An Introduction to Decidability of Higher-Order Matching

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Simply typed λ-calculus

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  - $\mathbf{0}$ single base type (for simplicity)
  - $A \to B$ type of functions from A to B

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- Order of $(A_1, \ldots, A_n, \mathbf{0})$ is $k + 1$ where k is maximum of orders of A_is

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1. if x (f) has type A then $x : A$ ($f : A$)
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3. if $t : A \rightarrow B$ and $u : A$ then $(tu) : B$

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- order of $t : A$ is order A
- closed $t : A$ no free variables
- $t, t' : A$ are α-equivalent renamings of each other

Dynamics: reduction

$$(\beta) \quad (\lambda x.t)v \rightarrow_\beta t\{v/x\} \qquad \{\cdot/\cdot\} \text{ Substitution}$$
$$(\eta) \quad \lambda x.(tx) \rightarrow_\eta t \qquad x \text{ not free in } t$$

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- Facts
  Strong normalisation and confluence (of $\to_\beta$, $\to_\eta$, $\to_{\beta\eta}$)
- Equivalence: $t =_{\beta\eta} t'$ if there are s, s'

$$\begin{array}{ccc} t & \to^*_{\beta\eta} & s \\ & & \shortparallel_\alpha \\ t' & \to^*_{\beta\eta} & s' \end{array}$$

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- t in η-long β-normal form if t is in β-normal form and there is no t' such that $t' \to_\eta t$
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- If t, v are in lnf and $tv \to_\beta^* s$ in β-normal form then s is in lnf
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- Inf $= \eta$-long β-normal form
- If t, v are in Inf and $tv \rightarrow^*_\beta s$ in β-normal form then s is in Inf
  No η-reductions when terms in Inf
- $T_A(C)$ is the set of closed Inf terms of type A whose constants belong to C.

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- $x : (\mathbf{0}, \mathbf{0})$ $\lambda x.x : ((\mathbf{0}, \mathbf{0}), \mathbf{0}, \mathbf{0})$ in β-normal form not Inf

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- Notation: $\lambda x_1 \ldots x_k.t$ abbreviates $\lambda x_1. \ldots .\lambda x_k.t$

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- Notation: $\lambda x_1 \ldots x_k.t$ abbreviates $\lambda x_1. \ldots \lambda x_k.t$
- Monster type $M = ((((\mathbf{0},\mathbf{0}),\mathbf{0}),\mathbf{0}),\mathbf{0},\mathbf{0})$ has order 5
- $\lambda xy.x(\lambda z_1.x(\lambda z_2.z_1 y)) \in T_M(\emptyset)$
  when $x : (((\mathbf{0},\mathbf{0}),\mathbf{0}),\mathbf{0})$, $z_i : (\mathbf{0},\mathbf{0})$ and $y : \mathbf{0}$.

Decision questions

- Higher-order unification
- $v = u$ contains free variables $x_1, \ldots, x_n$
- Solution $\theta = \{t_1/x_1, \ldots, t_n/x_n\}$ such that $v\theta =_{\beta\eta} u\theta$
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- Undecidable (even at order 2) [Huet 1972;Goldfarb 1981]

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- Decidable for all orders [Stirling 2006; 2009; 2012]

Matching is essentially monadic

- Given $v = u$ with $x_1 : A_1, \ldots, x_n : A_n$ in v
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- Conceptually simpler problem: just one free variable
- Called "interpolation": $x\, w_1 \ldots w_k = u$ where $x : (B_1, \ldots, B_k, \mathbf{0})$, $w_i : B_i$, $u : \mathbf{0}$ in Inf.
- Solution t in Inf such that $tw_1 \ldots w_k \rightarrow_\beta^* u$

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- Canonical solution $\lambda z.z\, t_1 \ldots t_n$ where t_is are closed
  Consequently $v\{t_1/x_1, \ldots, t_n/x_n\} \rightarrow^*_\beta u$
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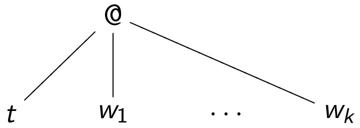
- Restrict constants in solution terms to be those in u plus fresh $b : \mathbf{0}$ Restricts to finitely many constants

Interpolation trees

- t (of the right type and in Inf) a potential solution to $xw_1 \dots w_k = u$ (Assume u does not have bound variables)
- Interpolation tree



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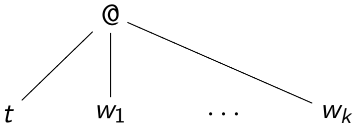

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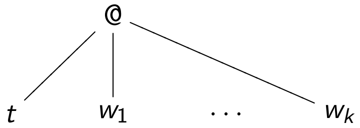

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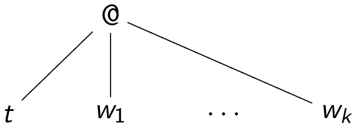

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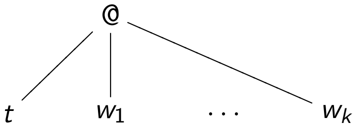

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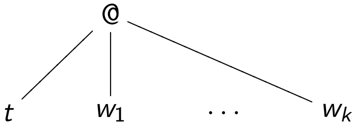

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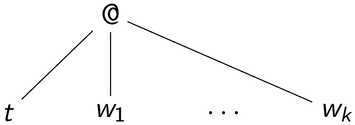

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- Based on [Kobayashi 2009, Kobayashi and Ong 2009]; use intersection types

$$
\begin{array}{lcll}
\theta & := & r \mid \tau \to \theta & r \text{ subterm } u \\
\tau & := & \bigwedge_{i_1 \in I_1} \theta_{i_1} \wedge \ldots \wedge \bigwedge_{i_m \in I_m} \theta_{i_m} & I_j \text{ finite}
\end{array}
$$

Typing rules

$$\Gamma \vdash a : a \qquad \Gamma, x : \bigwedge_{i \in I} \theta_i \vdash x : \theta_j \quad j \in I$$

$$\frac{\Gamma \vdash t_1 : r_1 \qquad \Gamma \vdash t_m : r_m}{\Gamma \vdash f\ t_1 \ldots t_m : f r_1 \ldots r_m}$$

$$\frac{\Gamma \vdash t : \bigwedge_{i_1 \in I_1} \theta_{i_1} \wedge \ldots \wedge \bigwedge_{i_m \in I_m} \theta_{i_m} \rightarrow r \qquad \Gamma \vdash v_j : \theta_{i_j} \text{ for all } i_j \in I_j}{\Gamma \vdash t v_1 \ldots v_m : r}$$

$$\frac{\Gamma, x_1 : \bigwedge_{i_1 \in I_1} \theta_{i_1}, \ldots, x_m : \bigwedge_{i_m \in I_m} \theta_{i_m} \vdash t : r}{\Gamma \vdash \lambda x_1 \ldots x_m . t : \bigwedge_{i_1 \in J_1} \theta_{i_1} \wedge \ldots \wedge \bigwedge_{i_m \in J_m} \theta_{i_m} \rightarrow r}\ J_i \subseteq I_i$$

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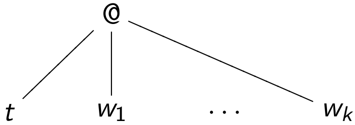
- Reduces matching to inhabitation problem for intersection types
- BUT, inhabitation is undecidable [Urzyczyn 1999]

Tree automata



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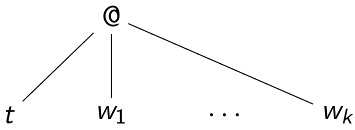

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- Tree automaton
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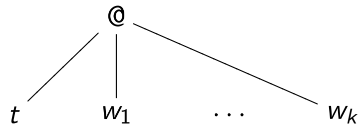

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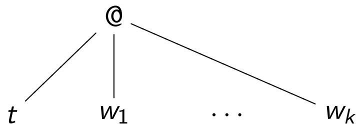

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- Non-emptiness decidable; sets recognised are regular

Tree automata for 4th-order [Comon and Jurski 1997]

- 4th-order $xw_1 \dots w_k = u$ and C constants in u plus b
- For finite alphabet Σ consider a potential solution term $t = \lambda x_1 \dots x_k.s$

$$s ::= z_j^n \mid x_i v_1 \dots v_m \mid f s_1 \dots s_m \qquad f \in C,\ m \geq 0$$
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- Node n of t is labelled with a variable/constant; let t^n be subterm rooted at n
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- n is accessible; look at evaluation of t^n
  which is normal form of $t^n\{w_1/x_1, \dots, w_k/x_k\}$
  this is a subterm of u when its subterms can be replaced by leaf variables z_j^i

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  which is normal form of $t^n\{w_1/x_1, \dots, w_k/x_k\}$
  this is a subterm of u when its subterms can be replaced by leaf variables z_j^i
- Guarantees finite alphabet for accessible nodes: reuse variables

Tree automata for 4th-order [Comon and Jurski 1997]

- Finite states:
  inaccessible nodes: $-$ for "doesn't matter"
  accessible nodes: U is subterms of u closed under replacement of subterms with leaves labelled with finitely many different variables $z^i_j : \mathbf{0}$

$$\{[e], [\lambda z^n_1 \dots z^n_m . e], [\lambda x_1 \dots x_k . u] \mid e \in U \cup \{-\}\}$$

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- Finite transitions: such as

$$\begin{array}{ll} z \Rightarrow [z] \quad z \Rightarrow [-] & a \Rightarrow [a] \quad c \Rightarrow [-] \\ \lambda x_1[ga] \Rightarrow [\lambda x_1.ga] & x_1[-][gz] \Rightarrow [gz] \\ \dots & \end{array}$$

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- Theorem The tree automaton associated with a 4th-order $xw_1 \dots w_k = u$ accepts t iff t solves $xw_1 \dots w_k = u$

Tree automata for 5th-order: PROBLEMS

- Finite alphabet ?

  can stack 2nd-order variables

  $\lambda xy.x(\lambda z_1.x(\lambda z_2 \ldots (\lambda z_n.z_n(z_{n-1}(\ldots z_1(y)) \ldots)) \ldots))$

  Need an infinite alphabet

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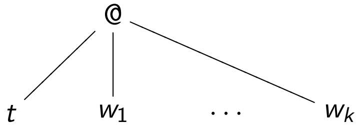
  Number of evaluations can be infinite

Interpolation trees

- t (of the right type and in Inf) a potential solution to $xw_1 \ldots w_k = u$ (Assume u does not have bound variables)
- Interpolation tree



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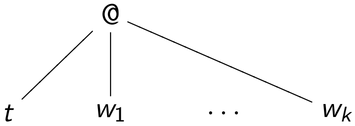

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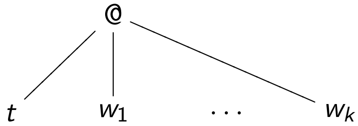
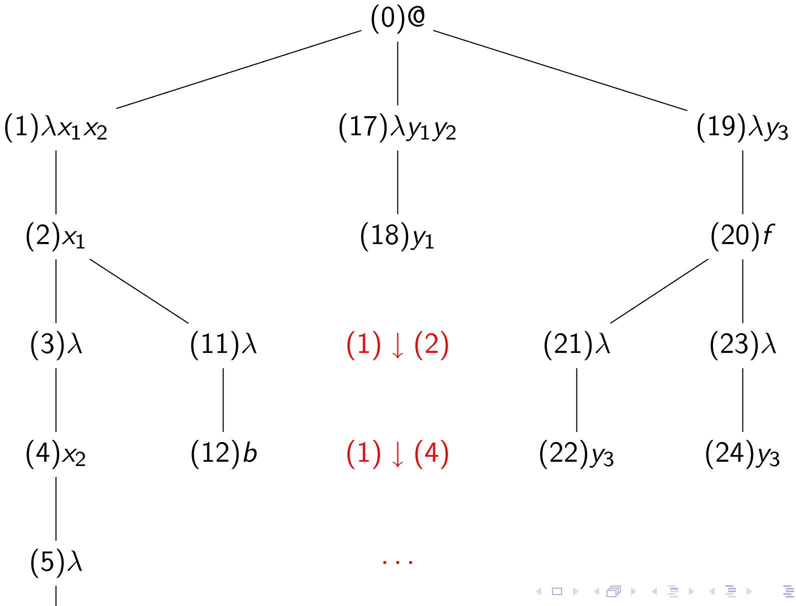


- Special representation of term trees
- $t, w_1, \ldots, w_k$ binding trees with dummy lambdas and binding relation $\downarrow$ between nodes
- $n \downarrow m$ if n labelled $\lambda \overline{y}$ binds y_j, label at m

Example: $x(\lambda y_1 y_2 . y_1)(\lambda y_3 . f y_3 y_3) = faa$



Tree automata for 5th-order

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  Fact For all A and finite C, there is a finite Σ such that every $t \in T_A(C)$ up to α-equivalence is a binding Σ-tree.

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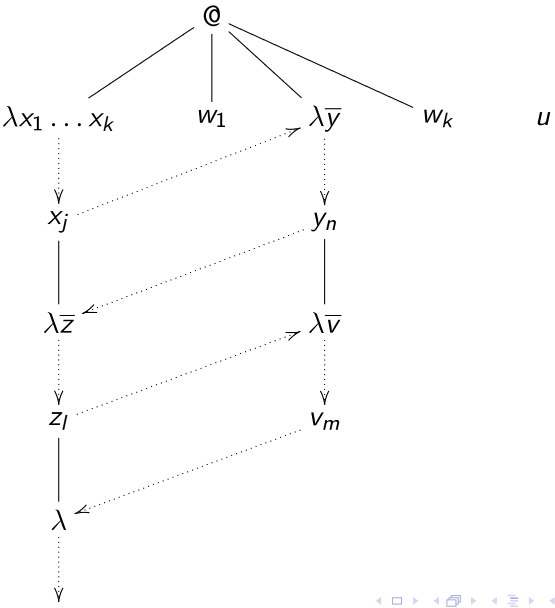
- Second problem ? need finer analysis than evaluation of an accessible node

Games: automaton moving around interpolation tree



Game/automaton $G(t, E)$

- Play: sequence $n_1 q_1 \theta_1, \dots, n_l q_l \theta_l$
  - n_i is a node of the interpolation tree,
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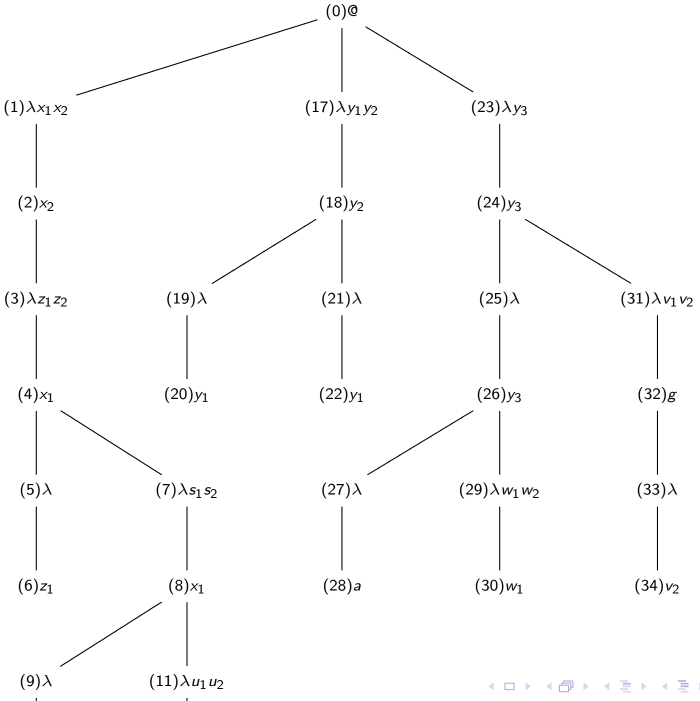
- Player $\forall$ loses every play in $G(t, E)$ iff t solves E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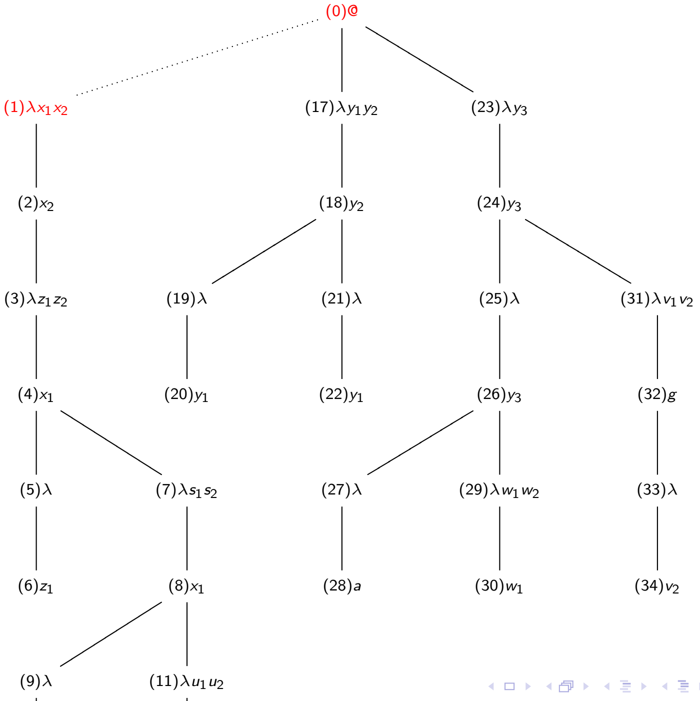
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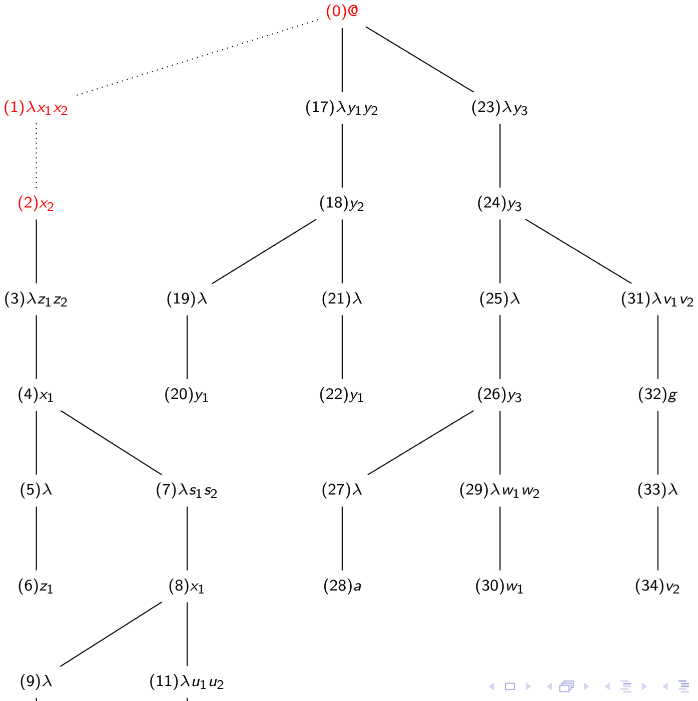

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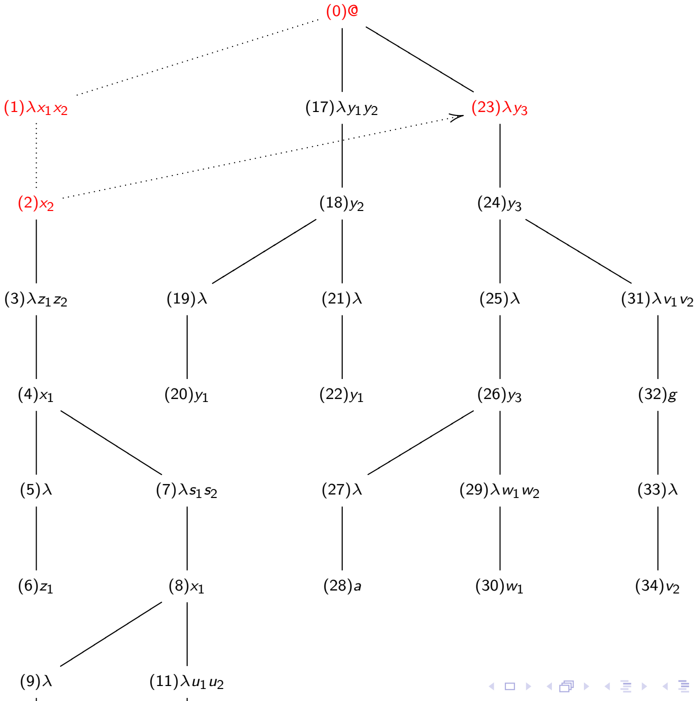
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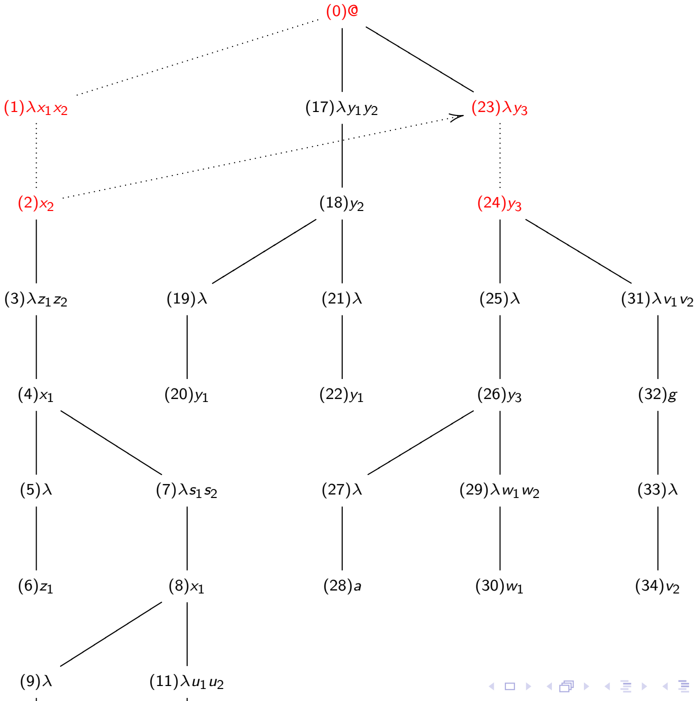
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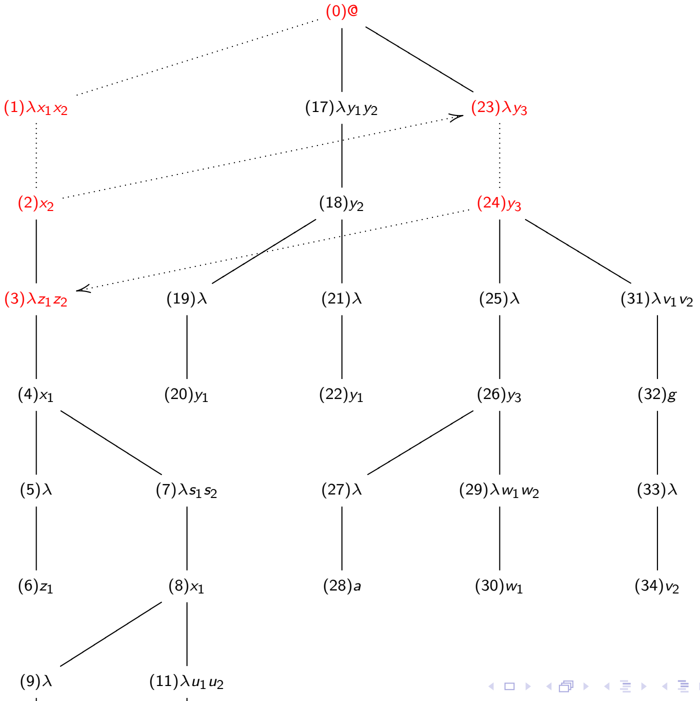
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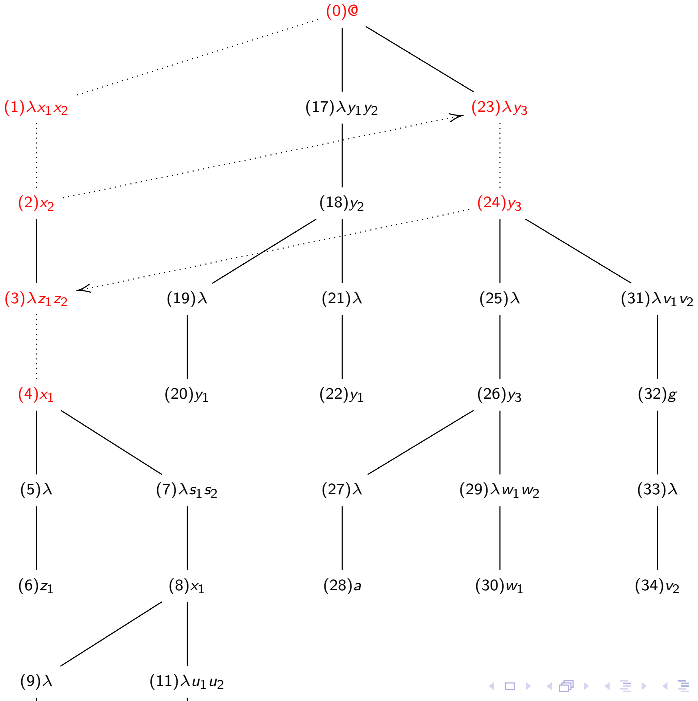
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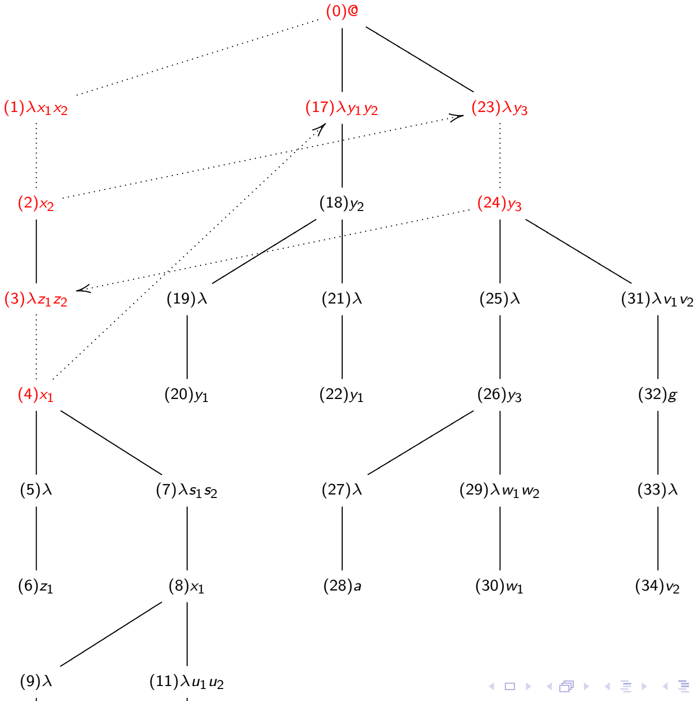
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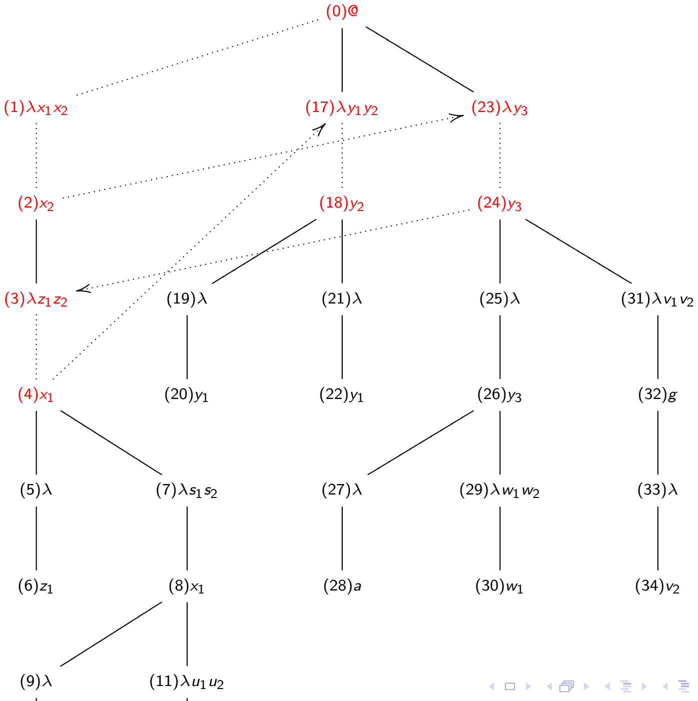
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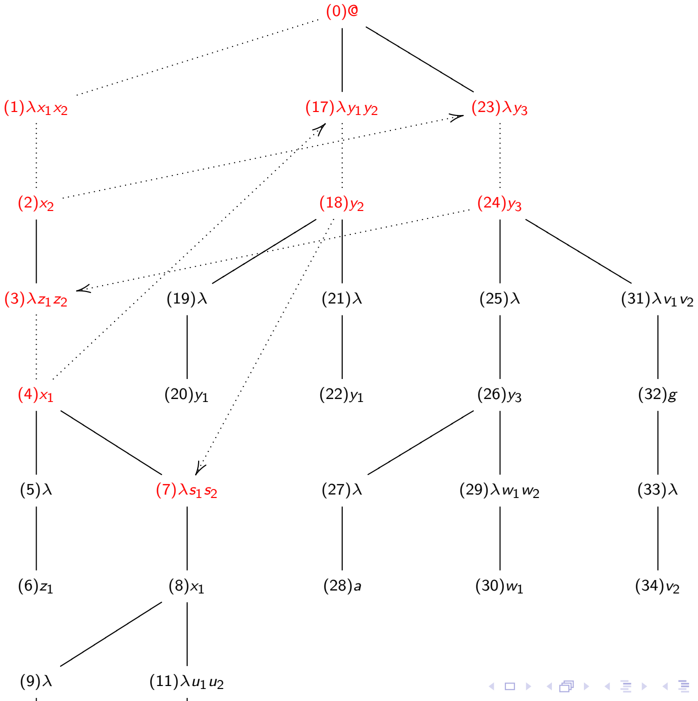
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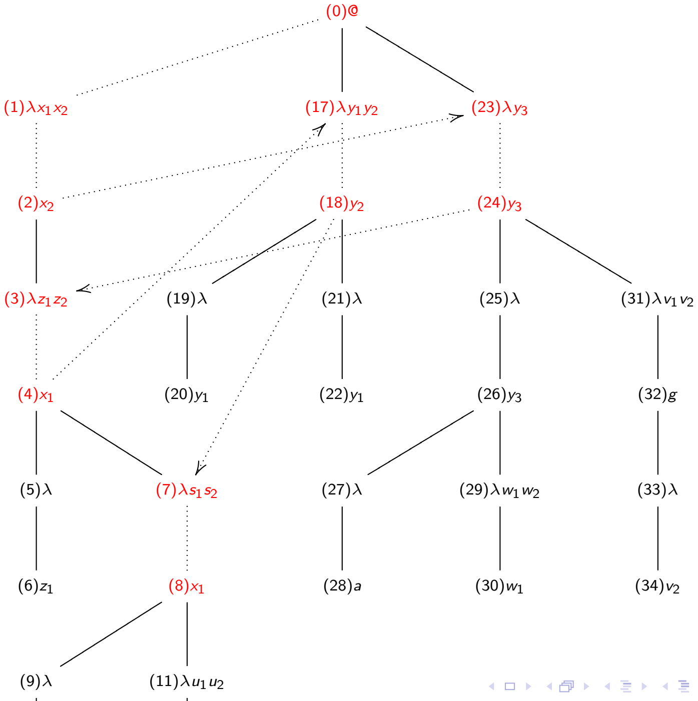
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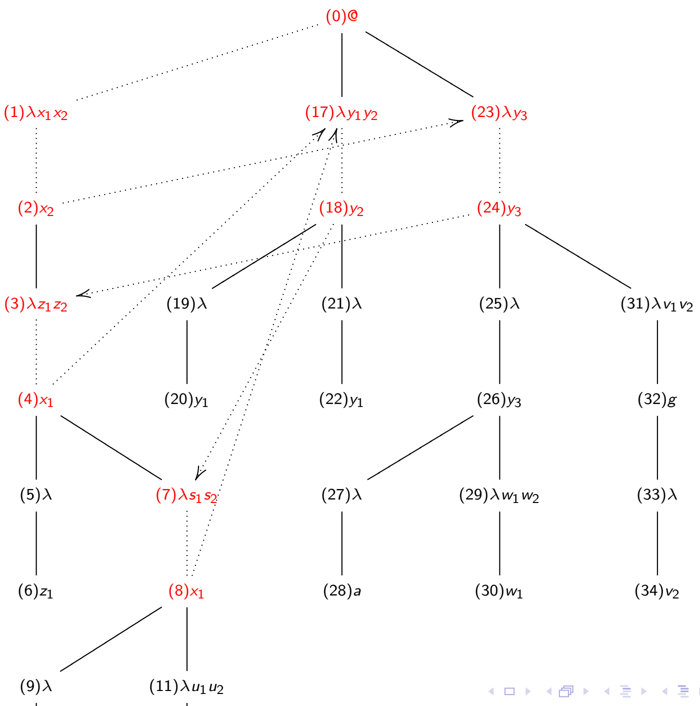
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(4)x_1
(20)y_1
(22)y_1
(26)y_3
(32)g
(5)λ
(7)$\lambda s_1 s_2$
(27)λ
(29)$\lambda w_1 w_2$
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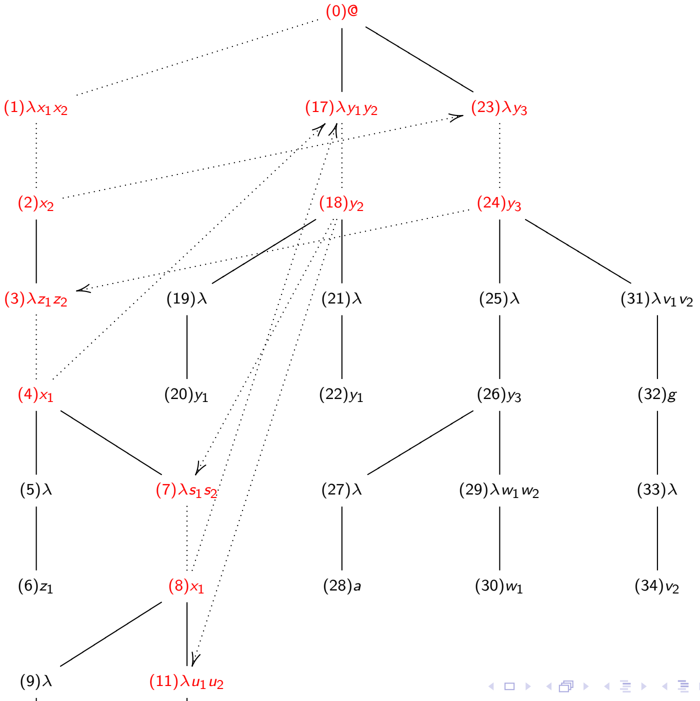
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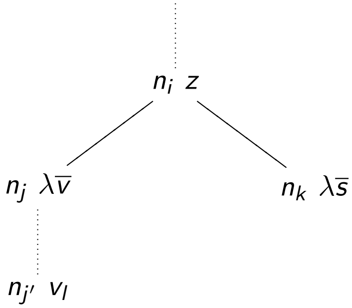
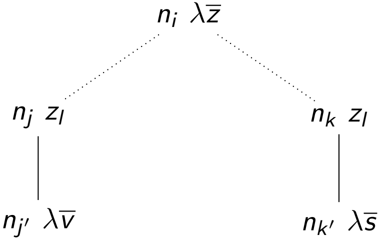
- Theorem The set of solutions of E is recognised by a special alternating binding tree automaton BUT non-emptiness of such automata undecidable [Stirling 2009; Ong, Tzevelekos 2009]

Uniformity properties of game playing I



- If play is at n_i then at n_j and then later at n_k then inbetween there must have been a position at an $n_{j'}$ bound by n_j

Uniformity properties of game playing II



- If play is at n_i then at n_j and then later at $n_{j'}$; then at n_k then there must be a corresponding sequence to that between n_j and $n_{j'}$ between n_k and $n_{k'}$ unless there is a different $\forall$ choice

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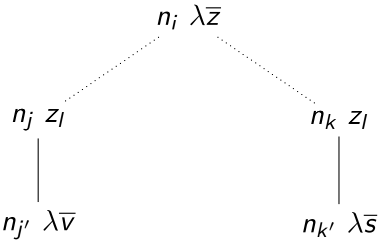

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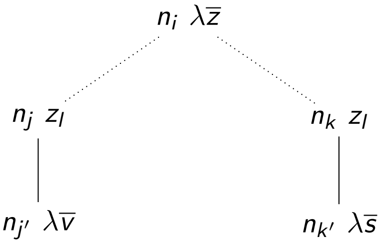

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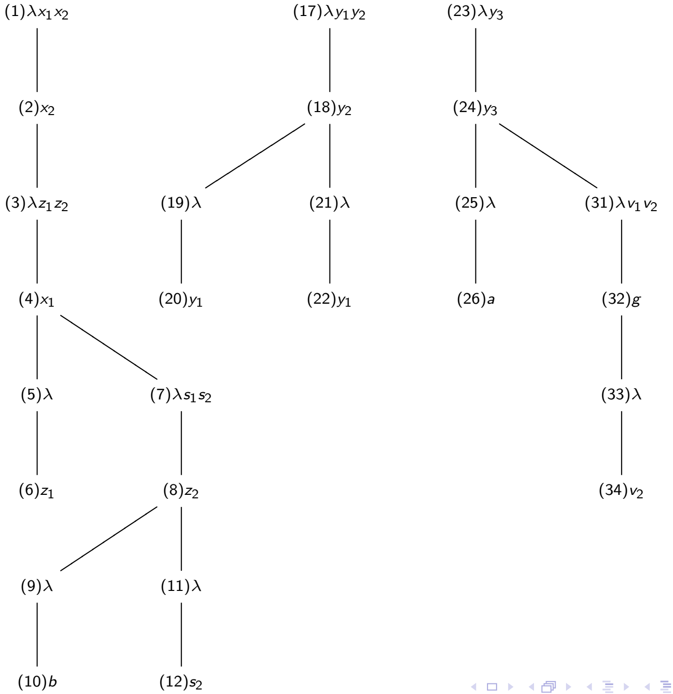
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(12)s_2
(17)$\lambda y_1 y_2$
(18)y_2
(19)λ
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(21)λ
(22)y_1
(23)λy_3
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(25)λ
(26)a
(31)$\lambda v_1 v_2$
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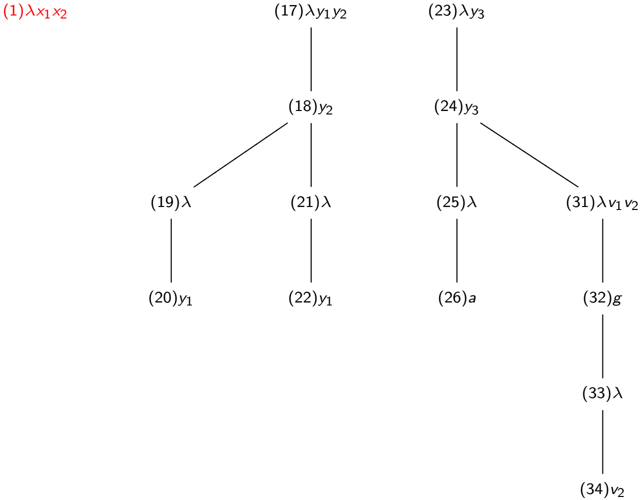
(33)λ
(34)v_2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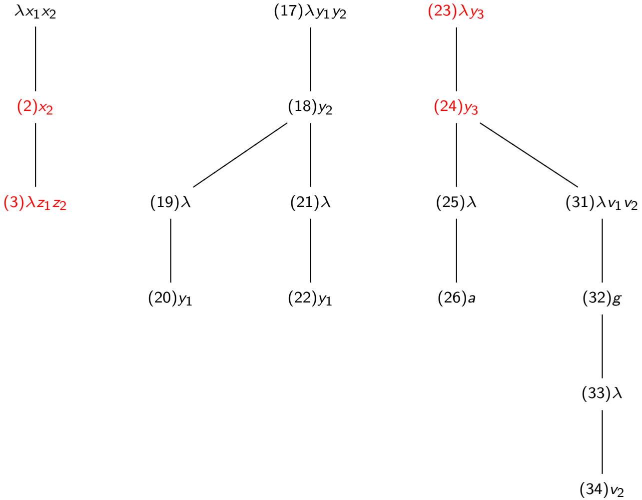
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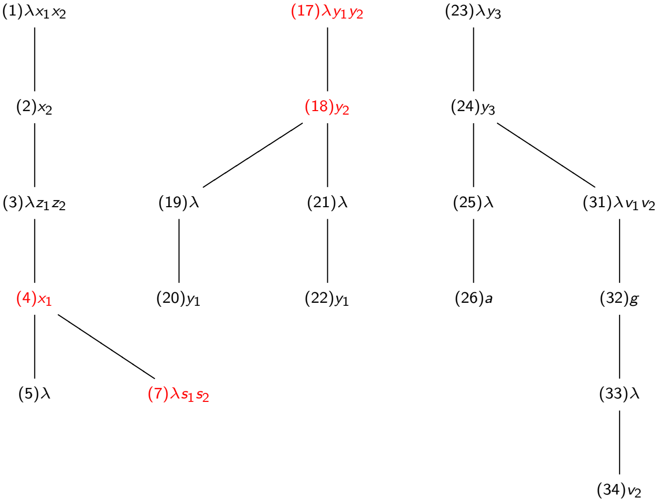
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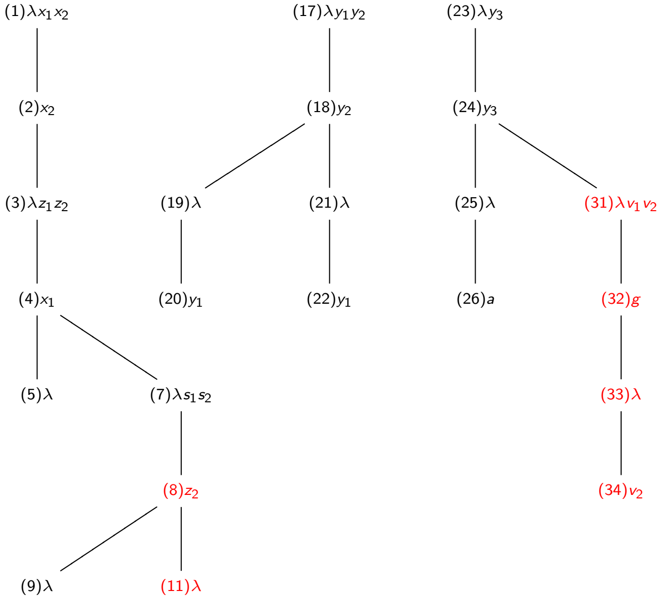
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(32)g
(33)λ
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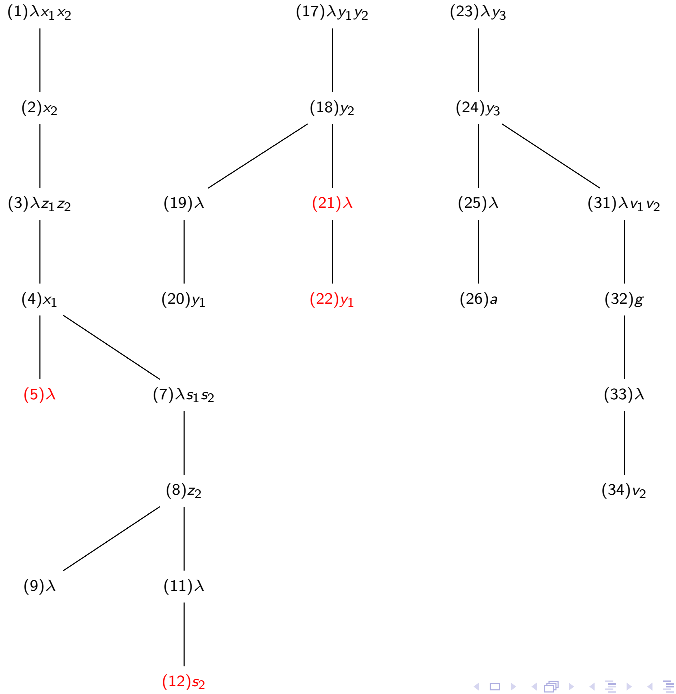

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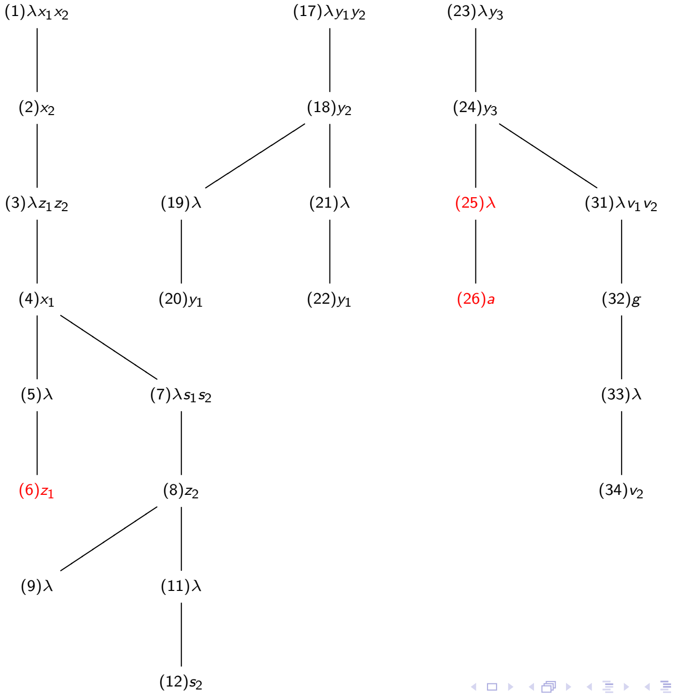
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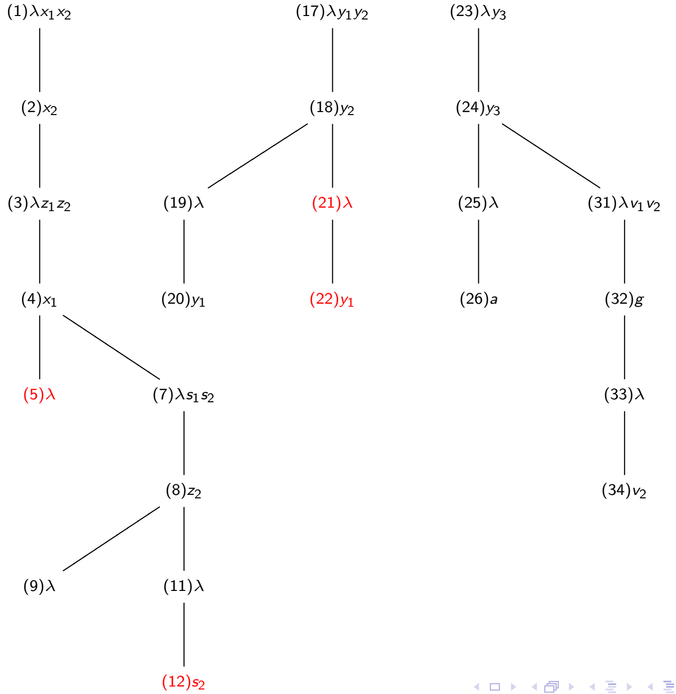

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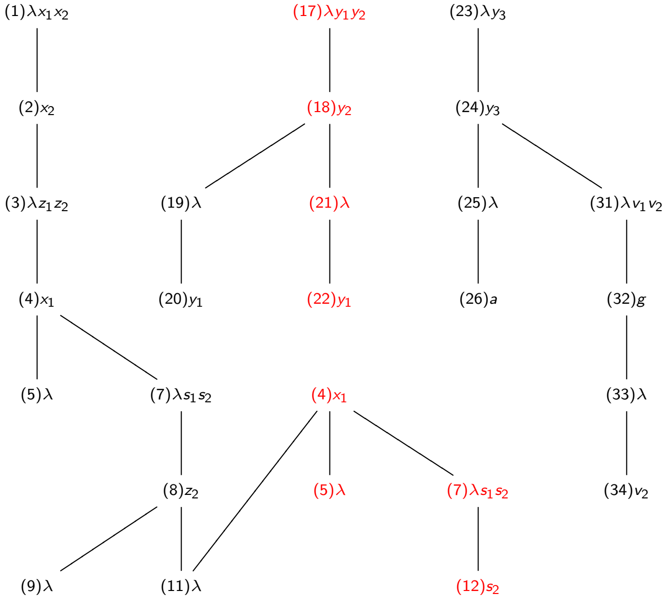
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(25)λ
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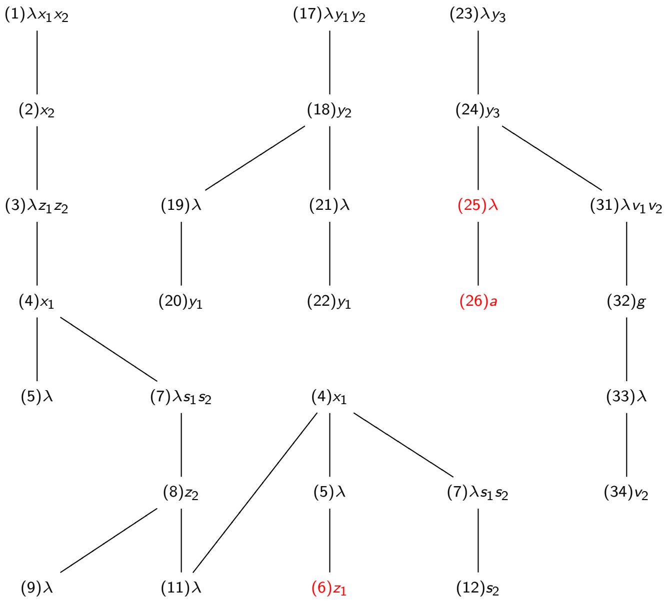

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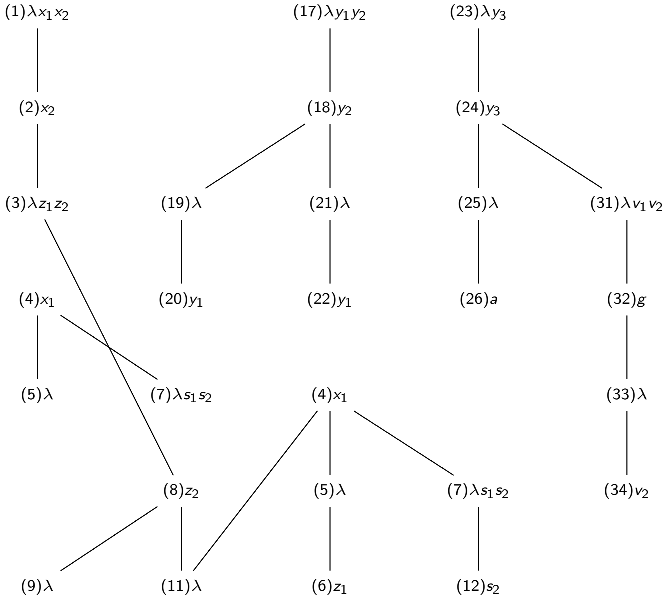
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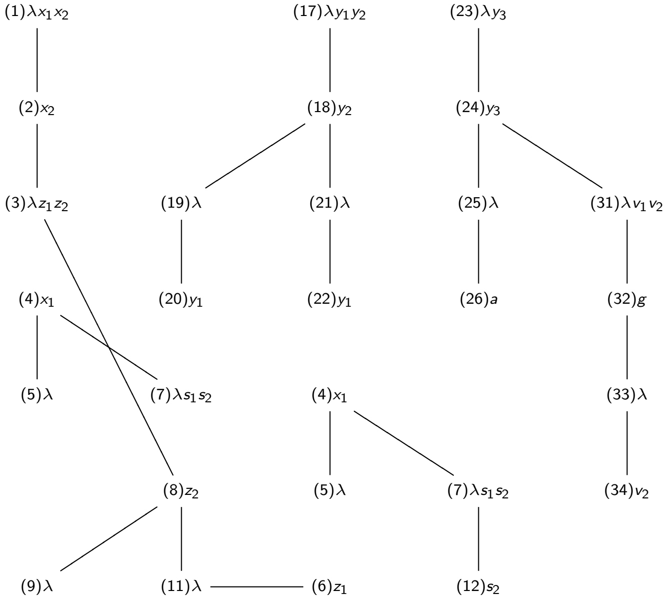

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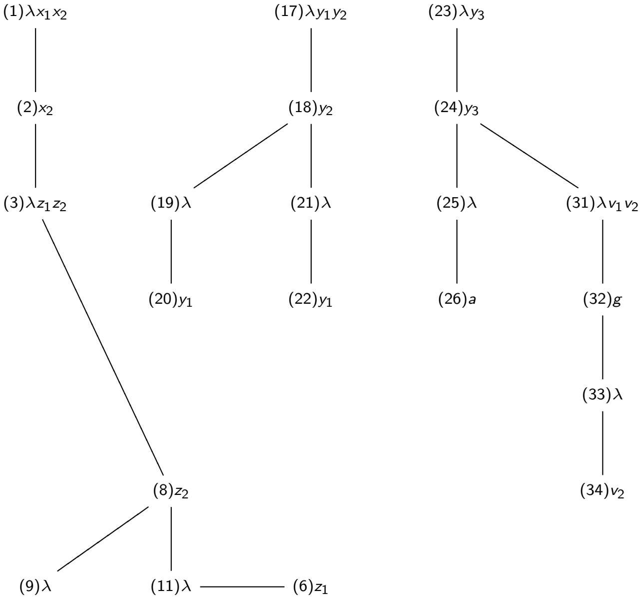
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The retract problem

Type A is a retract of B, if there are $t : A \longrightarrow B$ and $s : B \longrightarrow A$ such that $s(tx) =_{\beta\eta} x$

DECISION PROBLEM: given A, B, is A a retract of B ?

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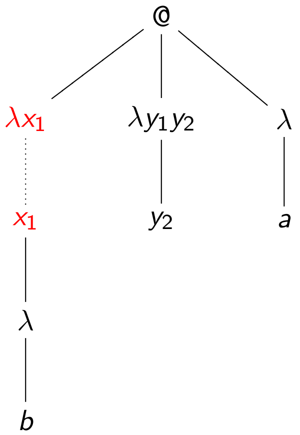
- Proof system for general case with EXPSPACE upper bound; soundness and completeness uses games [ICALP 13]

Games only work for long normal forms

- $x : ((\mathbf{0},\mathbf{0},\mathbf{0}),\mathbf{0},\mathbf{0})$ β-interpolation problem $x(\lambda y_1 y_2.y_2)a = a$



- $\lambda x_1.x_1 b(\lambda y_1 y_2.y_2)a \to_\beta^* a$
- No natural game?