$ = order A
- closed $t : A$ no free variables
- $t, t' : A$ are α-equivalent renamings of each other

Dynamics: reduction

$(\beta) \quad (\lambda x.t)v \rightarrow_\beta t\{v/x\}$ $\quad \{\cdot/\cdot\}$ Substitution

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- Facts Strong normalisation and confluence (of $\rightarrow_\beta$, $\rightarrow_\eta$, $\rightarrow_{\beta\eta}$)
- Equivalence: $t =_{\beta\eta} t'$ if there are s, s'

$$
\begin{array}{ccc}
t & \rightarrow^*_{\beta\eta} & s \\
 & & \shortparallel_\alpha \\
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- Inf $=$ η-long β-normal form
- If t, v are in Inf and $tv \to_\beta^* s$ and s in β-normal form then s is Inf

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- $T_A(C)$ is the set of closed Inf terms of type A whose constants belong to C.

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- Notation: $\lambda x_1 \ldots x_k.t$ abbreviates $\lambda x_1. \ldots. \lambda x_k.t$
- Monster type $M = ((((\mathbf{0}, \mathbf{0}), \mathbf{0}), \mathbf{0}), \mathbf{0}, \mathbf{0})$, has order 5
- $\lambda xy.x(\lambda z_1.x(\lambda z_2.z_1 y)) \in T_M(\emptyset)$
  when $x : (((\mathbf{0}, \mathbf{0}), \mathbf{0}), \mathbf{0})$, $z_i : (\mathbf{0}, \mathbf{0})$ and $y : \mathbf{0}$.

Decision questions

- Higher-order unification
- $v = u$ contains free variables $x_1, \ldots, x_n$
- Solution $\theta = \{t_1/x_1, \ldots, t_n/x_n\}$ such that $v\theta =_{\beta\eta} u\theta$
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- Undecidable (even at order 2) [Huet 1972;Goldfarb 1981]

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- Decidable for all orders [Stirling 2006; 2009]

Matching is essentially monadic

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- Conceptually simpler problem: just one free variable
- Called "interpolation": $x\, w_1 \ldots w_k = u$ where $x : (B_1, \ldots, B_k, \mathbf{0})$, $w_i : B_i$, $u : \mathbf{0}$ in Inf.
- Solution t in Inf such that $t w_1 \ldots w_k \rightarrow^*_\beta u$

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- Canonical solution $\lambda z.z\, t_1 \ldots t_n$ where t_is are closed
  Consequently $v\{t_1/x_1, \ldots, t_n/x_n\} \rightarrow_\beta^* u$
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- Restrict constants in solution terms to be those in u plus fresh $b : \mathbf{0}$ Restricts to finitely many constants

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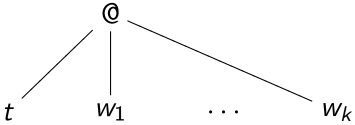
- t (of the right type and in Inf) a potential solution to $xw_1 \dots w_k = u$ (Assume u does not have bound variables)
- Interpolation tree



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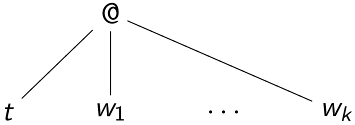

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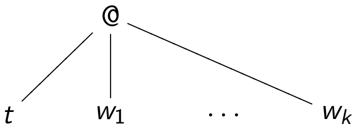

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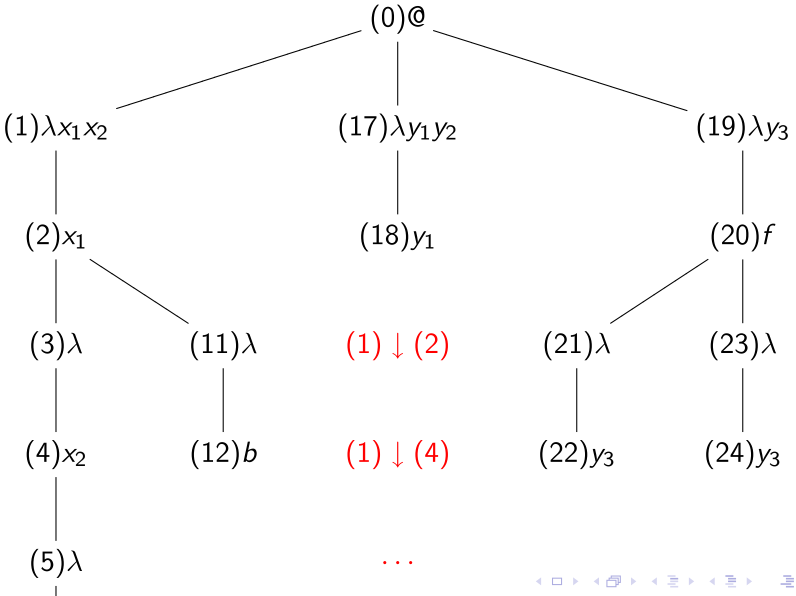
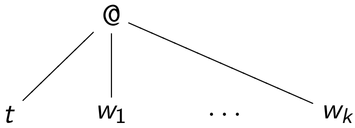
- $t, w_1, \ldots, w_k$ binding trees with dummy lambdas and binding relation $\downarrow$ between nodes
- $n \downarrow m$ if n labelled $\lambda\overline{y}$ binds y_j, label at m

Example: $x(\lambda y_1 y_2 . y_1)(\lambda y_3 . f y_3 y_3) = faa$



Tree automata



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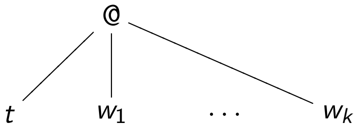

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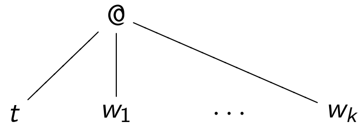

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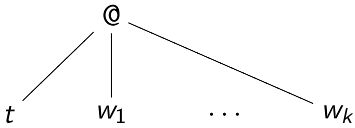

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- Non-emptiness decidable; sets recognised are regular

Tree automata for 4th-order [Comon,Jurski 97]

- 4th-order E, $xw_1 \dots w_k = u$ and C_E constants in u plus b
- For finite alphabet Σ consider a potential solution term $t = \lambda x_1 \dots x_k.s$

$$s ::= z_j^n \mid x_i v_1 \dots v_m \mid f s_1 \dots s_m \qquad f \in C_E,\ \ m \geq 0$$
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- Guarantees finite alphabet for accessible nodes: reuse variables

Tree automata for 4th-order [Comon,Jurski 97] II

- Finite states:
  inaccessible nodes: $-$ for "dont care"
  accessible nodes: U is subterms of u closed under replacement of subtrees with finitely many $z^i_j : \mathbf{0}$

$$\{[e], [\lambda e], [\lambda z^n_1 \dots z^n_m e], [\lambda x_1 \dots x_k u] \mid e \in U \cup \{-\}\}$$

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- Theorem The tree automaton associated with a 4th-order $xw_1 \dots w_k = u$ accepts t iff t solves $xw_1 \dots w_k = u$

Tree automata for 5th-order: PROBLEMS

- Finite alphabet ? can stack 2nd-order variables
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- Overcome the first problem: employ binding trees
  $\lambda xy.x(\lambda z.x(\lambda z \ldots x(\lambda z.z(\lambda.z(\ldots \lambda.z(\lambda.y))\ldots))\ldots))$
  $\downarrow$ from the first node labelled λz to the last node labelled z, and so on
  Fact For all A and finite C, there is a finite Σ such that every $t \in T_A(C)$ up to α-equivalence is a binding Σ-tree.

Tree automata for 5th-order: PROBLEMS

- Finite alphabet ? can stack 2nd-order variables
  $\lambda xy.x(\lambda z_1.x(\lambda z_2 \ldots (\lambda z_n.z_n(z_{n-1}(\ldots z_1(y))\ldots))\ldots))$
  Need an infinite alphabet
- Finite number of states ? stack same 2nd-order variable
  $z_n(z_n(\ldots(z_n a)\ldots))$
  Number of evaluations can be infinite
- Overcome the first problem: employ binding trees
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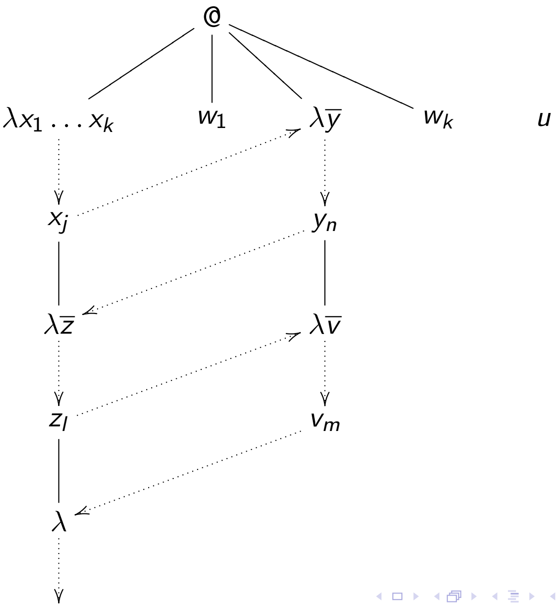
  Fact For all A and finite C, there is a finite Σ such that every $t \in T_A(C)$ up to α-equivalence is a binding Σ-tree.
- Second problem ? need finer analysis than evaluation of an accessible node

Games: automaton moving around interpolation tree



Game/automaton $G(t, E)$

- Play: sequence $n_1 q_1 \theta_1, \ldots, n_l q_l \theta_l$
  - n_i is a node of the interpolation tree,
  - q_i is a state $[u']$ where u' subterm of u or final
  - θ_i is look-up table: tells where to jump when at a variable

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- Current position is $n[r]\theta$; next position
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  - $\lambda\overline{y}$ then $n1[r]\theta$

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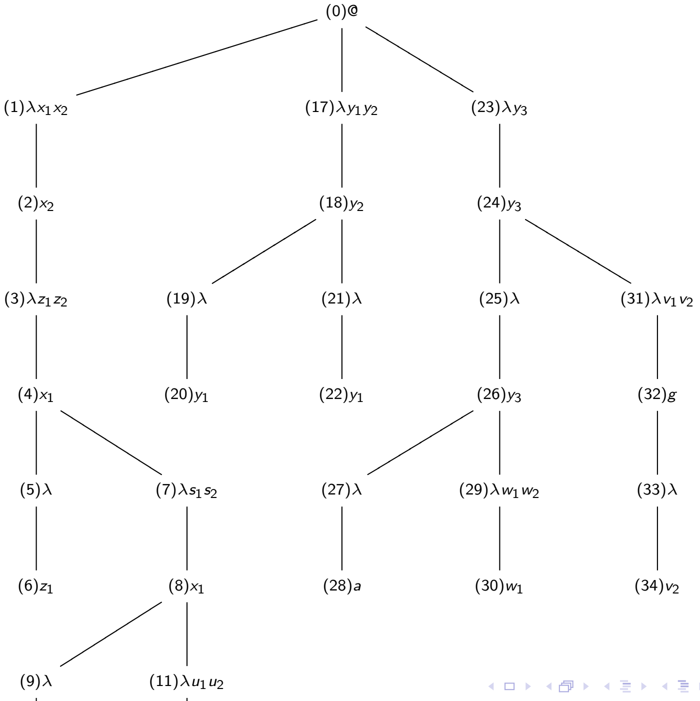
- Player $\forall$ loses every play in $G(t, E)$ iff t solves E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(19)λ
(21)λ
(25)λ
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(26)y_3
(32)g
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(9)λ
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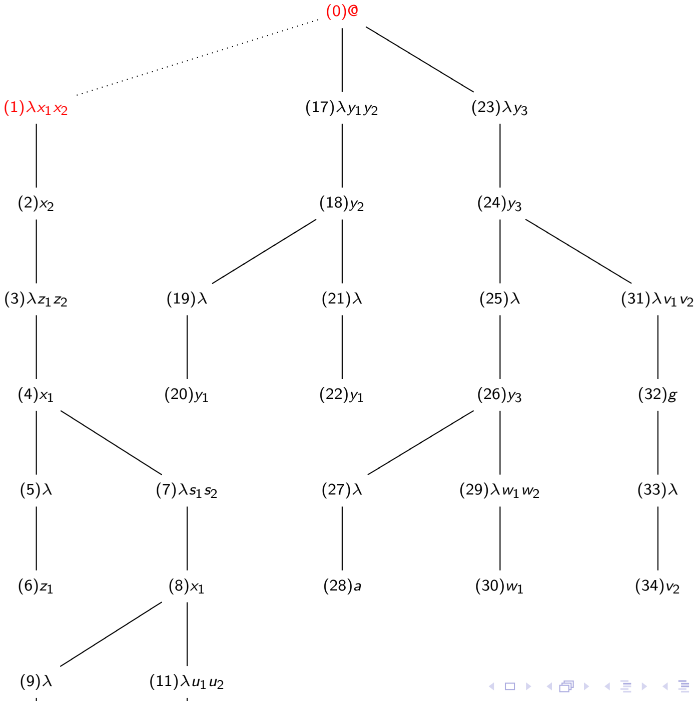
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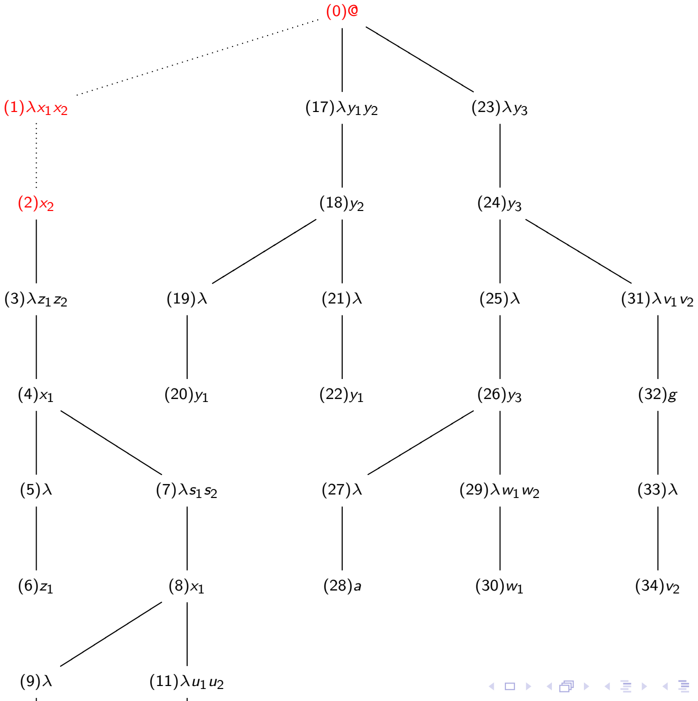
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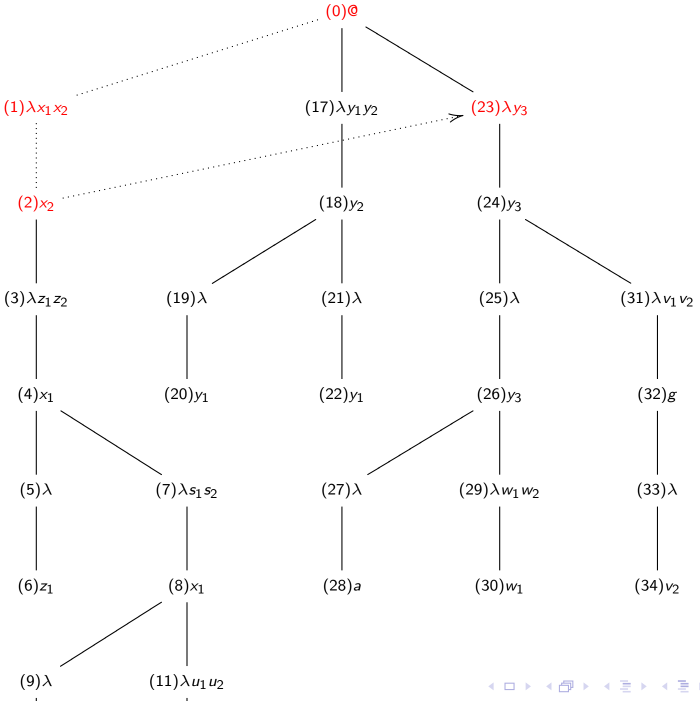
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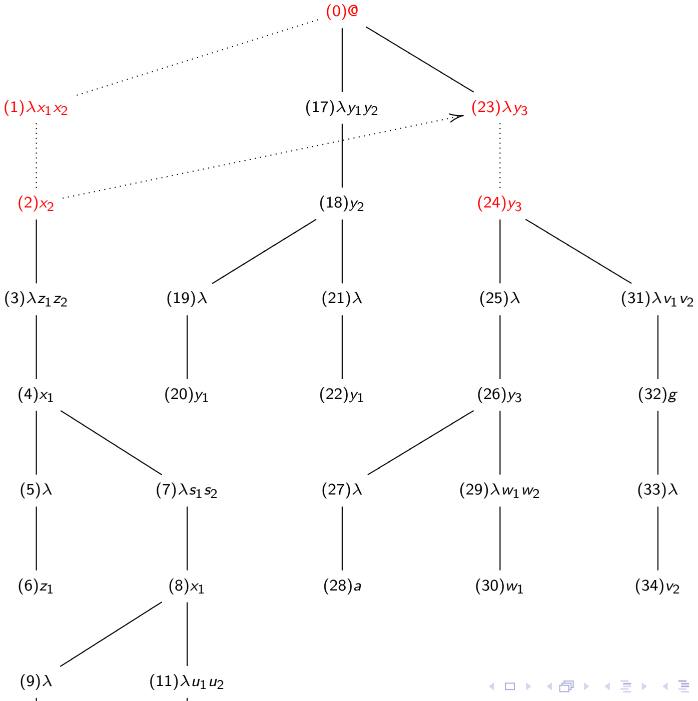
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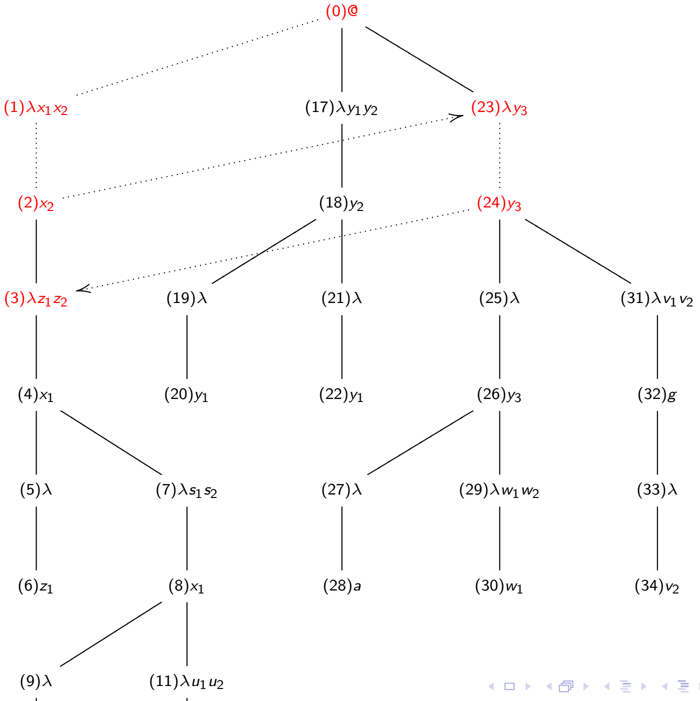
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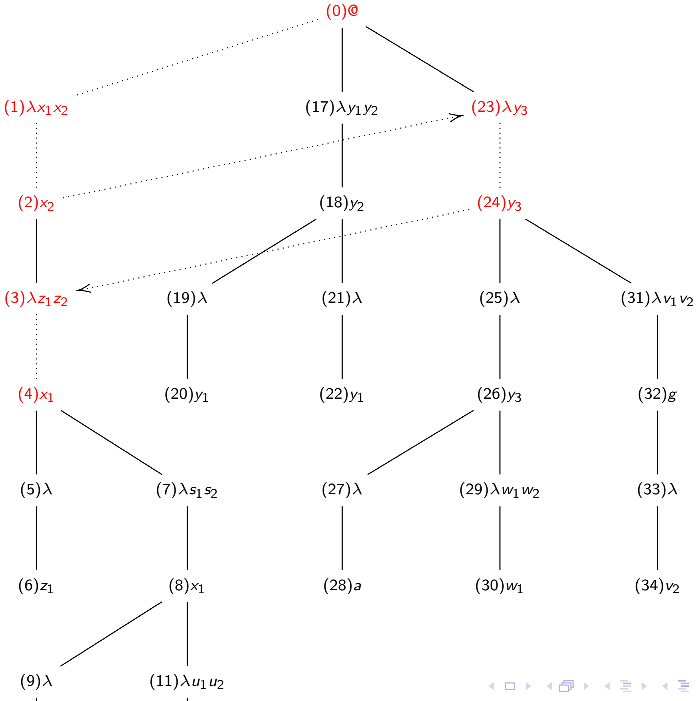
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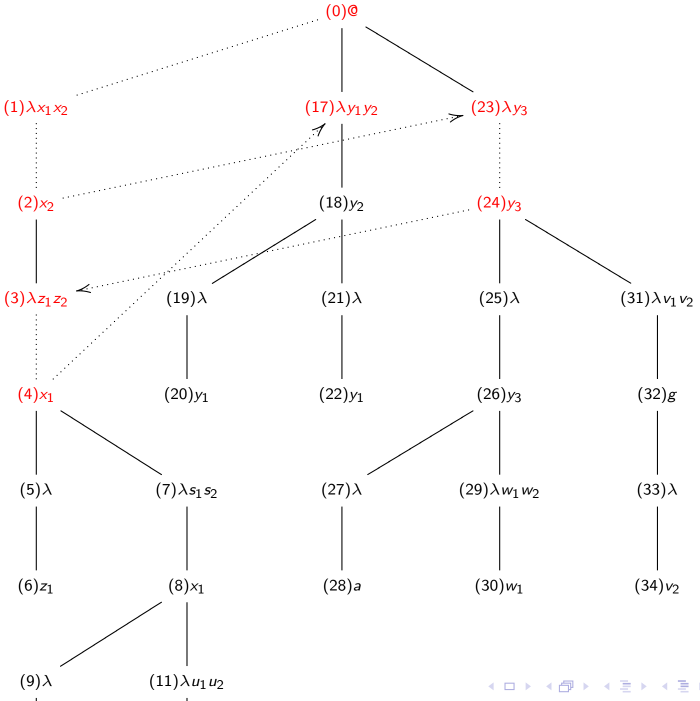
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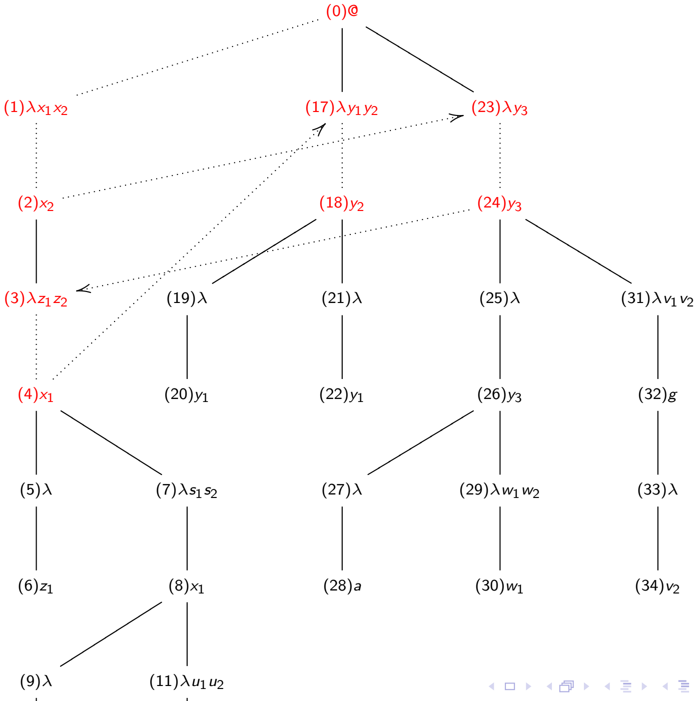
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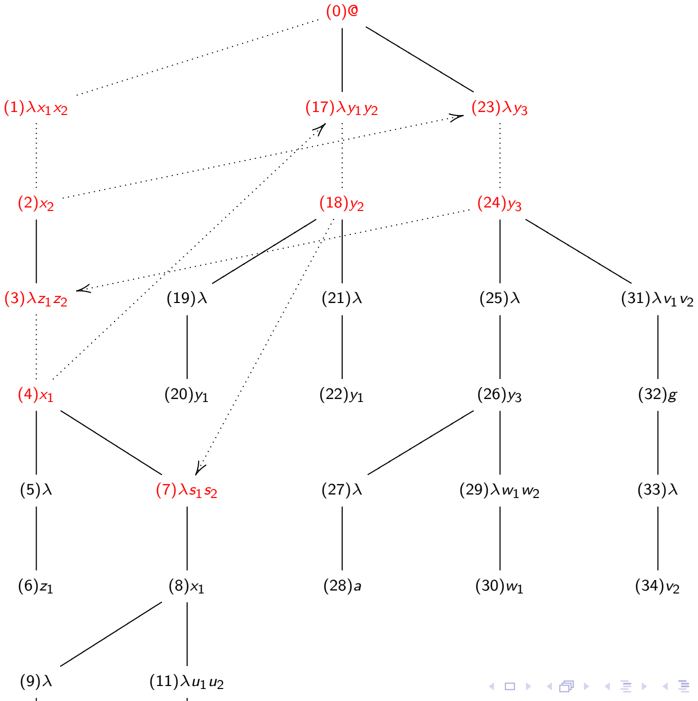
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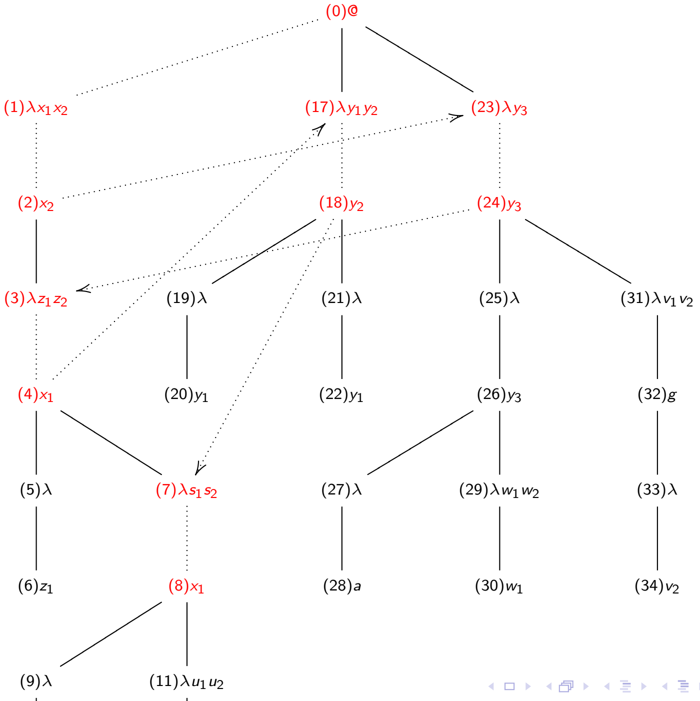

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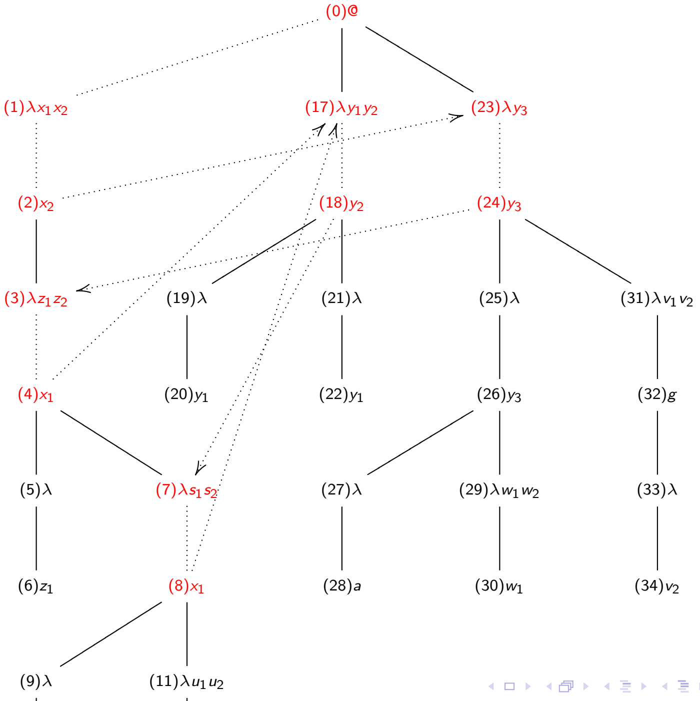

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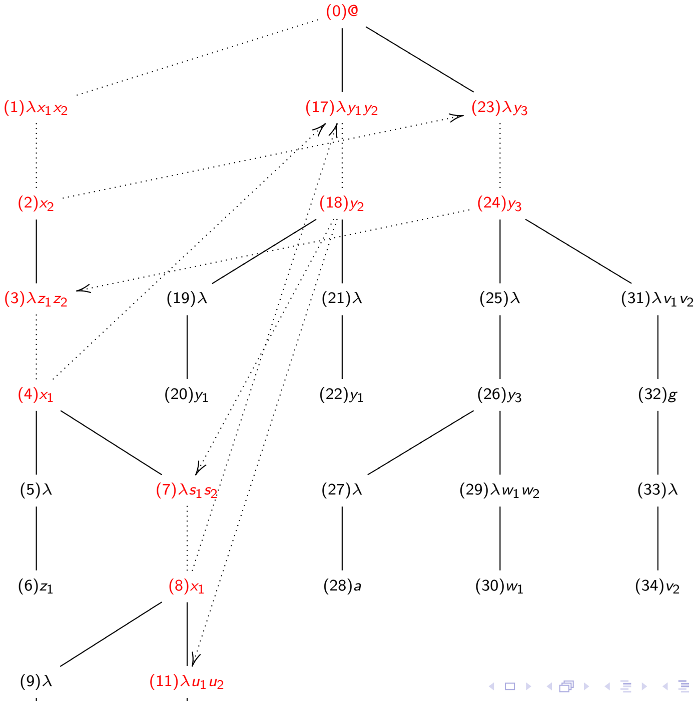
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Application of games to define tree automata

- Two problems extending Comon+ Jurski's result
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  2. Ensuring finite alphabet in automaton

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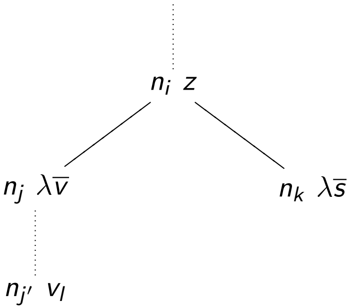
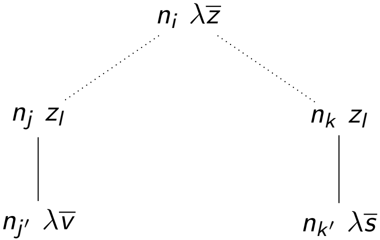
- Theorem The set of solutions of E is recognised by a special alternating binding tree automaton BUT non-emptiness of such automata undecidable [Stirling 2009; Ong, Tzevelekos 2009]

Uniformity properties of game playing I



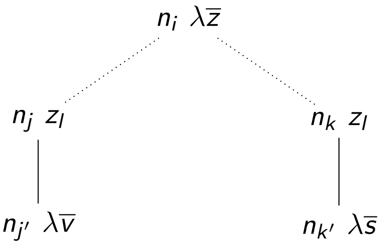
- If play is at n_i then at n_j and then later at n_k then inbetween there must have been a position at an $n_{j'}$ bound by n_j

Uniformity properties of game playing II



- If play is at n_i then at n_j and then later at $n_{j'}$; then at n_k then there must be a corresponding sequence to that between n_j and $n_{j'}$ between n_k and $n_{k'}$ unless there is a different $\forall$ choice

Uniformity properties of game playing II



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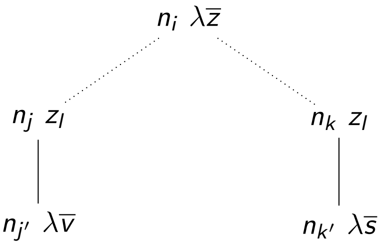
- Especially relevant if n_k is somewhere below n_j ("embedded")

Uniformity properties of game playing II



- If play is at n_i then at n_j and then later at $n_{j'}$; then at n_k then there must be a corresponding sequence to that between n_j and $n_{j'}$ between n_k and $n_{k'}$ unless there is a different $\forall$ choice
- Especially relevant if n_k is somewhere below n_j ("embedded")
- Property not enforced in a tree automaton

Application of games to decidability of matching

- Combinatorial argument based on uniformities of play

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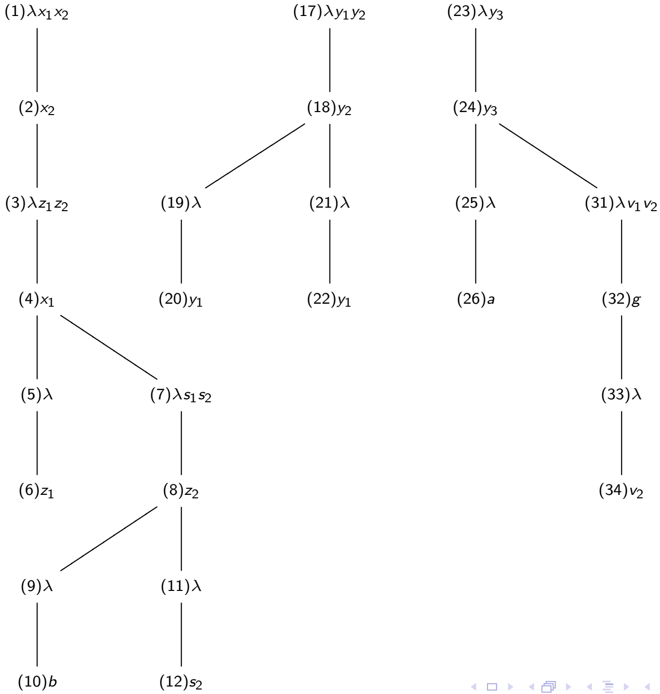
- 5th-order example

$$x(\lambda y_1 y_2 . y_2 y_1 y_1)(\lambda y_3 . y_3 a(\lambda v_1 v_2 . g v_2)) \;=\; ga$$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(2)x_2
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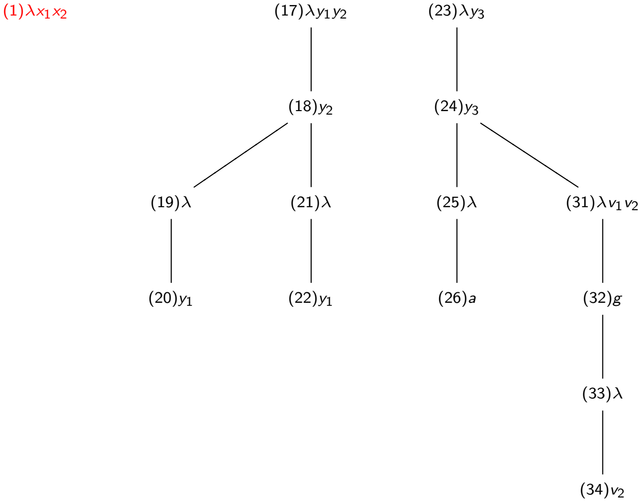
(6)z_1
(7)$\lambda s_1 s_2$
(8)z_2
(9)λ
(10)b
(11)λ
(12)s_2
(17)$\lambda y_1 y_2$
(18)y_2
(19)λ
(20)y_1
(21)λ
(22)y_1
(23)λy_3
(24)y_3
(25)λ
(26)a
(31)$\lambda v_1 v_2$
(32)g
(33)λ
(34)v_2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(21)λ
(22)y_1
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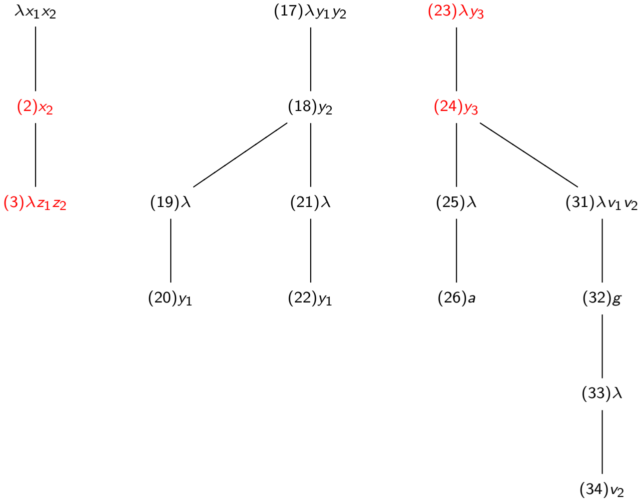
(24)y_3
(25)λ
(26)a
(31)$\lambda v_1 v_2$
(32)g
(33)λ
(34)v_2



λx1x2
(2)x2
(3)λz1z2
(17)λy1y2
(18)y2
(19)λ
(20)y1
(21)λ
(22)y1
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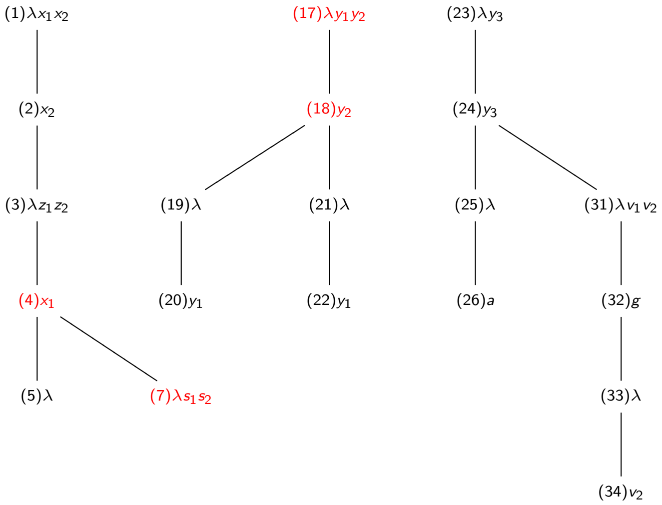
(24)y3
(25)λ
(26)a
(31)λv1v2
(32)g
(33)λ
(34)v2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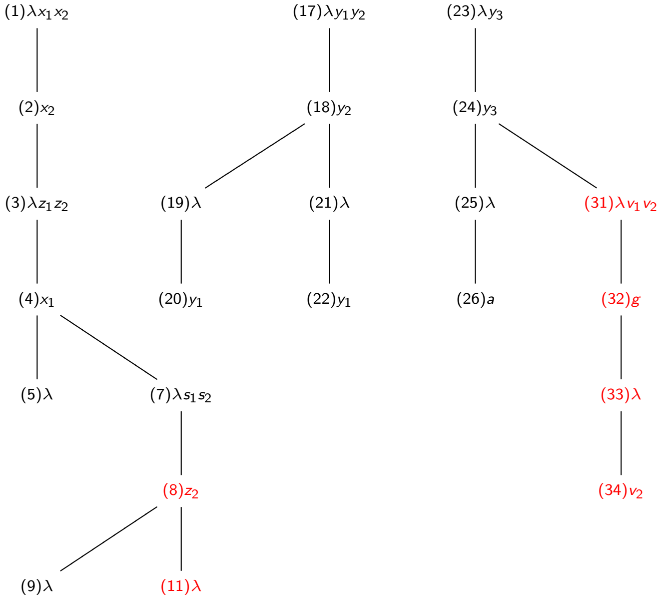
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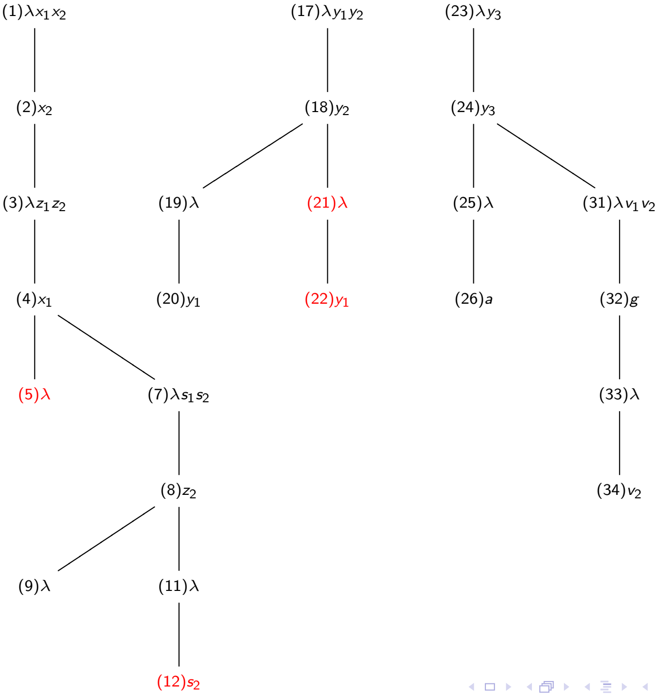
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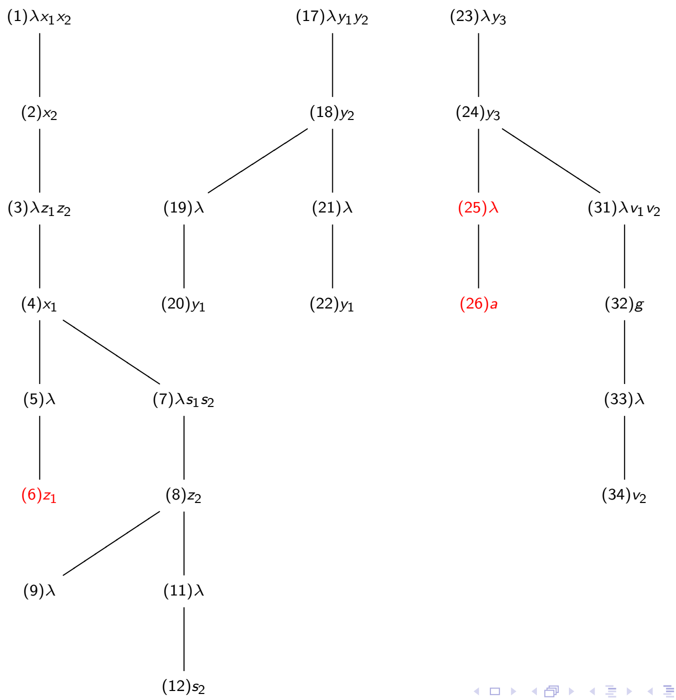
(1)λx1x2
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(5)λ
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(7)λs1s2
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(9)λ
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- Two steps in proof assuming an arbitrary solution term
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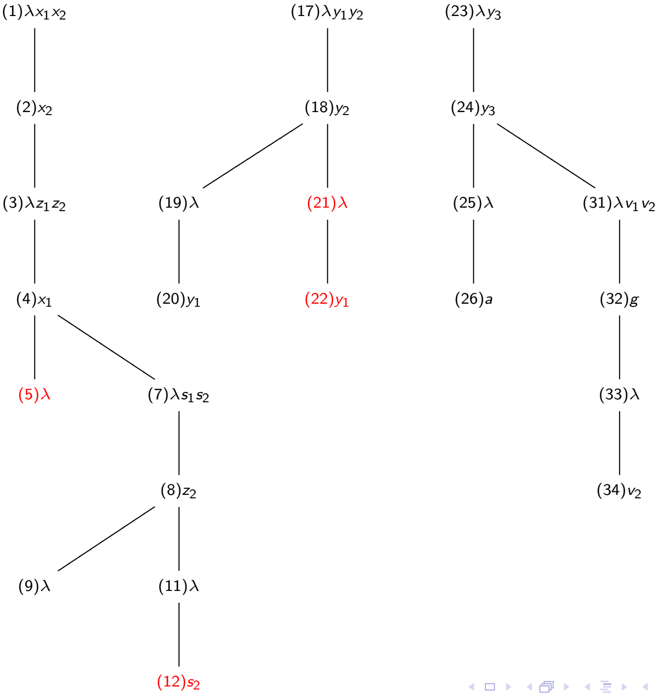
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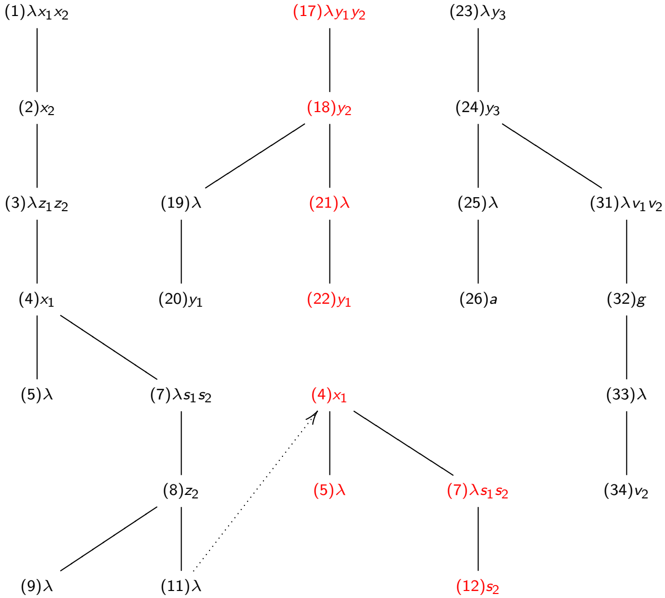
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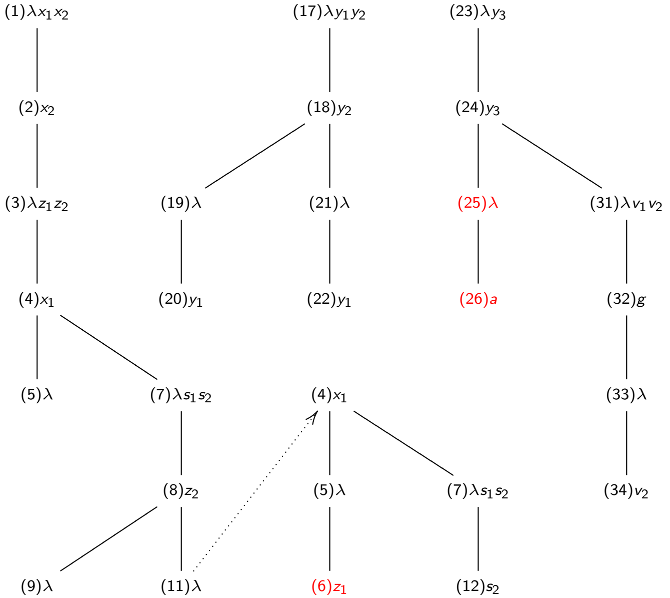

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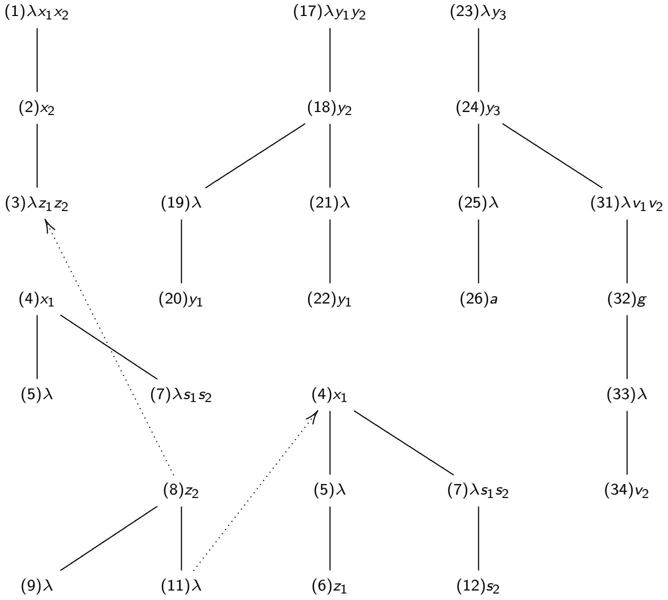

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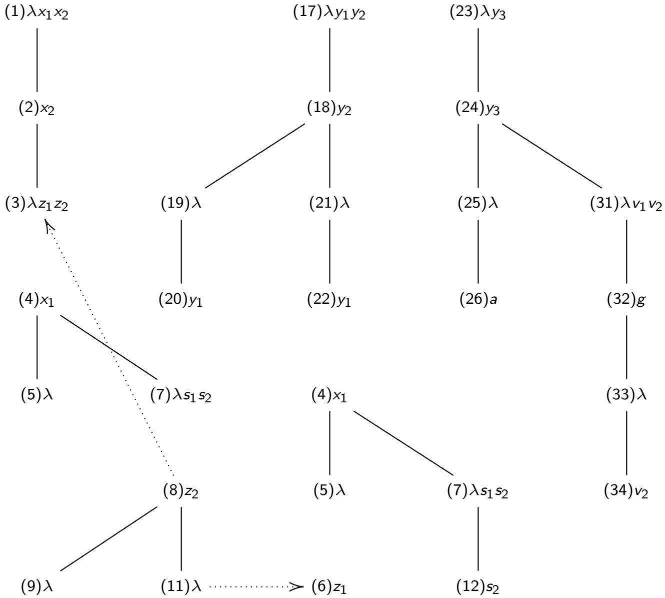
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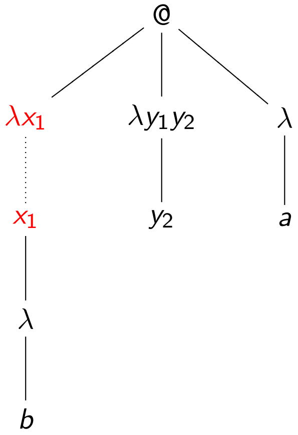
- THE END

Games only work for long normal forms

- $x : ((\mathbf{0},\mathbf{0},\mathbf{0}),\mathbf{0},\mathbf{0})$ β-interpolation problem $x(\lambda y_1 y_2 . y_2)a = a$



- $\lambda x_1 . x_1 b(\lambda y_1 y_2 . y_2)a \rightarrow^*_\beta a$
- No natural game?