Proof Systems for Retracts in Simply Typed Lambda Calculus

Colin Stirling
cps@inf.ed.ac.uk

LFCS
School of Informatics
University of Edinburgh

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  - if $M : \rho \to \sigma$ and $N : \rho$ then $(MN) : \sigma$

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  - if $M : \rho \to \sigma$ and $N : \rho$ then $(MN) : \sigma$
- closed $M : \sigma$ no free variables
- $M, M' : \sigma$ are α-equivalent renamings of each other

Dynamics: reduction

$$(\beta) \quad (\lambda x.M)N \to_\beta M\{N/x\} \qquad \{\cdot/\cdot\} \text{ Substitution}$$

$$(\eta) \quad \lambda x.(Mx) \to_\eta M \qquad x \text{ not free in } M$$

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$$\begin{array}{ccc} M & \rightarrow^*_{\beta\eta} & N \\ & & \shortparallel_\alpha \\ M' & \rightarrow^*_{\beta\eta} & N' \end{array}$$

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- β-equivalence: $M =_\beta M'$ similar; $\to^*_\beta$ replaces $\to^*_{\beta\eta}$

Retracts in simply typed lambda calculus

ρ is a retract of τ, if there are terms $C : \rho \to \tau$ and $D : \tau \to \rho$ such that $D(C(x^\rho)) =_{\beta\eta} x$

DECISION PROBLEM: given ρ, τ, is ρ a retract of τ ?

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- Here: proof system for general case with EXPSPACE upper bound

Some known properties: [LPS92, RU02]

Let $\rho = \rho_1 \to \ldots \to \rho_l \to a$ and $\tau = \tau_1 \to \ldots \to \tau_n \to a$

- Retracts must share "target" type a. So, ρ retract of τ if there is $D : \tau \to \rho$ and $C : \rho \to \tau$ and instantiating variables

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  - x occurs only once in $C(x^\rho)$
  - H is ε if ρ and τ are built from a single base type
  - if $T_i^{\rho_i}$ contains an occurrence of y_j then it is the head variable of $T_i^{\rho_i}$, z_i occurs in $S_j^{\tau_j}$ and $T_i^{\rho_i}$ contains no other occurrences of any y_k, $1 \leq k \leq n$

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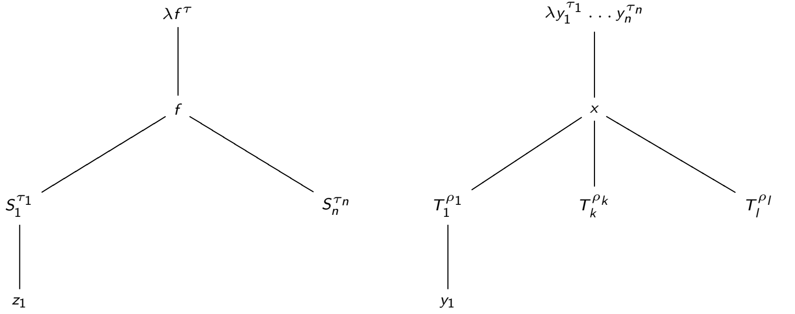

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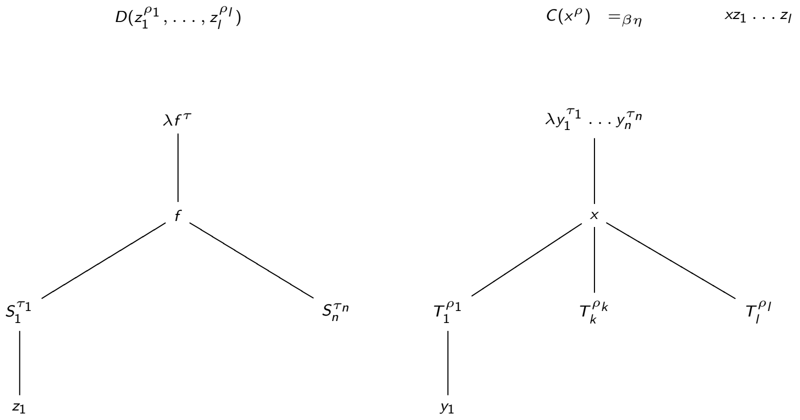

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- So, $T_1(\overline{v}_1)(S_1(x_1/z_1)) =_{\beta\eta} x_1\overline{v}_1$; i.e., ρ_1 is retract of τ_1 and $\rho_2 \to \ldots \to \rho_l \to a$ is retract of $\tau_2 \to \ldots \to \tau_n \to a$

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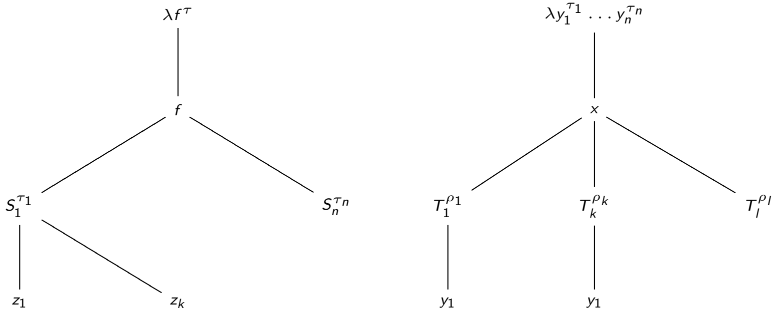

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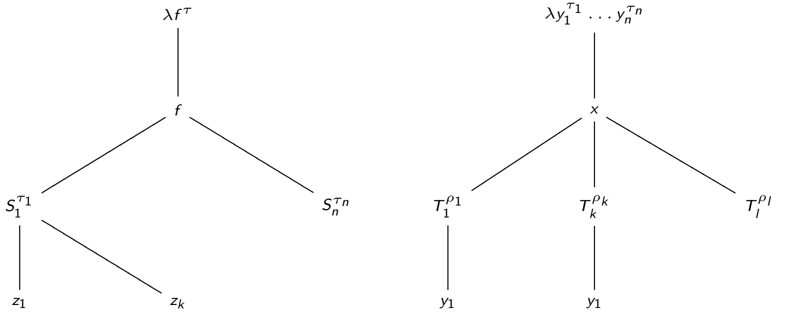

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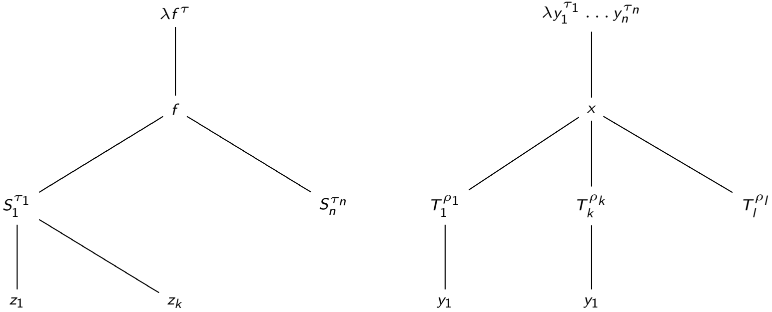

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- May not guarantee ρ_1 is a retract of τ_1 and ... and ρ_k is a retract of τ_1; do have $\rho_{k+1} \to \ldots \to \rho_l \to a$ is retract of $\tau_2 \to \ldots \to \tau_n \to a$

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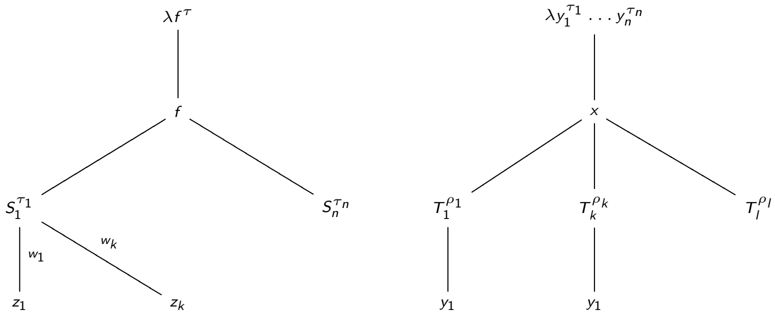

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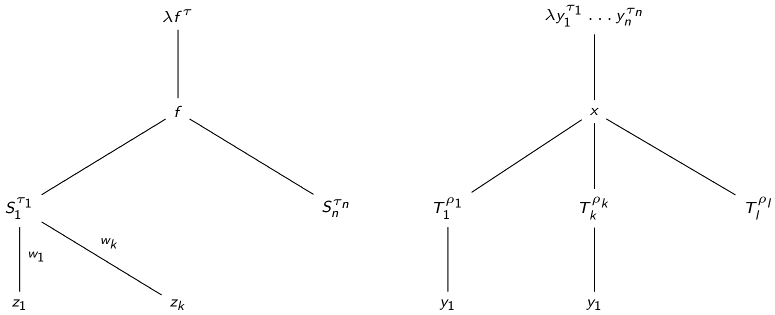

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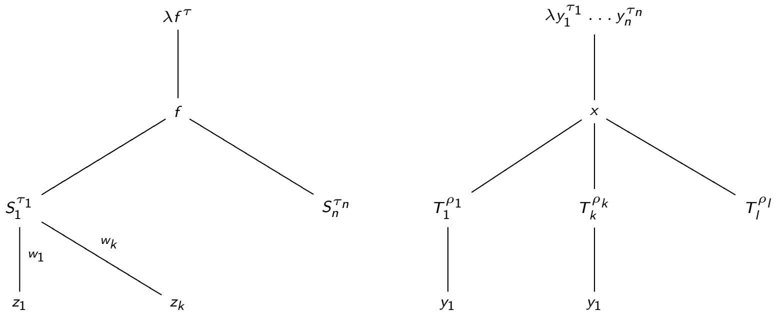

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- Captured using operator $\tau_1 \upharpoonright w_i$
- Guarantees ρ_i is a retract of $\tau_1 \upharpoonright w_i$

Goal directed proof system (for singleton B)

$$I \quad \rho \trianglelefteq \rho$$

$$W \; \frac{\rho \trianglelefteq \sigma \to \tau}{\rho \trianglelefteq \tau}$$

$$C \; \frac{\delta \to \rho \trianglelefteq \sigma \to \tau}{\delta \trianglelefteq \sigma \qquad \rho \trianglelefteq \tau}$$

$$P_1 \; \frac{\rho_1 \to \ldots \to \rho_k \to \rho \trianglelefteq \sigma \to \tau}{[\rho_1, \ldots, \rho_k] \trianglelefteq \sigma \qquad \rho \trianglelefteq \tau}$$

$$P_2 \; \frac{[\rho_1, \ldots, \rho_k] \trianglelefteq \sigma}{\rho_1 \trianglelefteq \sigma \restriction w_1 \quad \ldots \quad \rho_k \trianglelefteq \sigma \restriction w_k}$$

$w_1 \sqsubset \ldots \sqsubset w_k$ are k-minimal realisable paths of type σ

Example proof tree

$$
\dfrac{\dfrac{(\sigma \to o) \to (\sigma \to o) \to o \trianglelefteq (\sigma \to (o \to o \to o) \to o) \to o}{[\sigma \to o, \sigma \to o] \trianglelefteq \sigma \to (o \to o \to o) \to o} \quad o \trianglelefteq o}{\sigma \to o \trianglelefteq \sigma \to o \quad \sigma \to o \trianglelefteq \sigma \to o}
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- Let $\sigma' = \sigma \to (o \to o \to o) \to o$
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- In both cases the paths preclude second component of σ'

General case: multiple base types

- Further operation on paths: $w(\sigma)$ subtype after w

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- Proof system just differs in one rule

$$P_2' \; \frac{[\rho_1, \ldots, \rho_k] \trianglelefteq \sigma}{\rho_1 \trianglelefteq v_1(\sigma) \restriction w_1 \qquad \ldots \qquad \rho_k \trianglelefteq v_k(\sigma) \restriction w_k}$$

where the realisable paths are $v_1 w_1, \ldots, v_k w_k$

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- Define a canonical presentation of paths in terms
- Uniformity poperties of game play underpin combinatorics on paths: notions of realisable (families of) paths, k-minimal paths, ...
- Main proofs of soundness and completeness are then inductive: see full version

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- BUT checking that a subgoal obeys the side conditions in the case of the product rules requires exponential space
- So, EXPSPACE decision procedure
- Can this be reduced to PSPACE?

Examples of when ρ is retract of τ

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