Language Theory and Infinite Graphs

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Characterising DPDA

- A DPDA can be converted into a determinized SDG, and configuration $p\alpha$ is converted into sum configuration $\text{sum}(p\alpha)$ of the SDG where
  
  $$G^c(p\alpha) \sim G^d(\text{sum}(p\alpha))$$

- (Conversely, a determinized SDG can be transformed into a DPDA)

- DPDA equivalence problem = to deciding whether two admissible configurations of a determinized SDG are bisimulation equivalent

\[
\alpha_1 + \ldots + \alpha_n \sim \beta_1 + \ldots + \beta_m \ ?
\]
Notation

- $E, F, G, \ldots$ range over admissible configurations

- $+$ can be extended: if $E$ and $F$ are admissible, $E \cup F$ is admissible, $E$ and $F$ are disjoint, $E \cap F = \emptyset$, then $E + F$ is the admissible configuration $E \cup F$

- Also use sequential composition, if $E$ and $F$ are admissible, then $EF$ is the admissible configuration $\{\beta \gamma : \beta \in E \text{ and } \gamma \in F\}$

- $E = E_1 G_1 + \ldots + E_n G_n$ is a head/tail form, if the head $E_1 + \ldots + E_n$ is admissible and at least one $E_i \neq \emptyset$, and each tail $G_i \neq \emptyset$

  If we write $E = E_1 G_1 + \ldots + E_n G_n$ then $E$ is a head/tail form
Congruence

- Decision question $E \sim F$ ?

- for all $j$, $G_j \sim H_j$ iff $E_1G_1 + \ldots + E_nG_n \sim E_1H_1 + \ldots + E_nH_n$

- We can tear apart configurations and replace parts with equivalent parts
Notation

- For $X \in S$, $w(X)$ is the unique shortest word $v \in A^+$ such that $X \xrightarrow{v} \varepsilon$

- $w(E)$ is the unique shortest word for configuration $E$

- $M$ is the maximum norm of the SDG

- “$E$ following $u$”, written $E \cdot u$, is the unique $F$ such that $E \xrightarrow{u} F$
Interested in showing, for instance, that $YX + Z \sim YYX + YZ + Z$
DPDA decision procedure

• Given by a deterministic tableau proof system where goals are reduced to subgoals

• Goals and subgoals have the form $E \equiv F$, is $E \sim F$? $E, F$ admissible

• Rules are (generalisations of) unfold, UNF, and balance rules, BAL(L) and BAL(R). (CUT free)
Proof rules: UNF

\[
\begin{align*}
E \vdash F \\
(E \cdot a_1) \vdash (F \cdot a_1) & \quad \ldots \quad (E \cdot a_k) \vdash (F \cdot a_k) \\
A & = \{a_1, \ldots, a_k\}
\end{align*}
\]
Example

\[
\begin{align*}
X & \xrightarrow{a} X & Y & \xrightarrow{a} \epsilon & Z & \xrightarrow{a} \emptyset \\
Z & \xrightarrow{b} \epsilon & X & \xrightarrow{c} X & Y & \xrightarrow{c} YY & Z & \xrightarrow{c} YZ + Z
\end{align*}
\]

\[
YX + Z = YYX + YZ + Z
\]

\[
\begin{align*}
X &= YX + Z & \epsilon &= \epsilon & YYX + YZ + Z &= YYYX + YYZ + YZ + Z
\end{align*}
\]
Proof rules: BAL(L)

\[ X_1H_1 + \ldots + X_kH_k \vdash F \]

\[ : \quad \vdash \quad C \]

\[ E_1H_1 + \ldots + E_kH_k \vdash F' \]

\[ \vdash \quad E_1(F \cdot w(X_1)) + \ldots + E_k(F \cdot w(X_k)) \vdash F' \]

where C is the condition

1. Each \( E_i \neq \varepsilon \) and at least one \( H_i \neq \varepsilon \)

2. There are \( \max\{ |w(X_i)| : E_i \neq \emptyset \text{ for } 1 \leq i \leq k \} \) applications of UNF between the top goal and the bottom goal, and no application of any other rule
Proof rules: BAL(R)

\[ F \vdash X_1H_1 + \ldots + X_kH_k \]

\[ \vdots \]

\[ F' \vdash E_1H_1 + \ldots + E_kH_k \]

\[ \vdash F' \vdash E_1(F \cdot w(X_1)) + \ldots + E_k(F \cdot w(X_k)) \]

where \( C \) is the condition

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Example

\[
X \xrightarrow{a} X \quad Y \xrightarrow{a} \epsilon \quad Z \xrightarrow{a} \emptyset \quad X \xrightarrow{b} \epsilon \quad Y \xrightarrow{b} \emptyset \\
Z \xrightarrow{b} \epsilon \quad X \xrightarrow{c} X \quad Y \xrightarrow{c} YY \quad Z \xrightarrow{c} YZ + Z
\]

\[
\begin{align*}
YX + Z &= Y Y X + YZ + Z \quad (F) \\
\text{UNF} &
\end{align*}
\]

\[
\begin{align*}
\text{BAL(L)} &
\end{align*}
\]

\[
w(Y) = a, \ w(Z) = b \text{ and } (F \cdot w(Y)) = Y X + Z, \ (F \cdot w(Z)) = \epsilon
\]
Procedure

- Start with an initial goal, $E \vdash F$, and deterministically build a proof tree by applying the tableau rules (there is an ordering on the rules)

- Goals are thereby reduced to subgoals

- Need to define final goals. Rules are not applied to final goals

A successful tableau is a finite proof tree all of whose leaves are successful final goals

Otherwise a tableau is unsuccessful
Unsuccessful final goals

\[ E \equiv F \quad (\text{exactly one of } E, F \text{ is } \emptyset) \]
Infinite proof tree?

\[
X X^5 = A A^5
\]

\[
\frac{Y X X^5 = C A^5}{Y X A^5 = C A^5} \quad \text{BAL(L)}
\]

\[
\frac{X X A^5 = A A^6}{Y X^5 = A^5} \quad \text{UNF}
\]

\[
\frac{X X A^5 = A A^6}{Y X A^6 = C A^6} \quad \text{BAL(L)}
\]

\[
X A^5 = A^6 \quad \text{UNF}
\]
Extensions

- If \( E = E_1 G_1 + \ldots + E_n G_n \), \( F = F_1 H_1 + \ldots + F_m H_m \), then
  \[ F \text{ is a tail extension of } E \text{ provided that for each } i : 1 \leq i \leq m, \]
  \[ H_i = K_1^i G_1 + \ldots + K_n^i G_n \]
- The extension \( e \) is the \( m \)-tuple \((K_1^1 + \ldots + K_n^1, \ldots, K_1^m + \ldots + K_n^m)\) without the \( G_i \)'s, the width of \( e \) is \( m \), and \( F \) is said to extend \( E \) with \( e \)
- Extensions are matrices, written in a linear notation
- If \( E = E_1 G_1 + \ldots + E_n G_n \) and \( F = F_1 G_1 + \ldots + F_n G_n \), then \( E \) extends \( F \) by \( e = (\varepsilon + \emptyset + \ldots + \emptyset, \ldots, \emptyset + \emptyset + \ldots + \varepsilon) \). Identity extension \((\varepsilon)\)
Composing extensions

If

\[ E = E_1 G_1 + \ldots + E_l G_l \] and

\[ E' = E'_1 G'_1 + \ldots + E'_{m} G'_{m} \] and

\[ E'' = E''_1 G''_1 + \ldots + E''_{n} G''_{n} \] and

\[ E' \] extends \( E \) by \( e = (J_1^1 + \ldots + J_l^1, \ldots, J_1^m + \ldots + J_l^m) \) and

\[ E'' \] extends \( E' \) by \( f = (K_1^1 + \ldots + K_m^1, \ldots, K_1^n + \ldots + K_m^n) \),

then \( E'' \) extends \( E \) by \( e f = (H_1^1 + \ldots + H_l^1, \ldots, H_1^n + \ldots + H_l^n) \)

where \( H_j^i = K_j^i J_j^1 + \ldots + K_j^m J_j^m \)
Example

\[ E = YG_1 + ZG_2 \]
\[ E' = YG'_1 + ZG'_2 \]
\[ E'' = YG''_1 + ZG''_2 \]

where \( G_1 = X \) and \( G_2 = \varepsilon \)
where \( G'_1 = YX + Z \) and \( G'_2 = \varepsilon \)
where \( G''_1 = YYX + YZ + Z \) and \( G''_2 = \varepsilon \)

\( E' \) extends \( E \) by \( e = (Y + Z, \emptyset + \varepsilon) \) and
\( E'' \) extends \( E' \) by \( f = e = (Y + Z, \emptyset + \varepsilon) \)

Therefore, \( E'' \) extends \( E \) by \( ef = (YY + (YZ + Z), \emptyset + \varepsilon) \).
Notation

Assume goals with the same heads

\[\text{goal } g \quad (E) \quad E_1 G_1 + \ldots + E_n G_n \doteq F_1 G_1 + \ldots + F_n G_n \quad (F)\]

\[\text{goal } h \quad (E') \quad E_1 H_1 + \ldots + E_n H_n \doteq F_1 H_1 + \ldots + F_n H_n \quad (F')\]

\(h\) extends \(g\) by \(e\), if \(E'\) extends \(E\) by \(e\) (and \(F'\) extends \(F\) by \(e\))
Repeating patterns

Width is 3. Assume \( g(i), h(i), i : 1 \leq i \leq 8 \)

\[
E_1G_1^i + E_2G_2^i + E_3G_3^i = F_1G_1^i + F_2G_2^i + F_3G_3^i
\]

\[
E_1H_1^i + E_2H_2^i + E_3H_3^i = F_1H_1^i + F_2H_2^i + F_3H_3^i
\]

Assume extensions \( e_1, e_2 \) and \( e_3 \) for \( g(i) \) and \( h(i) \)

If each \( g(i), 1 \leq i \leq 8 \), and each \( h(i), 1 \leq i < 8 \), \( m \)-true then \( h(8) \) is \( m \)-true
General case

If there are two families of goals $g(i), h(i), 1 \leq i \leq 2^n$

\[
E_1 G^i_1 + \ldots + E_n G^i_n = F_1 G^i_1 + \ldots + F_n G^i_n
\]

\[
E_1 H^i_1 + \ldots + E_n H^i_n = F_1 H^i_1 + \ldots + F_n H^i_n
\]

and extensions $e_1, \ldots, e_n$ such that for each $e_j$ and $i \geq 0$

\[
g(2^j i + 2^{j-1} + 1) \text{ extends } g(2^j i + 2^{j-1}) \text{ by } e_j
\]

\[
h(2^j i + 2^{j-1} + 1) \text{ extends } h(2^j i + 2^{j-1}) \text{ by } e_j.
\]

and each $g(i), i : 1 \leq i \leq 2^n$, and each $h(j), j : 1 \leq j < 2^n$, is $m$-true, then $h(2^n)$ is $m$-true.
Successful final goals

\[ E \dot{=} F \quad \text{the root goal} \]
\[ \vdots \]
\[ \vdots \quad \text{there are goals } g(i) \; h(i) \ldots \]
\[ \vdots \quad h(2^n) \text{ is } E' \dot{=} F' \]

\[ E \dot{=} E \quad E' \dot{=} F' \]
Decidability

Propositions

- Every tableau is finite

- $E \sim F$ iff the tableau with root $E \models F$ is successful

Complexity upper bound is primitive recursive (because of extensions). The number of goals involved may grow with the number of states when applying extension theorem as it is width of a goal that counts

**Corollary:** $E \sim F$ iff $E \sim_{f(n)} F$

where $f$ is the primitive recursive function, $n$ is the size of the DPDA, $E$ and $F$
Higher-order pushdown automata

Basic transitions of the form for order $n$

\[ pX \xrightarrow{a} q \text{ Op} \quad \text{where Op} \in \{\text{swap}(a), \text{push}(i), \text{pop}(i) \mid 1 < i \leq n\} \]

Stacks of stacks of stacks . . .

Example at order 2

\[
\begin{align*}
p'Z & \xrightarrow{a} pZ \\
pZ & \xrightarrow{a} pXZ \\
qX & \xrightarrow{b} q\epsilon \\
rX & \xrightarrow{c} r\epsilon \\
pX & \xrightarrow{a} pXX \\
pX & \xrightarrow{b} q \text{ push} \\
pX & \xrightarrow{b} q \text{ push} \\
qZ & \xrightarrow{c} r \text{ pop} \\
rZ & \xrightarrow{c} r\epsilon
\end{align*}
\]
Results/Open Questions

• The languages generable by order 2 pushdown automata are indexed languages (mildly context-sensitive languages) introduced by Aho 1968

• The monadic second-order theory of any graph generated by a higher-order automaton is decidable (Knapik, Niwinski and Urzyczyn 2002)
  Proof nice mixture of Rabin’s theorem and geometry of interaction

• Are languages generable by order $n$, $n > 2$, context-sensitive?

• Is language equivalence decidable for determinisitc higher-order pda?
  Intimately related to higher-order schema problem open since 1960s
Thanks for listening