Dependency Tree Automata

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FOSSACS 2009, York, March 23rd

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- Equivalence checking: is state s equivalent to t ?
- Mostly computing dyadic fixed points e.g. bisimulations to solve it. May need algebraic/combinatorial properties of reachability sets/generators of graph

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3. $\vdots \quad \vdots \quad \vdots$
4. Application of tree automata to higher-order matching
   [Comon + Jurski 1997, Stirling 2007, work described here]

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- order of $\mathbf{0}$ is 1;
- order of $(A_1, \ldots, A_n, \mathbf{0})$ is $1 + \max\{\text{order of } A_i\text{s}\}$

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- $t, t' : A$ are α-equivalent renamings of each other

Reduction and all

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\begin{array}{lll}
(\beta) & (\lambda x.t)v \rightarrow_\beta t\{v/x\} & \{\cdot/\cdot\}\ \text{Substitution} \\
(\eta) & \lambda x.(tx) \rightarrow_\eta t & x \text{ not free in } t
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- When η-long forms $=_\beta$ is $=_{\beta\eta}$

Monadic decision questions

- Given a type A and property ϕ, is the following set non-empty?

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- BUT: PROBLEM

Monster type $M = ((((\mathbf{0}, \mathbf{0}), \mathbf{0}), \mathbf{0}), \mathbf{0}, \mathbf{0})$, order 5

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- To write down this subset of terms up to α-equivalence requires an alphabet of unbounded size
- There are automata with infinite alphabets but none applicable e.g.[Segoufin 2006 survey: "Automata and logics for words and trees over an infinite alphabet"]

Binding Trees

- $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ is alphabet, each $s \in \Sigma$ has arity $\mathrm{ar}(s) \geq 0$. Σ_1 are binders (with arity 1); Σ_2 are (bound) variables; Σ_3 other symbols (such as constants)

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- Compare nested words/trees [Alur + Chaudhuri + Madhusudan, 2006]

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- Finite alphabet $+$ edge relation $\downarrow$

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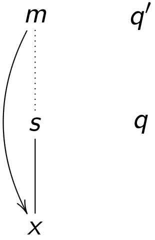

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- OPEN PROBLEM IN PAPER: is non-emptiness decidable for the alternating automata? (Undecidable shown in [Ong + Tzevelekos 2009], to appear at LICS 09)

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- Decision question: given $v = u$, does it have a solution ?
- Order is max order of the x_is

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- Undecidable for $=_\beta$ [Loader 2003]

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- Restrict constants in solution terms to be those in u plus fresh $c : \mathbf{0}$: finite alphabet

Example

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- A canonical solution has the form
  $\lambda x.x(\lambda y.y(\lambda y_1^1 \dots y(\lambda y_1^k.s) \dots))$
  where s is the constant a or one of the variables y_1^j, $1 \leq j \leq k$.

Padovani's decidability proof for order 4 and tree automata

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- In the proof states of automaton built from $\equiv_{A'}^{u'}$ representatives where u' subterm of u, A' subtype of A.

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     $\leq$ 4th-order: only need to examine finitely many terms to build $\equiv^{u'}_{A'}$ classes

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  [Stirling 2007]
  - States of automaton built from variable profiles (based on [Ong 2006] schema paper). Proof uses game-theoretic characterisation of interpolation from [Stirling 2005,2006]

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  - OPEN QUESTION: is there a class of tree automata between dependency tree automata and alternating tree automata that can recognise solutions to matching.