Applying automata and games to simply typed lambda calculus

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- model checking + equivalence checking
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- Equivalence checking: is state s equivalent to t ?
- Mostly computing dyadic fixed points e.g. bisimulations to solve it. May need algebraic/combinatorial properties of reachability sets/generators of graph

Active research goal: transfer these techniques to

finite/infinite state systems with binding

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4. Application of tree automata and games to deciding higher-order matching
   [Comon, Jurski 1997; Stirling 2005-2012, work described here]

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- Order of $\mathbf{0}$ is 1;
- Order of $(A_1, \ldots, A_n, \mathbf{0})$ is $k + 1$ where k is maximum of orders of A_is

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- order of $t : A$ = order A
- closed $t : A$ no free variables
- $t, t' : A$ are α-equivalent renamings of each other

Dynamics: reduction

$$(\beta) \quad (\lambda x.t)v \to_\beta t\{v/x\} \qquad \{\cdot/\cdot\} \text{ Substitution}$$
$$(\eta) \quad \lambda x.(tx) \to_\eta t \qquad x \text{ not free in } t$$

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  Strong normalisation and confluence (of $\to_\beta$, $\to_\eta$, $\to_{\beta\eta}$)
- Equivalence: $t =_{\beta\eta} t'$ if there are s, s'

$$\begin{array}{ccc} t & \to^*_{\beta\eta} & s \\ & & \|_\alpha \\ t' & \to^*_{\beta\eta} & s' \end{array}$$

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- If t, v are in Inf and $tv \rightarrow_\beta^* s$ in β-normal form then s is in Inf
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- $T_A(C)$ is the set of closed Inf terms of type A whose constants belong to C.

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- Notation: $\lambda x_1 \ldots x_k.t$ abbreviates $\lambda x_1.\ldots.\lambda x_k.t$
- Monster type $M = ((((\mathbf{0},\mathbf{0}),\mathbf{0}),\mathbf{0}),\mathbf{0},\mathbf{0})$, has order 5
- $\lambda xy.x(\lambda z_1.x(\lambda z_2.z_1 y)) \in T_M(\emptyset)$
  when $x : (((\mathbf{0},\mathbf{0}),\mathbf{0}),\mathbf{0})$, $z_i : (\mathbf{0},\mathbf{0})$ and $y : \mathbf{0}$.

Decision questions

- Higher-order unification
- $v = u$ contains free variables $x_1, \ldots, x_n$
- Solution $\theta = \{t_1/x_1, \ldots, t_n/x_n\}$ such that $v\theta =_{\beta\eta} u\theta$
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- Undecidable (even at order 2) [Huet 1972;Goldfarb 1981]

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- Decidable for all orders [Stirling 2006; 2009; 2012]

Matching is essentially monadic

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  where $x : ((A_1, \ldots, A_n, \mathbf{0}), \mathbf{0})$
- Conceptually simpler problem: just one free variable
- Called "interpolation": $x\, w_1 \ldots w_k = u$
  where $x : (B_1, \ldots, B_k, \mathbf{0})$, $w_i : B_i$, $u : \mathbf{0}$ in Inf.
- Solution t in Inf such that $t w_1 \ldots w_k \rightarrow_\beta^* u$

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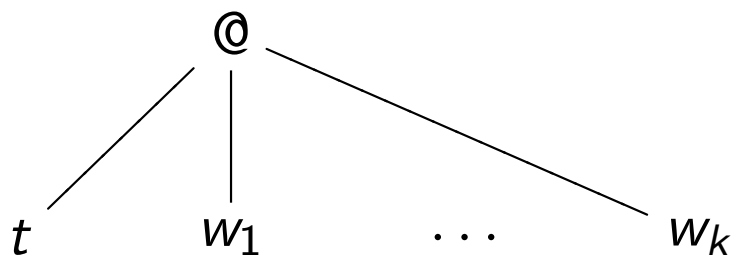
- Restrict constants in solution terms to be those in u plus fresh $b : \mathbf{0}$ Restricts to finitely many constants

Interpolation trees

- t (of the right type and in Inf) a potential solution to $xw_1 \ldots w_k = u$ (Assume u does not have bound variables)
- Interpolation tree



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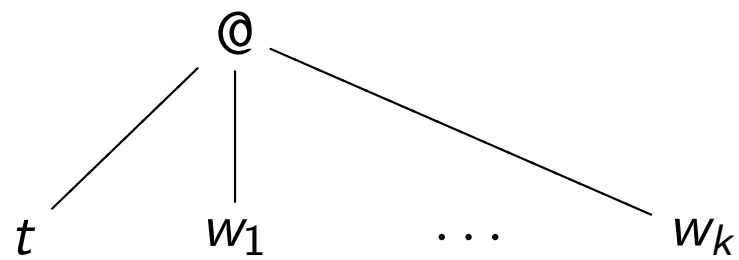

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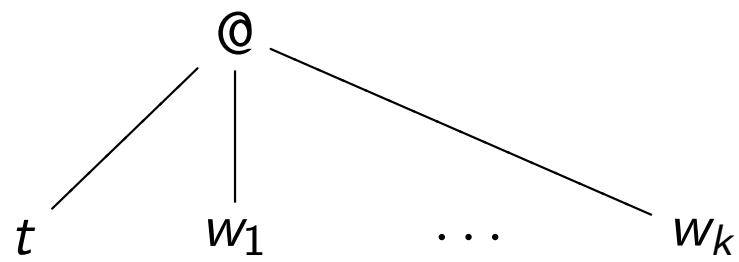

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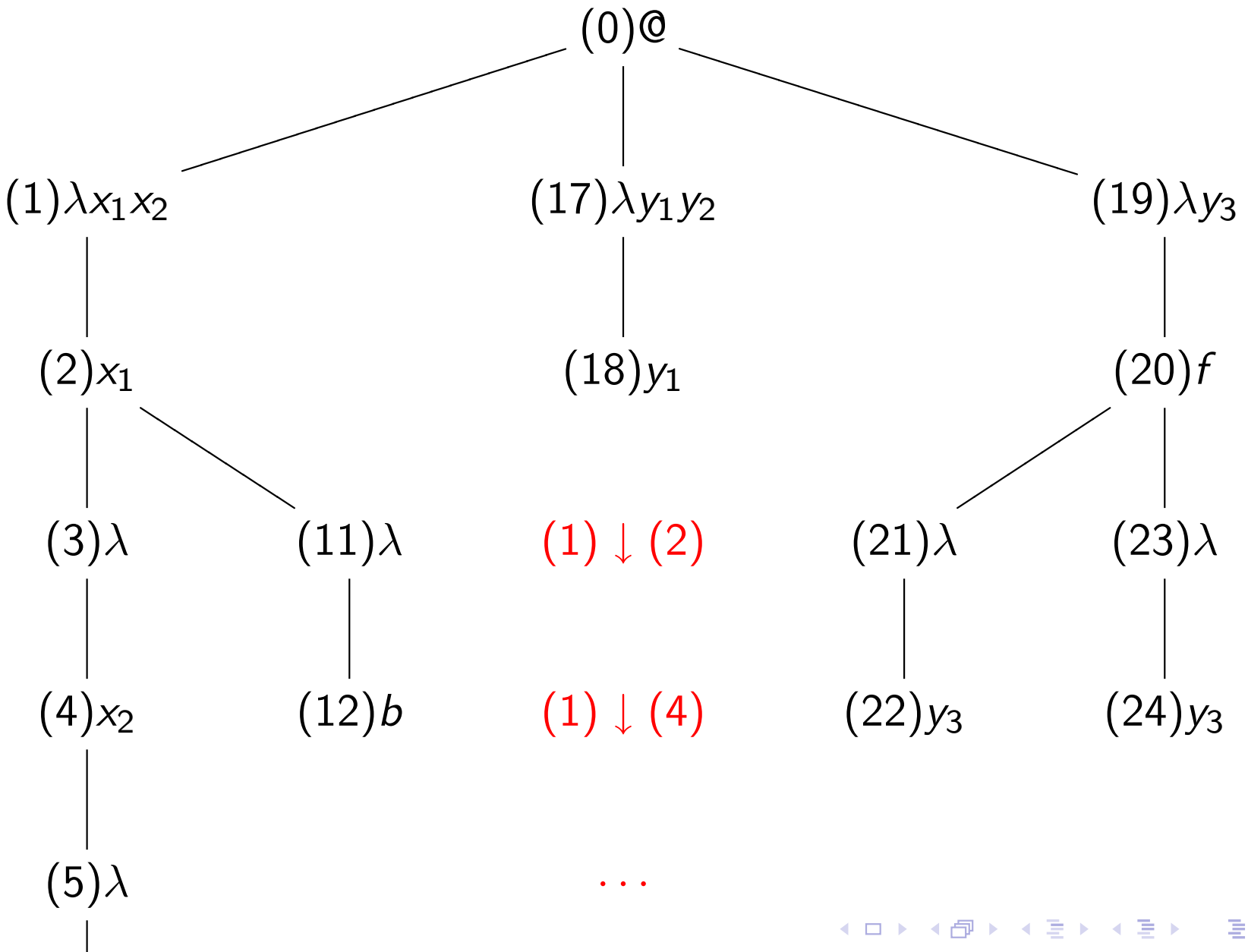
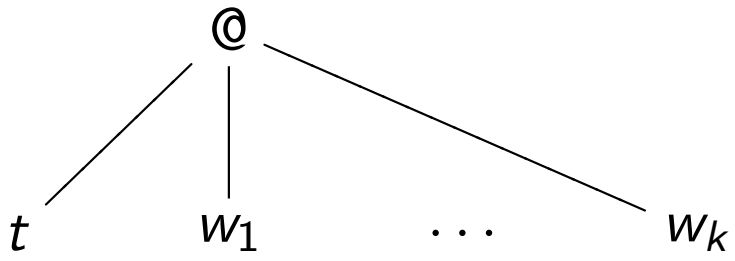
- $t, w_1, \dots, w_k$ binding trees with dummy lambdas and binding relation $\downarrow$ between nodes
- $n \downarrow m$ if n labelled $\lambda\overline{y}$ binds y_j, label at m

Example: $x(\lambda y_1 y_2.y_1)(\lambda y_3.fy_3y_3) = faa$



Tree automata



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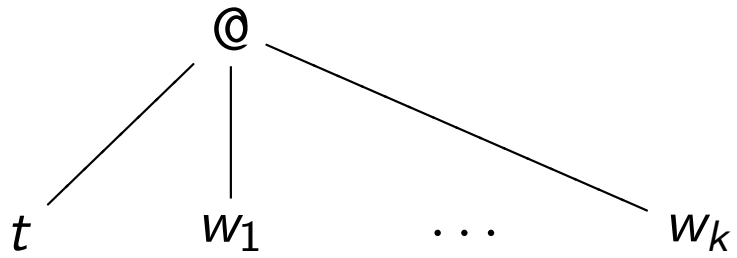

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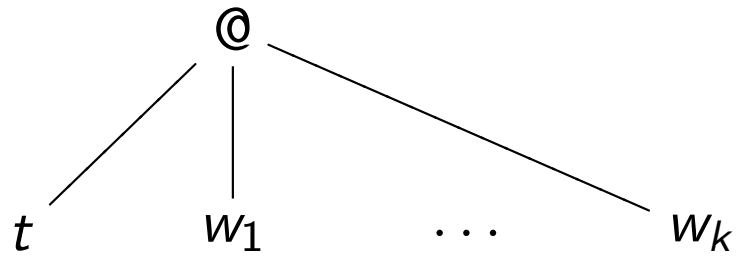

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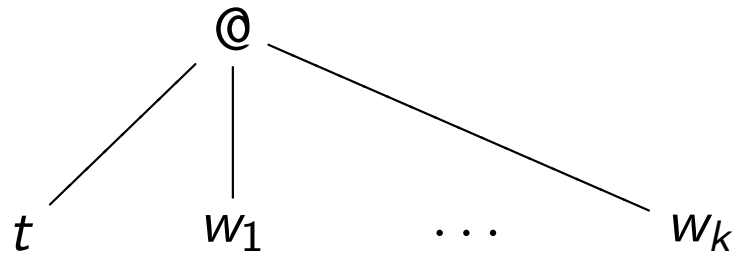

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- Non-emptiness decidable; sets recognised are regular

Tree automata for 4th-order [Comon,Jurski 97]

- 4th-order E, $xw_1 \ldots w_k = u$ and C_E constants in u plus b
- For finite alphabet Σ consider a potential solution term $t = \lambda x_1 \ldots x_k.s$

$$s ::= z_j^n \mid x_i v_1 \ldots v_m \mid f s_1 \ldots s_m \quad f \in C_E,\ m \geq 0$$
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- Guarantees finite alphabet for accessible nodes: reuse variables

Tree automata for 4th-order [Comon,Jurski 97] II

- Finite states:
  inaccessible nodes: $-$ for "dont care"
  accessible nodes: U is subterms of u closed under replacement of subtrees with leaves labelled with finitely many different variables $z_j^i : \mathbf{0}$

$$\{[e], [\lambda.e], [\lambda z_1^n \dots z_m^n.e], [\lambda x_1 \dots x_k.u] \mid e \in U \cup \{-\}\}$$

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- Finite transitions: such as

$$z \Rightarrow [z] \quad z \Rightarrow [-] \quad a \Rightarrow [a] \quad c \Rightarrow [-]$$
$$\lambda x_1[ga] \Rightarrow [\lambda x_1.ga] \quad x_1[\lambda.-][\lambda.gz] \Rightarrow [gz]$$
$$\dots$$

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- Theorem The tree automaton associated with a 4th-order $xw_1 \dots w_k = u$ accepts t iff t solves $xw_1 \dots w_k = u$

Tree automata for 5th-order: PROBLEMS

- Finite alphabet ? can stack 2nd-order variables

  $\lambda xy.x(\lambda z_1.x(\lambda z_2 \ldots (\lambda z_n.z_n(z_{n-1}(\ldots z_1(y))\ldots))\ldots))$

  Need an infinite alphabet

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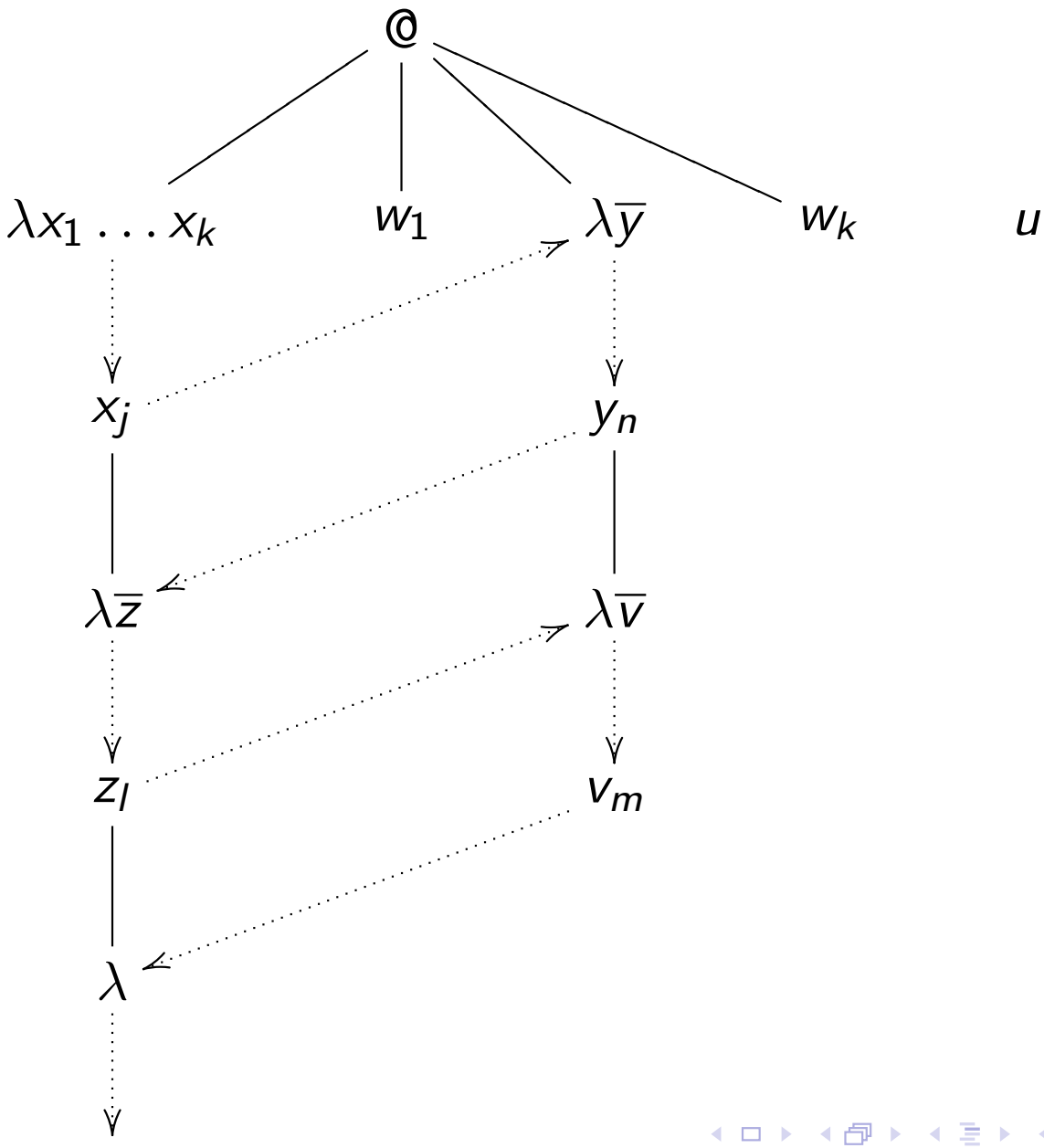
- Second problem ? need finer analysis than evaluation of an accessible node

Games: automaton moving around interpolation tree



Game/automaton $\mathsf{G}(t, E)$

- Play: sequence $n_1 q_1 \theta_1, \ldots, n_l q_l \theta_l$
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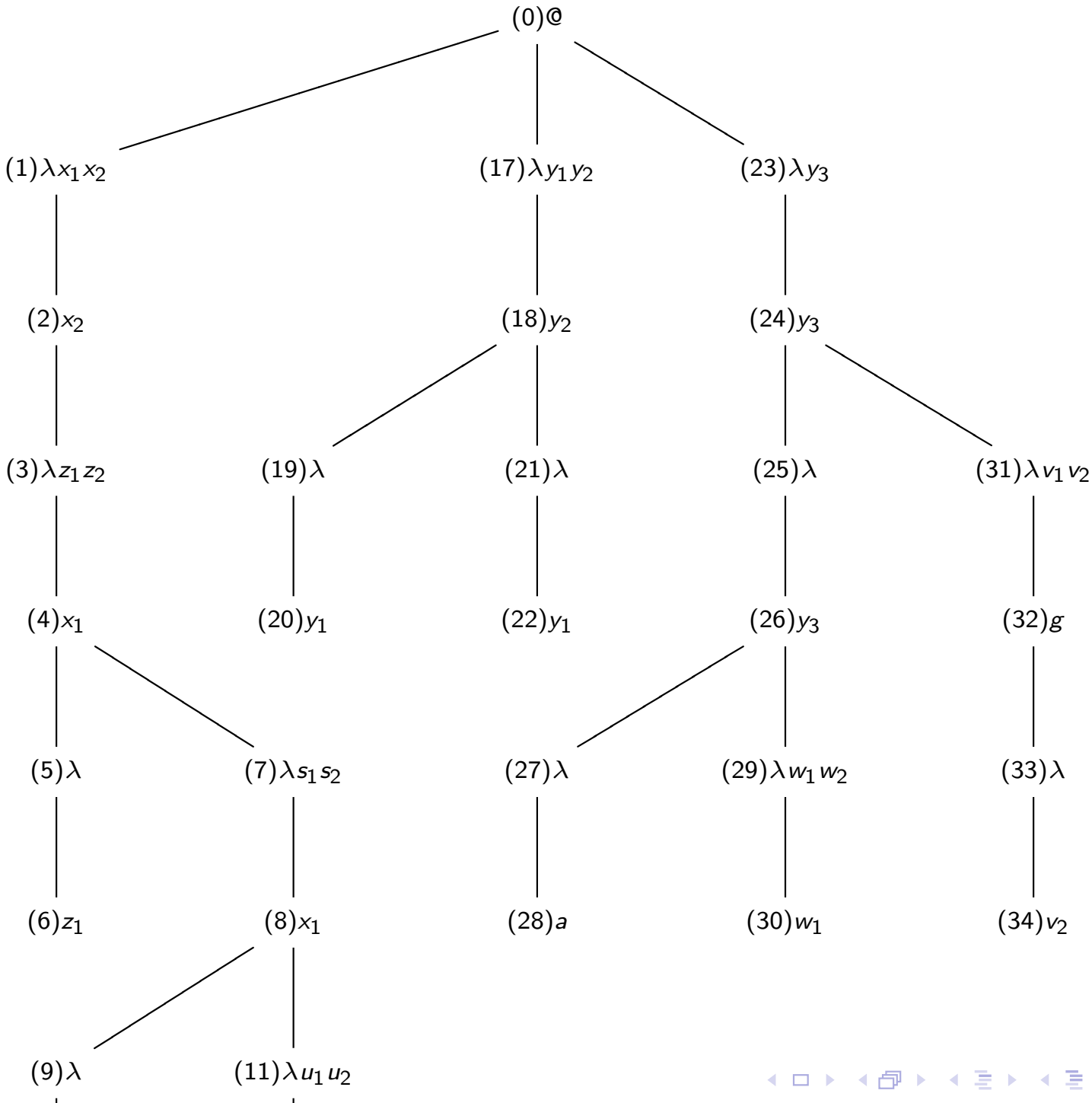
- Player $\forall$ loses every play in $\mathsf{G}(t, E)$ iff t solves E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(21)λ
(25)λ
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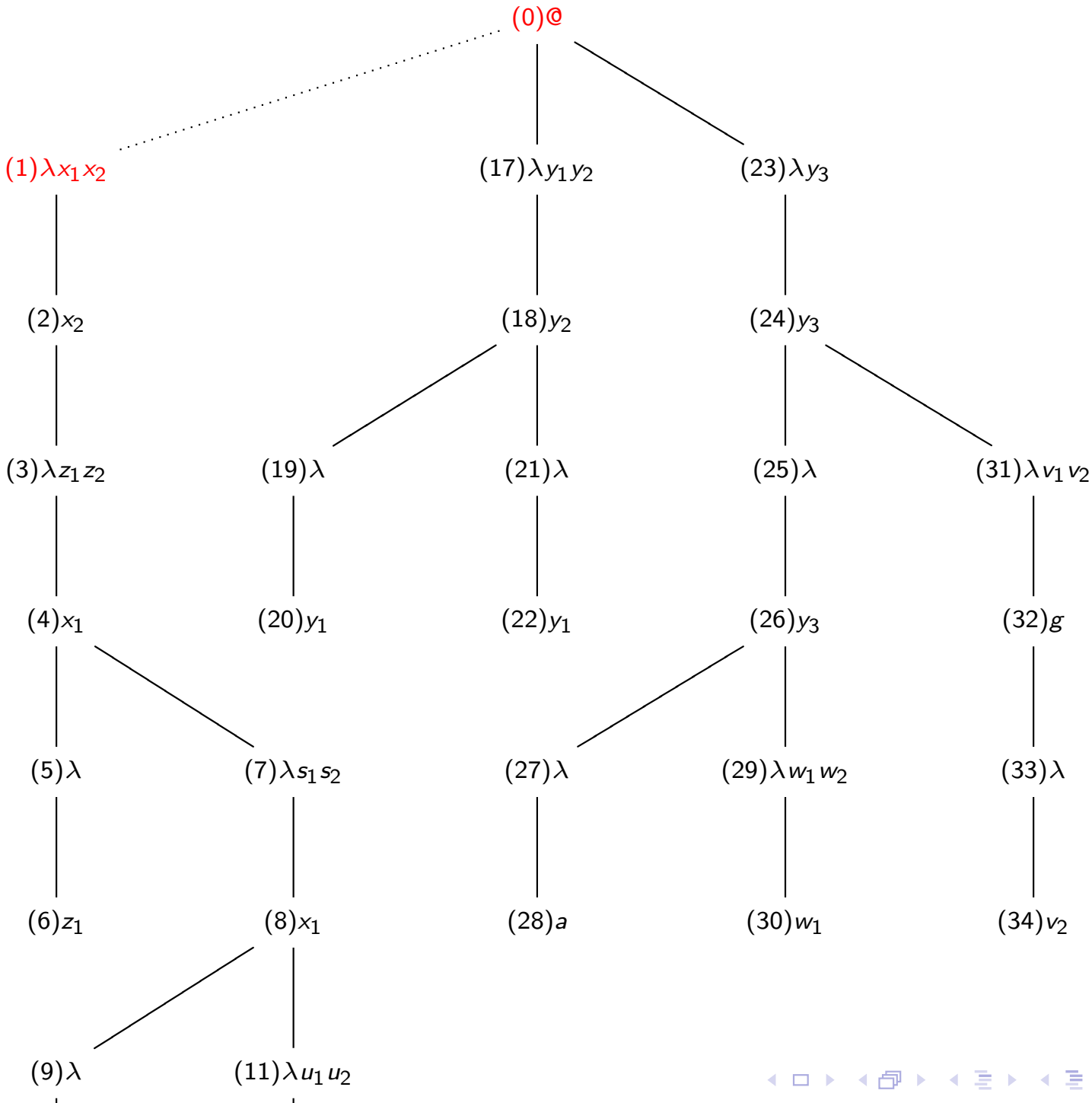
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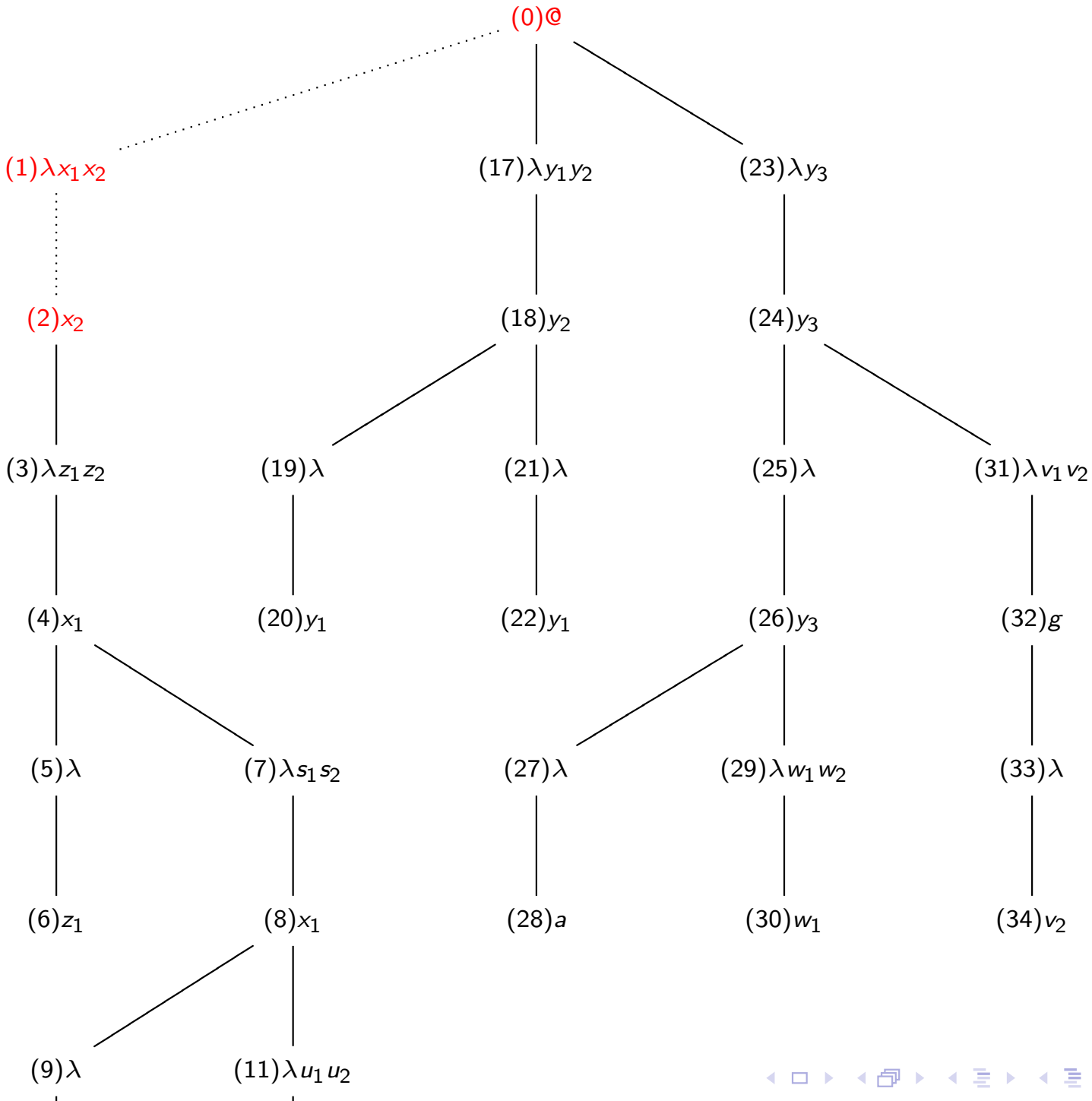
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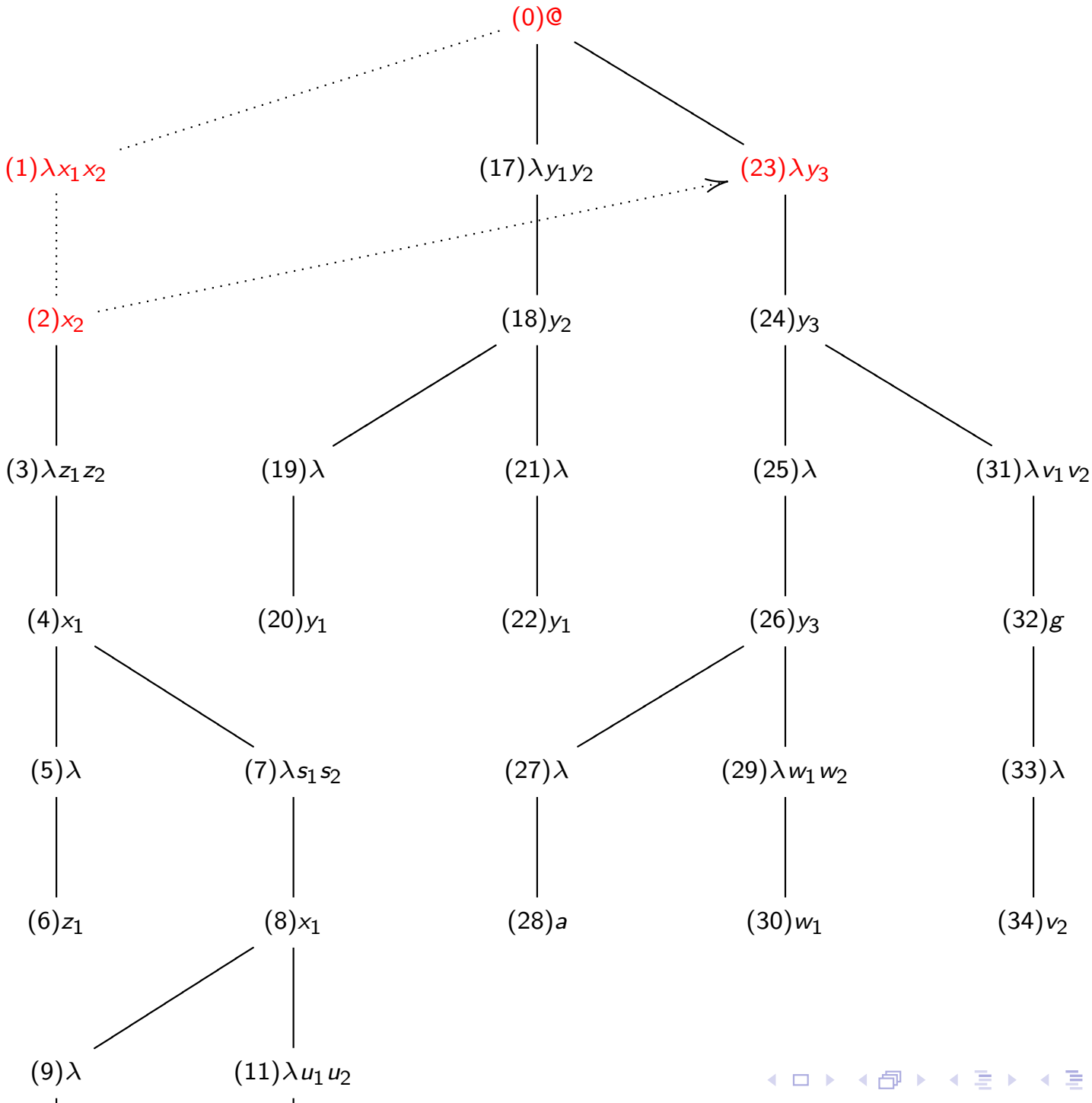
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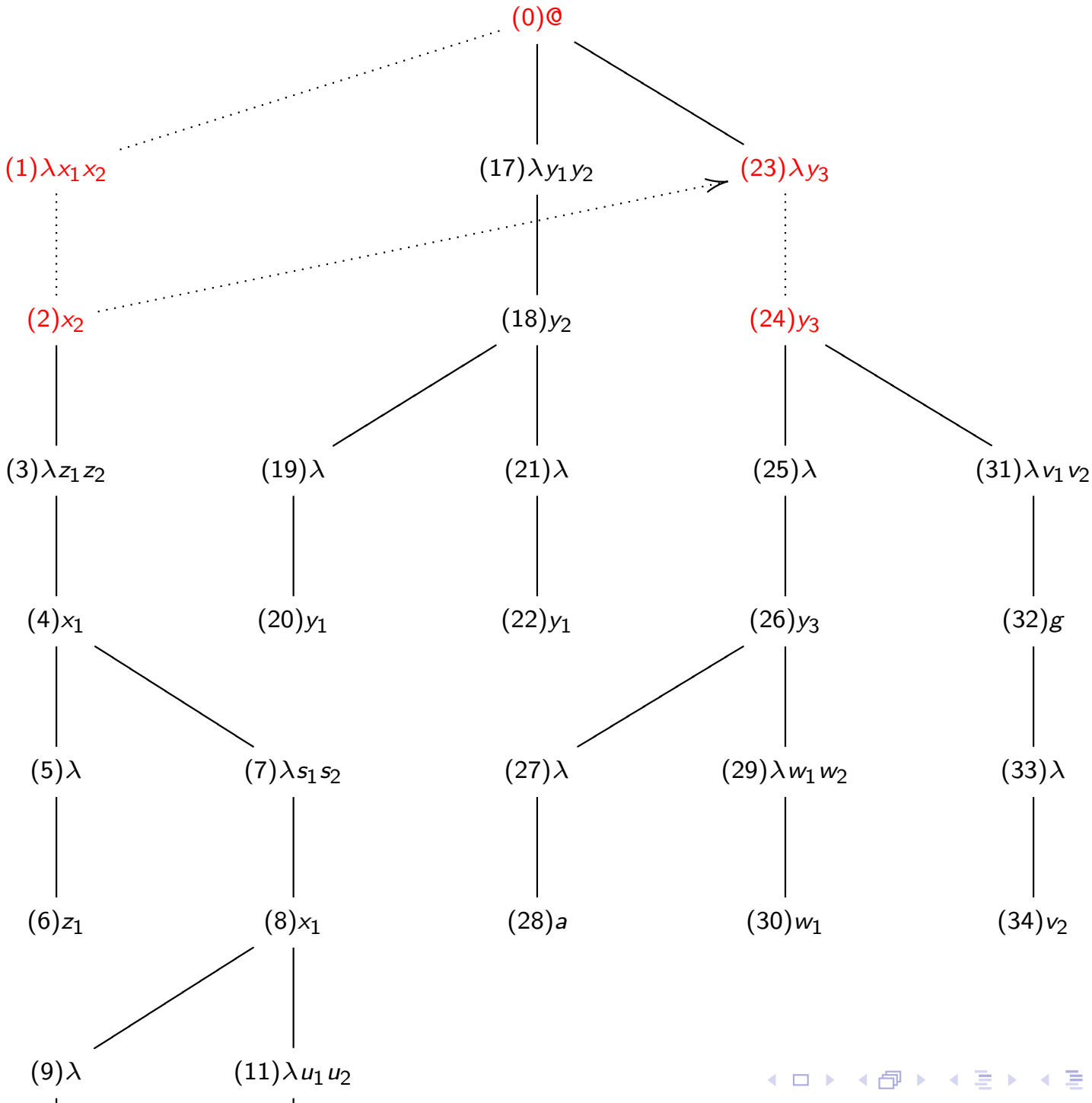
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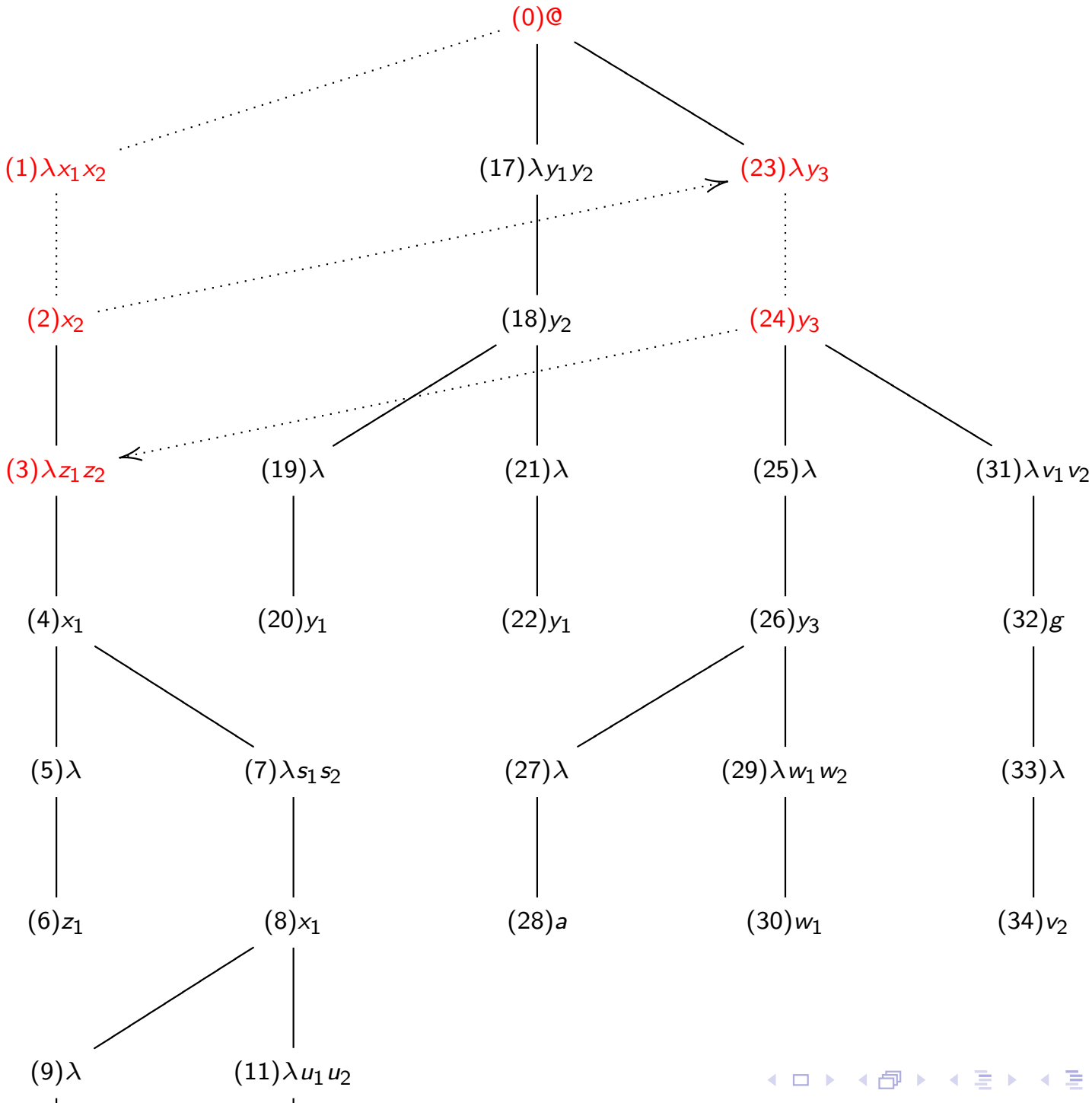
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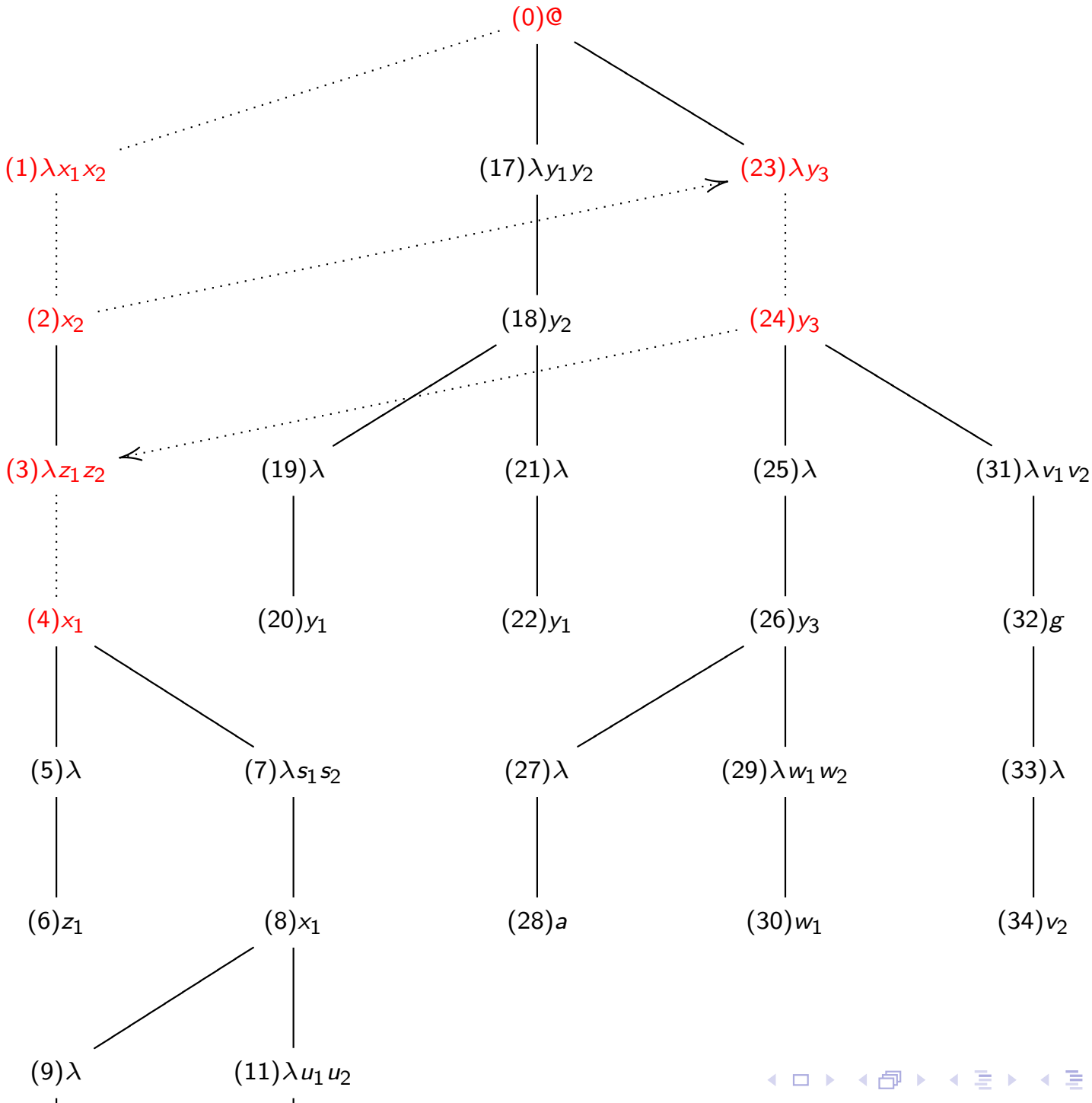
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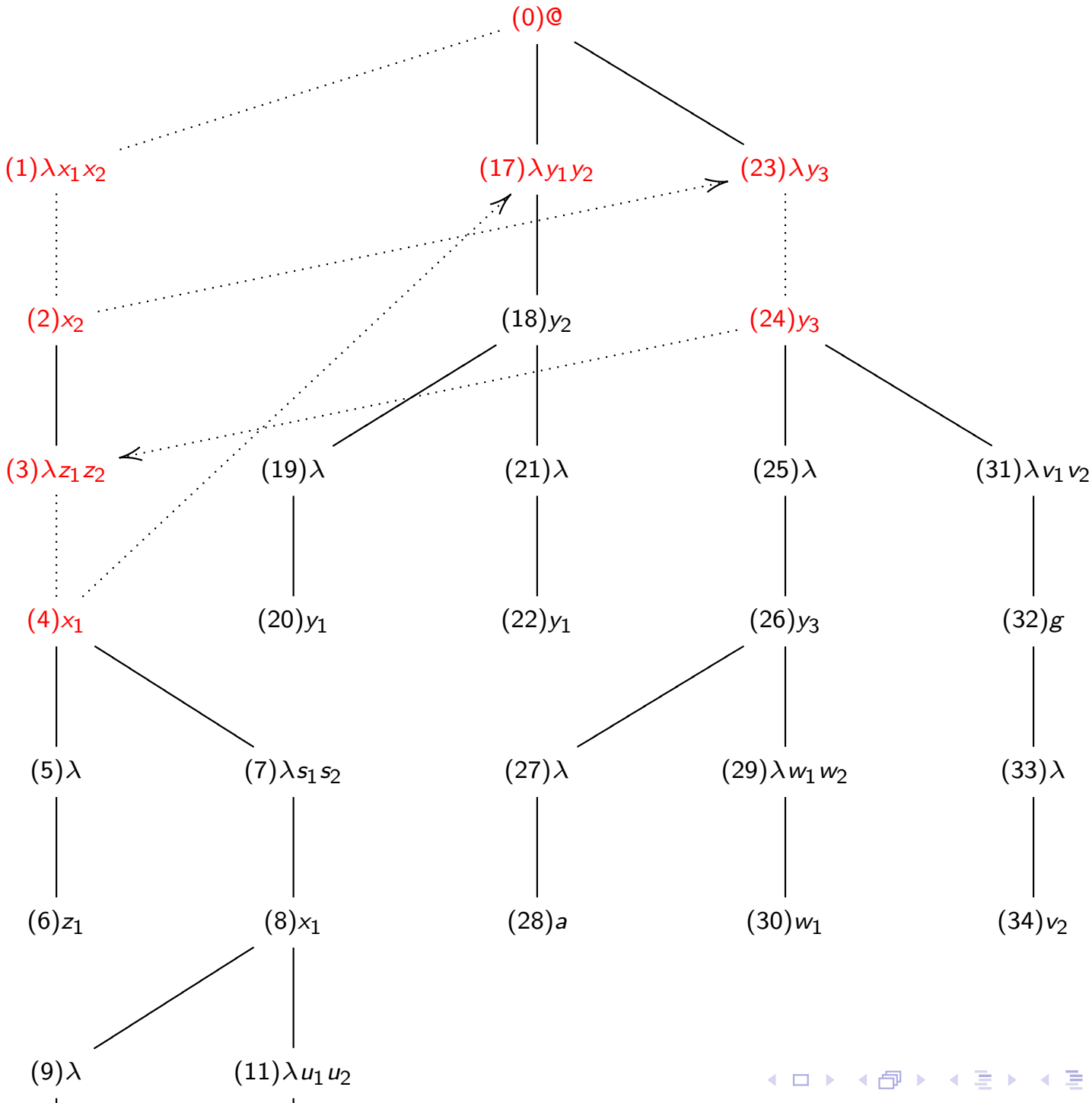
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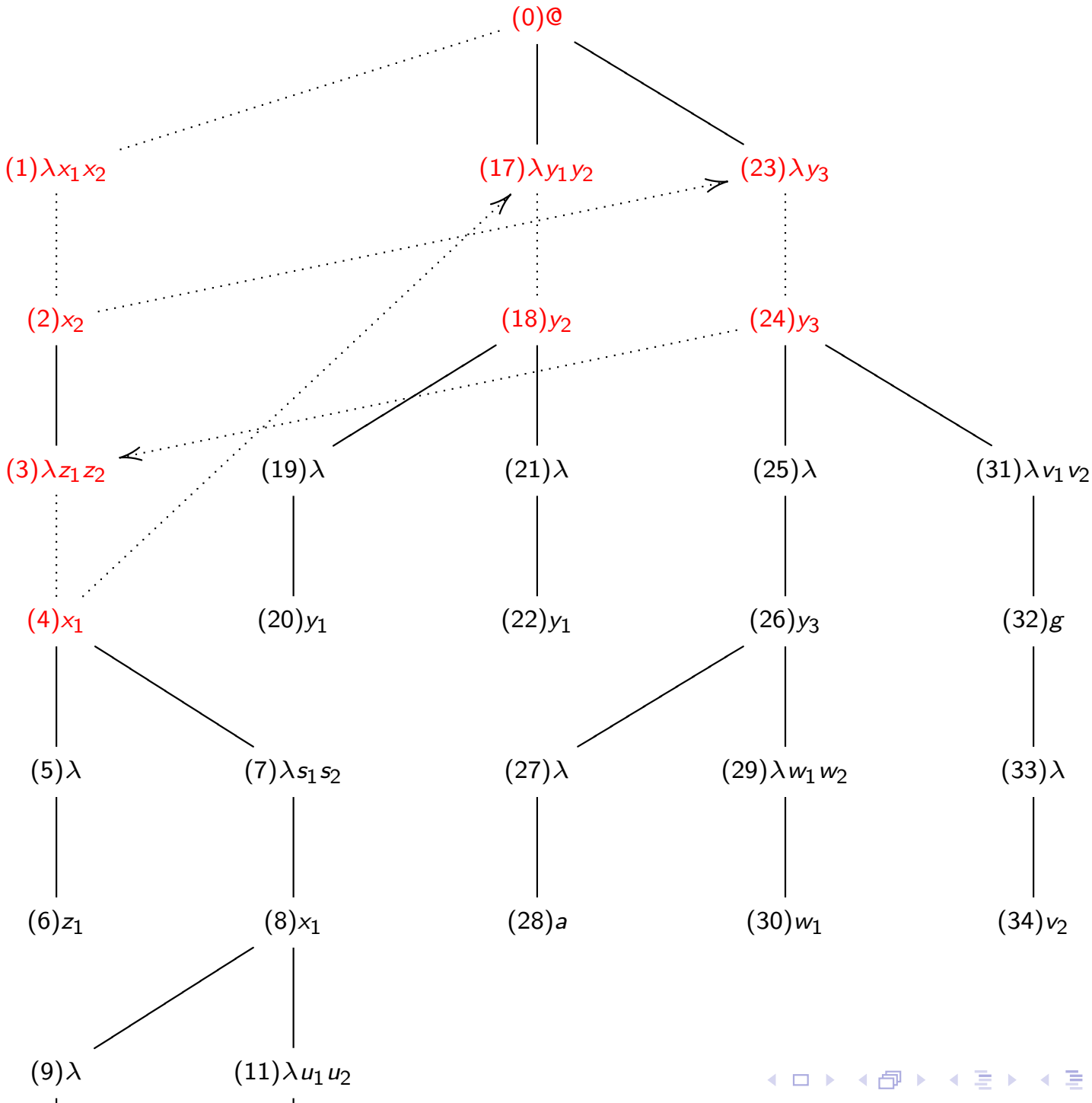
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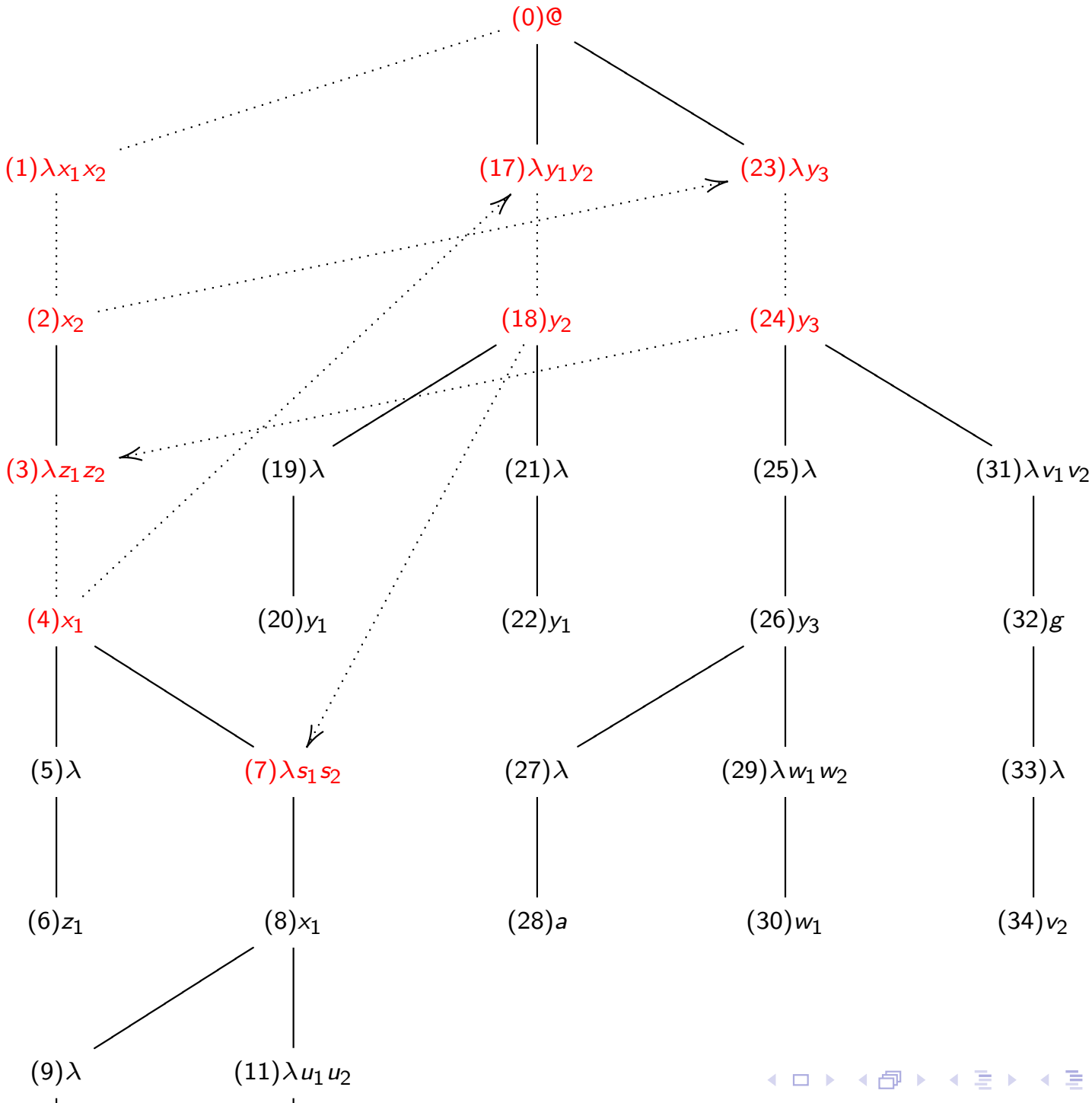
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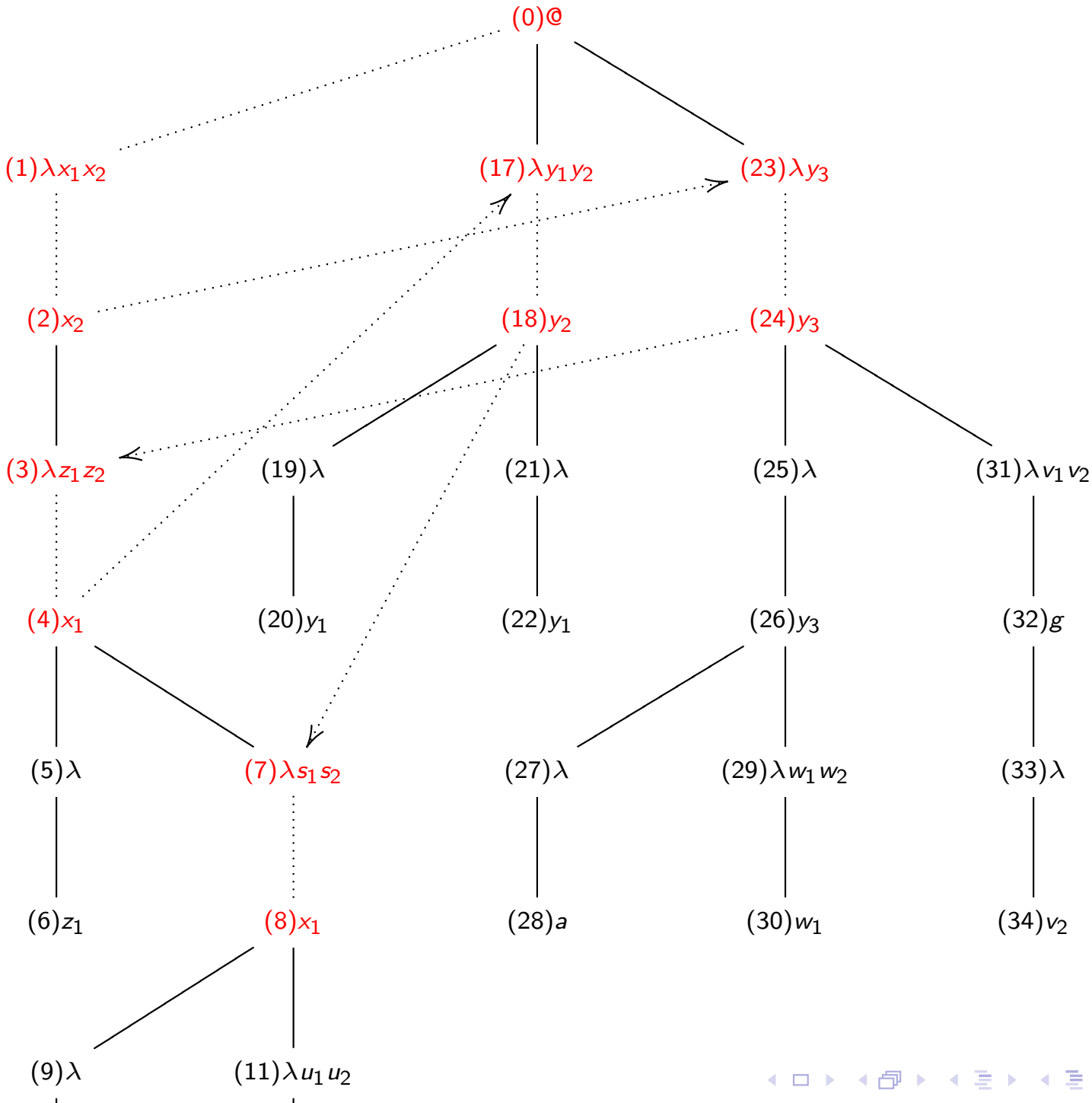
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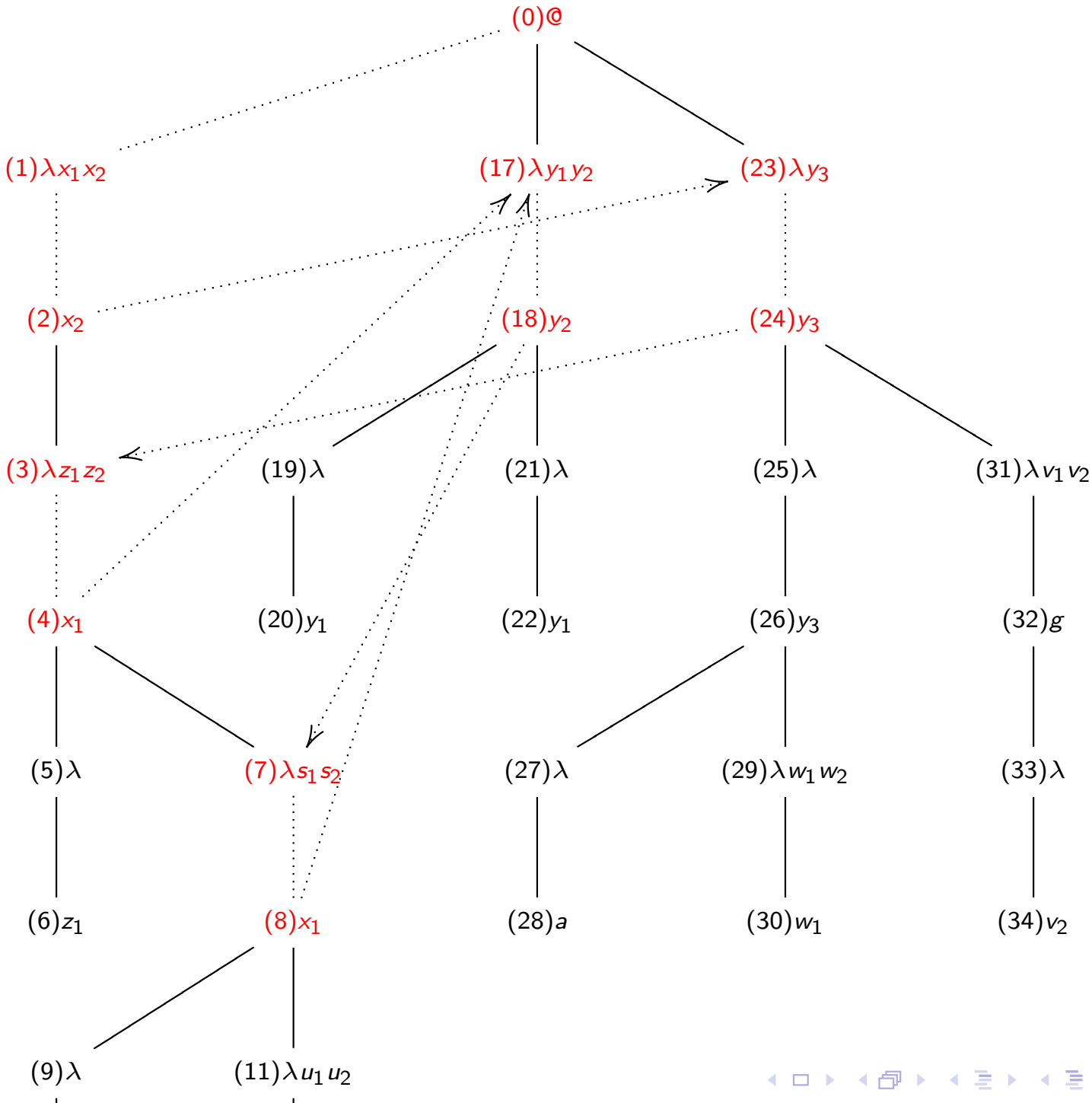
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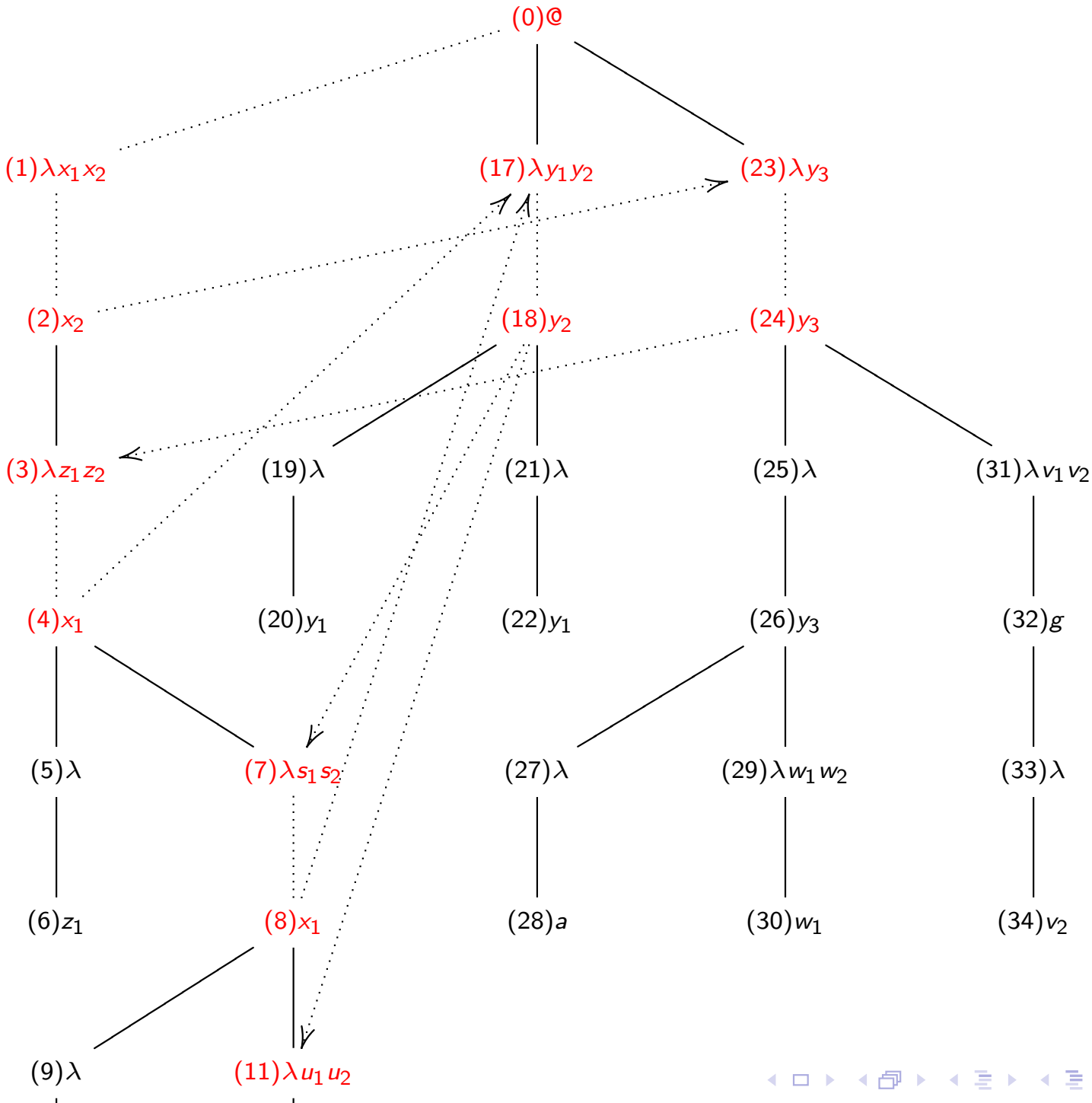
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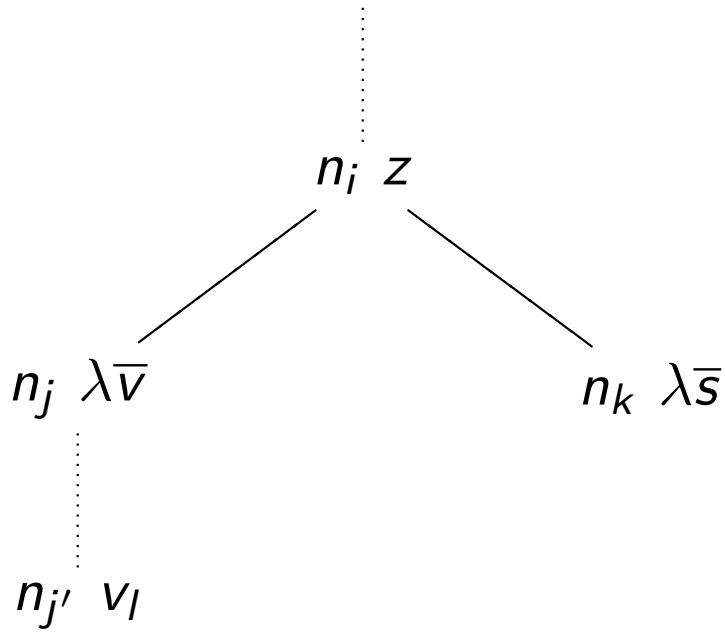
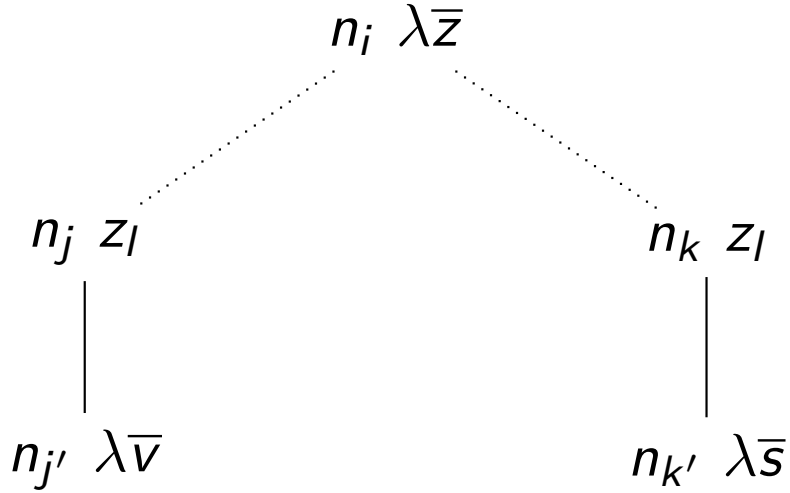
- Theorem The set of solutions of E is recognised by a special alternating binding tree automaton BUT non-emptiness of such automata undecidable [Stirling 2009; Ong, Tzevelekos 2009]

Uniformity properties of game playing I



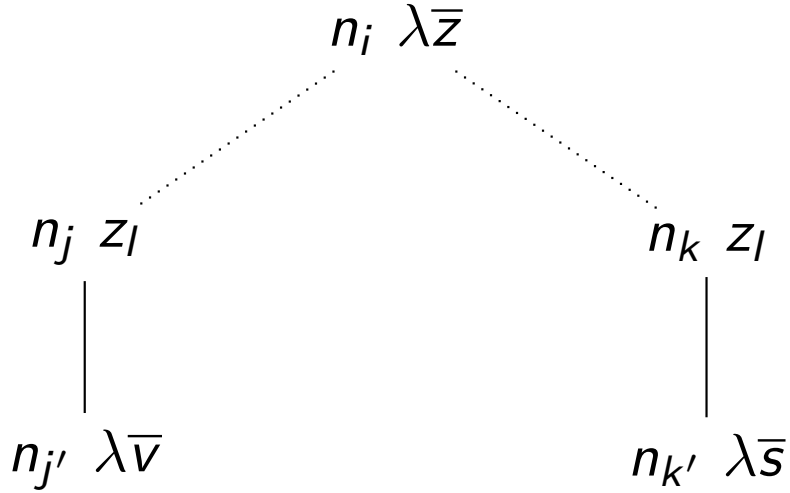
- If play is at n_i then at n_j and then later at n_k then inbetween there must have been a position at an $n_{j'}$ bound by n_j

Uniformity properties of game playing II



- If play is at n_i then at n_j and then later at $n_{j'}$; then at n_k then there must be a corresponding sequence to that between n_j and $n_{j'}$ between n_k and $n_{k'}$ unless there is a different $\forall$ choice

Uniformity properties of game playing II



- If play is at n_i then at n_j and then later at $n_{j'}$; then at n_k then there must be a corresponding sequence to that between n_j and $n_{j'}$ between n_k and $n_{k'}$ unless there is a different $\forall$ choice
- Especially relevant if n_k is somewhere below n_j ("embedded")

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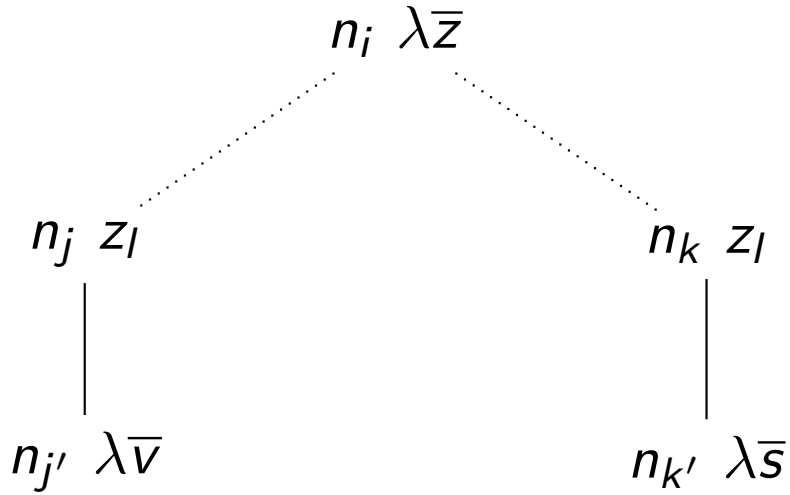

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- Property not enforced in a tree automaton

Application of games to decidability of matching

- Combinatorial argument based on uniformities of play

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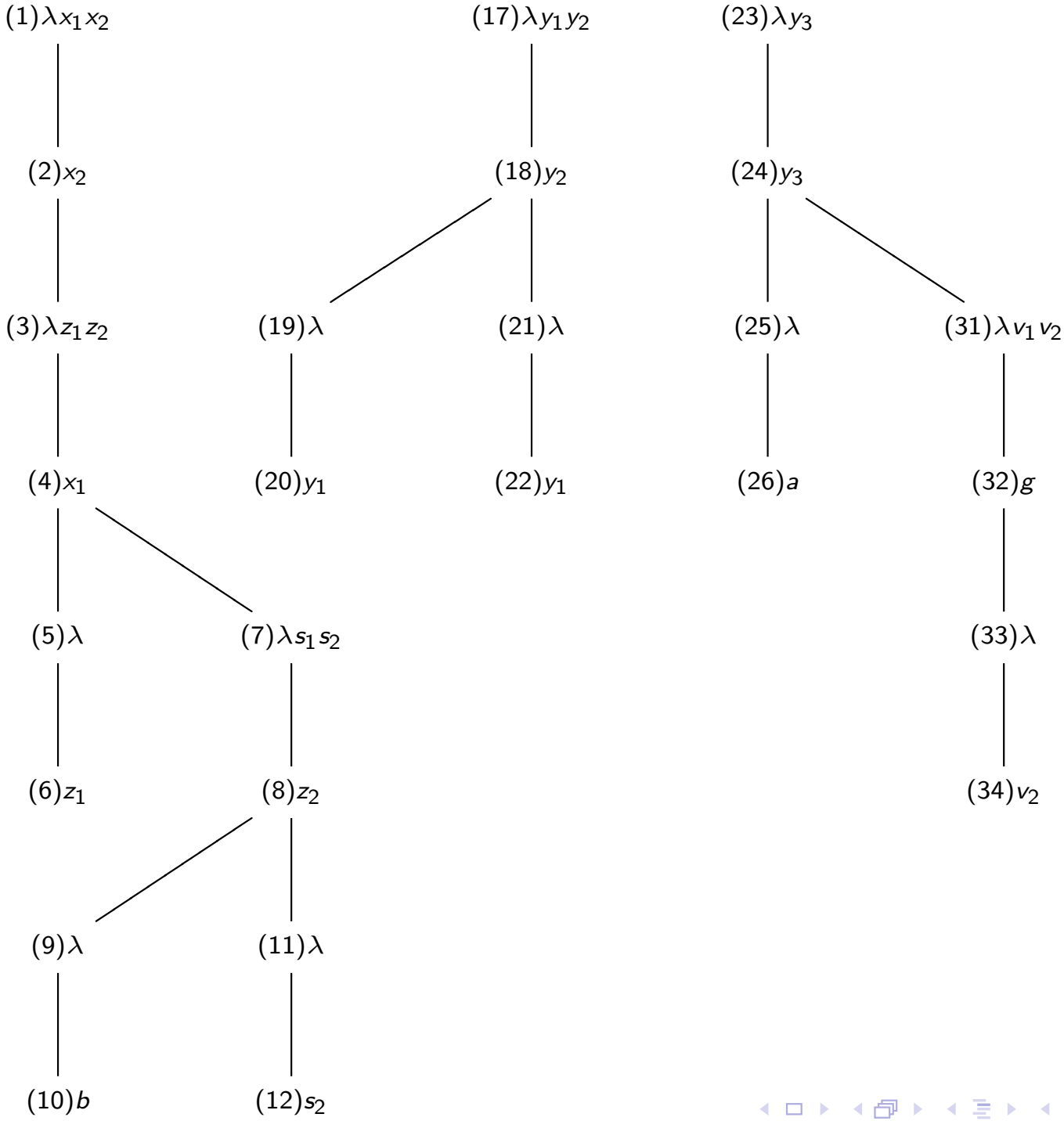
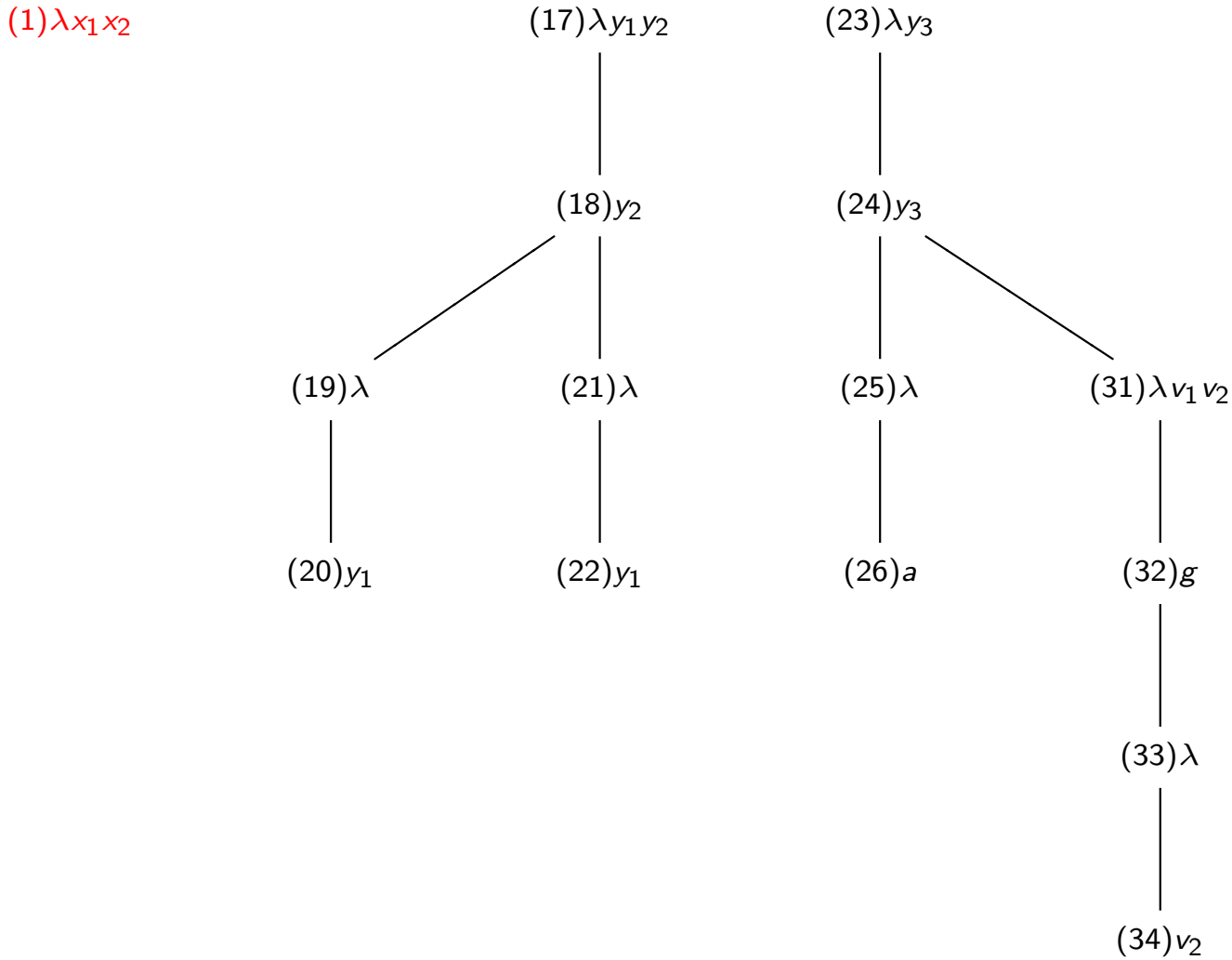
$$x(\lambda y_1 y_2 . y_2 y_1 y_1)(\lambda y_3 . y_3 a(\lambda v_1 v_2 . g v_2)) \;=\; ga$$


(1)λx1 x2
(2)x2
(3)λz1 z2
(4)x1
(5)λ
(6)z1
(7)λs1 s2
(8)z2
(9)λ
(10)b
(11)λ
(12)s2
(17)λy1 y2
(18)y2
(19)λ
(20)y1
(21)λ
(22)y1
(23)λy3
(24)y3
(25)λ
(26)a
(31)λv1 v2
(32)g
(33)λ
(34)v2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(24)y3
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(21)λ
(25)λ
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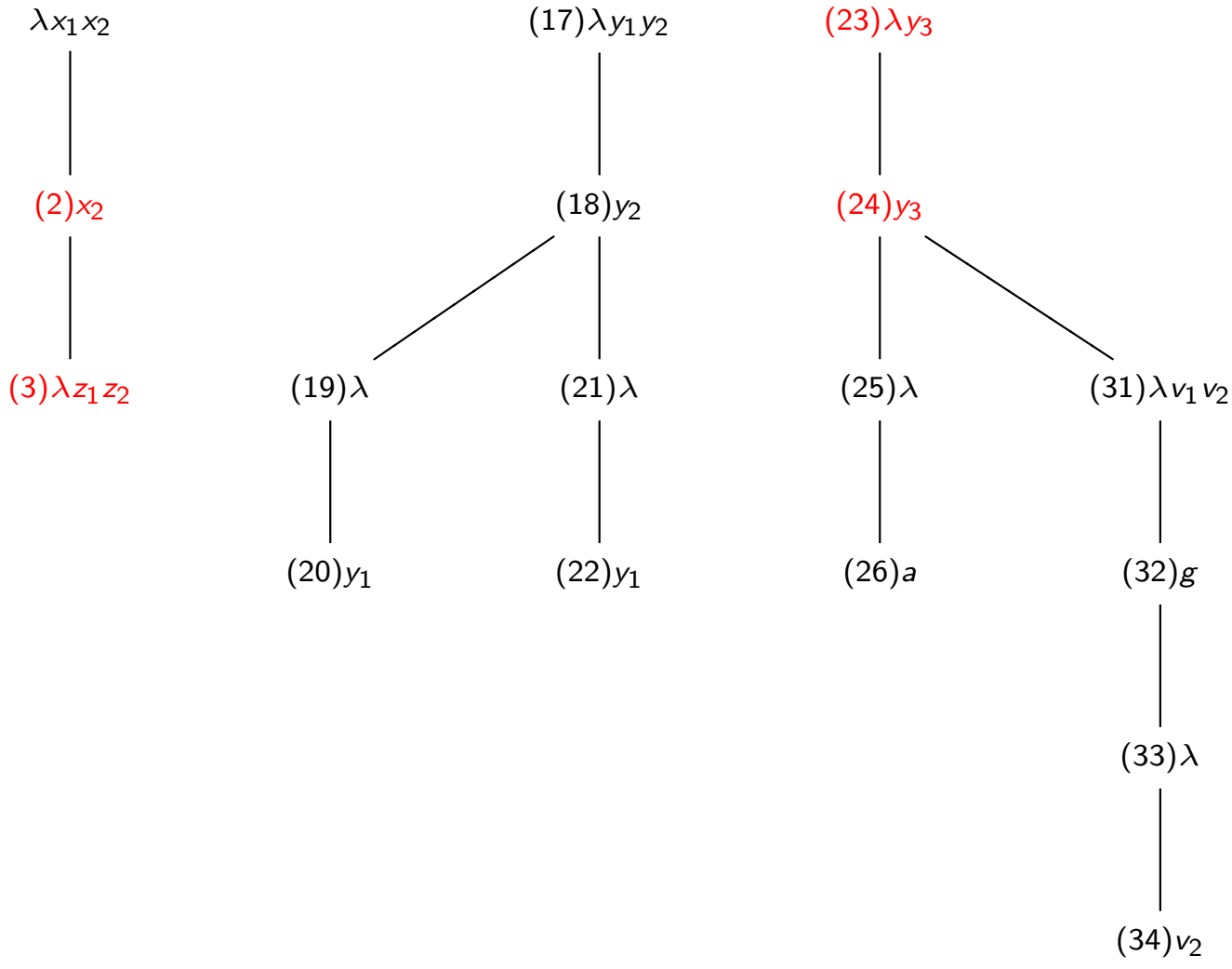
(22)y1
(26)a
(32)g
(33)λ
(34)v2


$\lambda x_1 x_2$
$(2)x_2$
$(3)\lambda z_1 z_2$
$(17)\lambda y_1 y_2$
$(18)y_2$
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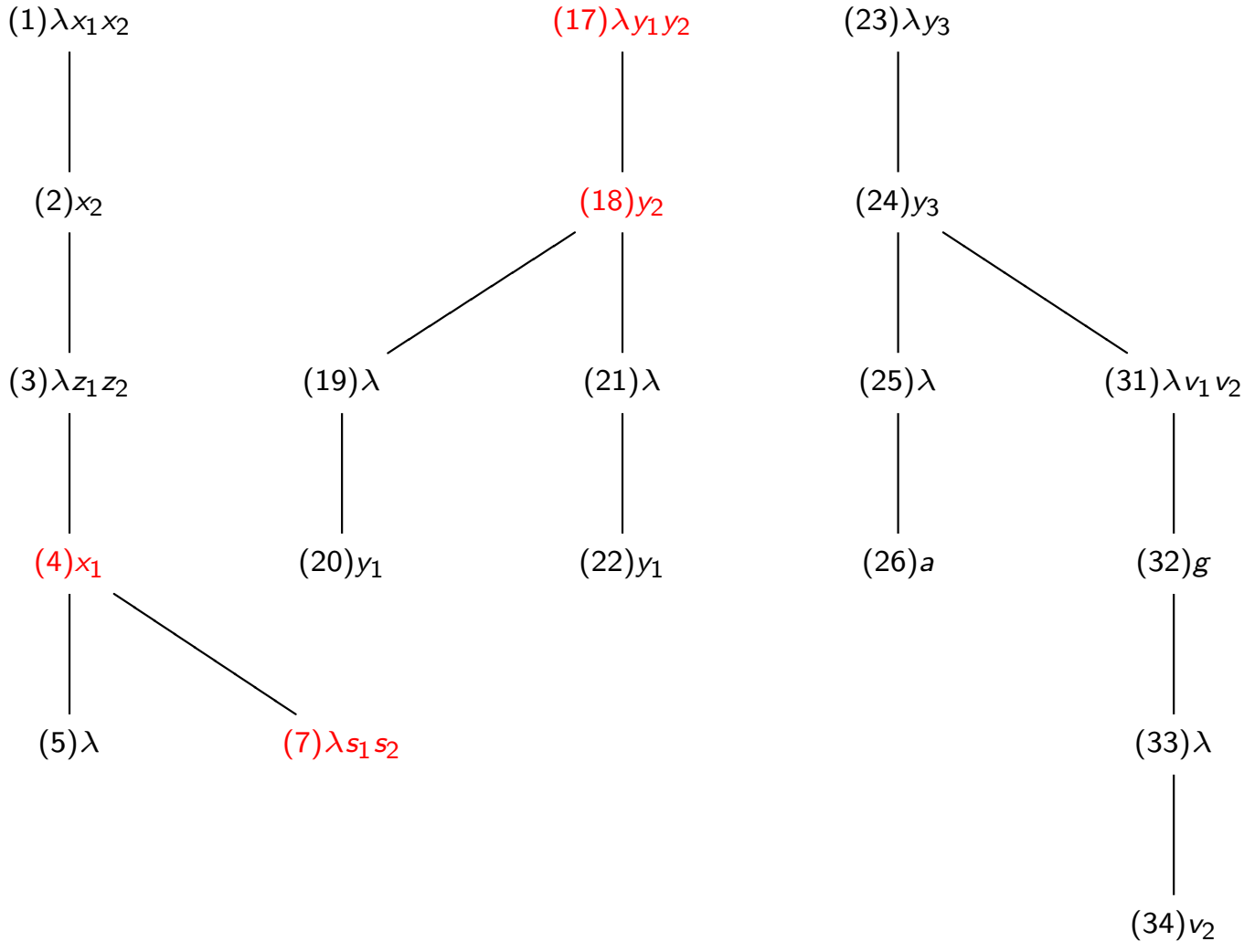
$(20)y_1$
$(21)\lambda$
$(22)y_1$
$(23)\lambda y_3$
$(24)y_3$
$(25)\lambda$
$(26)a$
$(31)\lambda v_1 v_2$
$(32)g$
$(33)\lambda$
$(34)v_2$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(3)λz1 z2
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(5)λ
(7)λs1 s2
(17)λy1 y2
(18)y2
(19)λ
(20)y1
(21)λ
(22)y1
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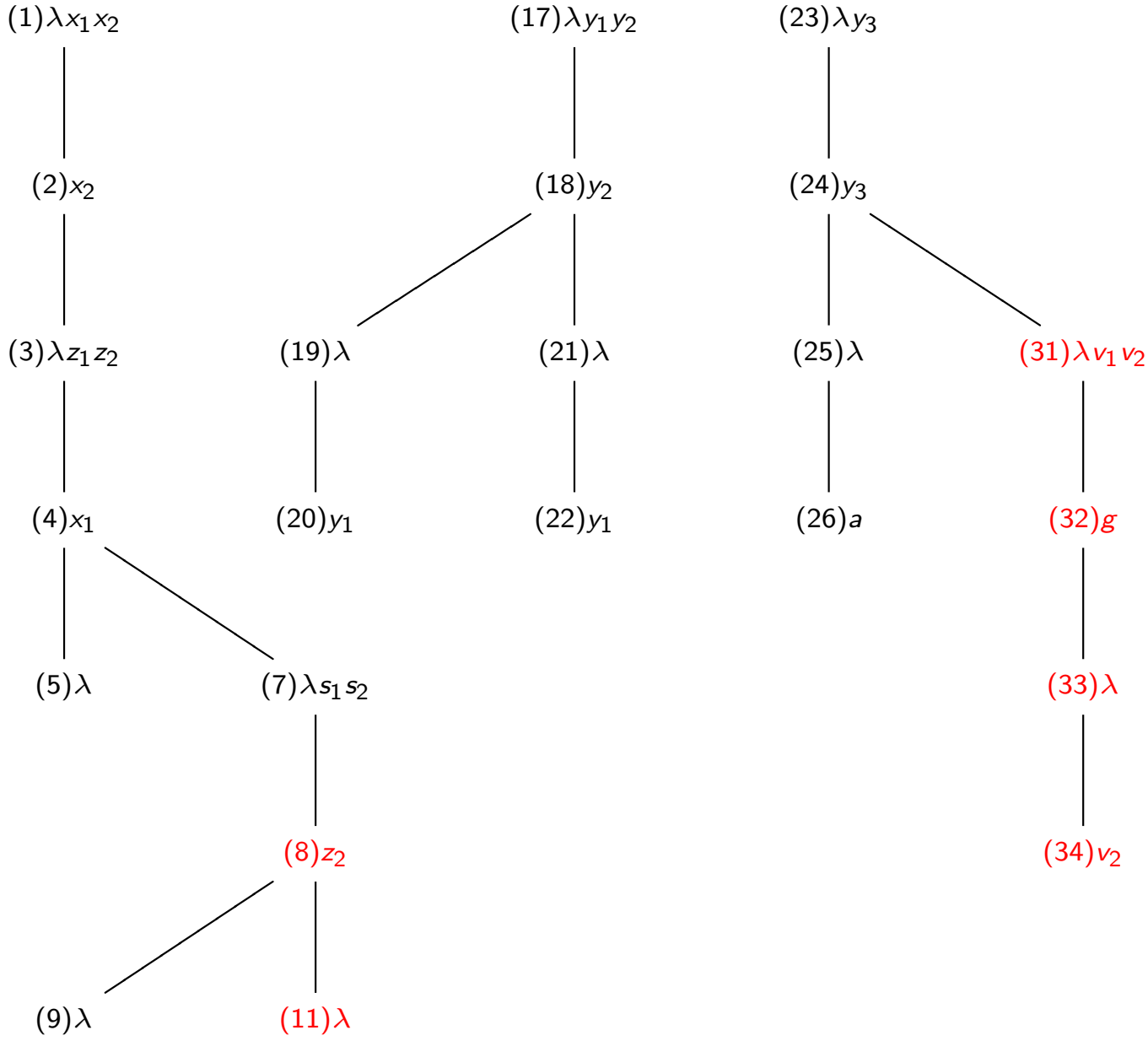
(31)λv1 v2
(32)g
(33)λ
(34)v2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(11)λ
(17)λy1 y2
(18)y2
(19)λ
(21)λ
(20)y1
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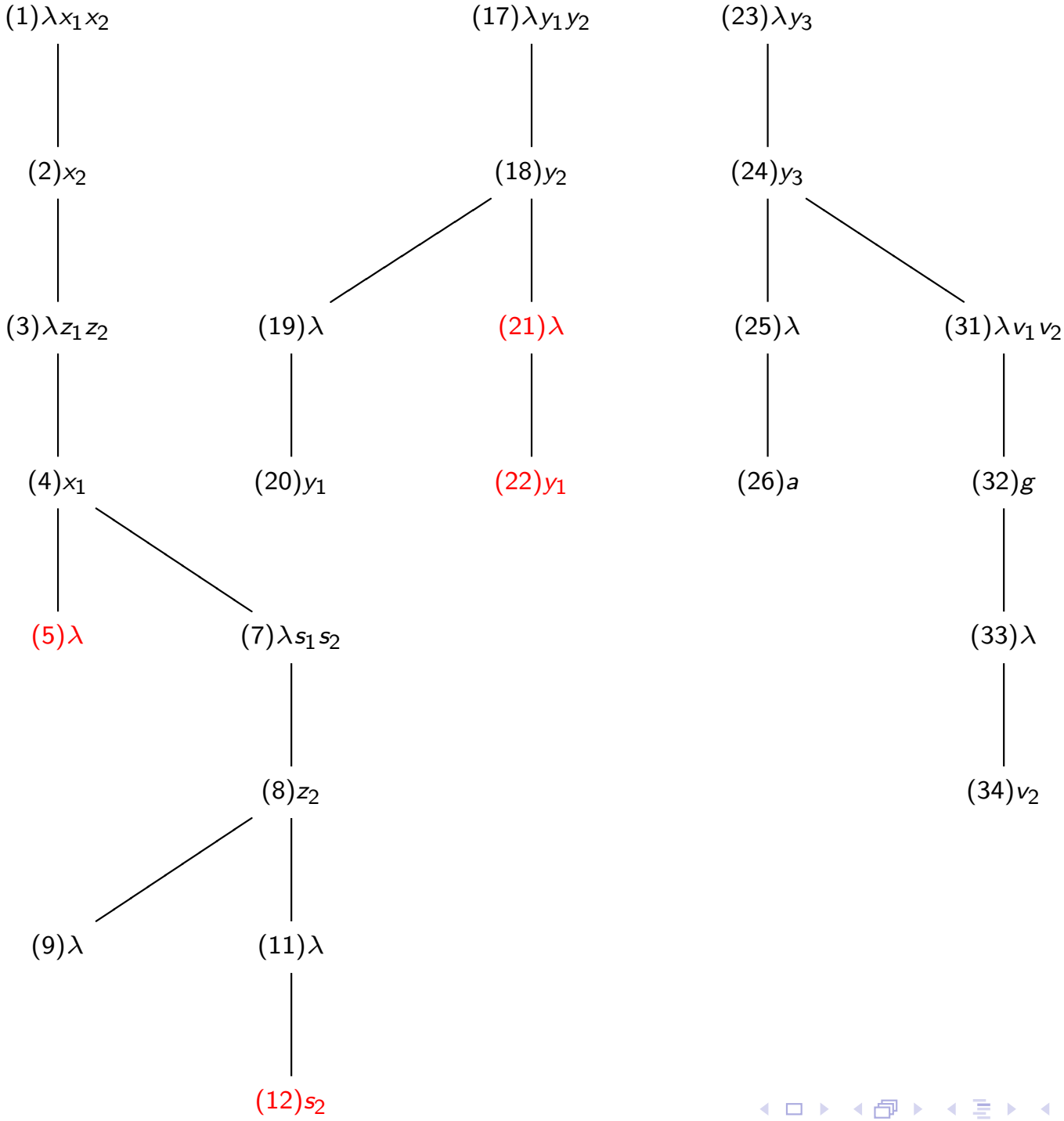
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(11)λ
(12)s2
(17)λy1 y2
(18)y2
(19)λ
(20)y1
(21)λ
(22)y1
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(26)a
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(33)λ
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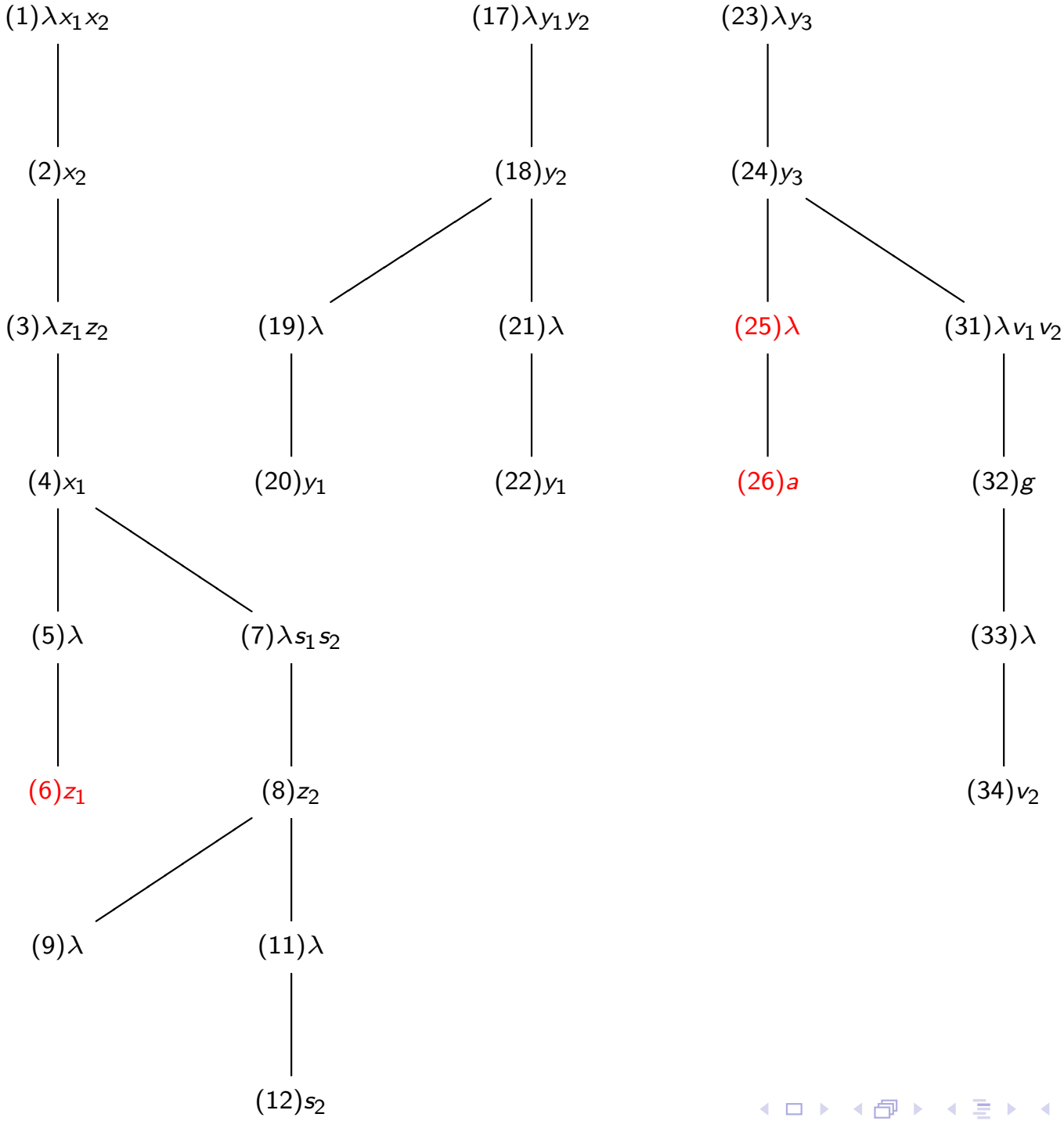
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(4)x1
(5)λ
(6)z1
(7)λs1s2
(8)z2
(9)λ
(11)λ
(12)s2
(17)λy1y2
(18)y2
(19)λ
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(21)λ
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(25)λ
(26)a
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Application of games to decidability of matching

- Combinatorial argument based on uniformities of play.
- Two steps in proof assuming an arbitrary solution term
  1. partition each play

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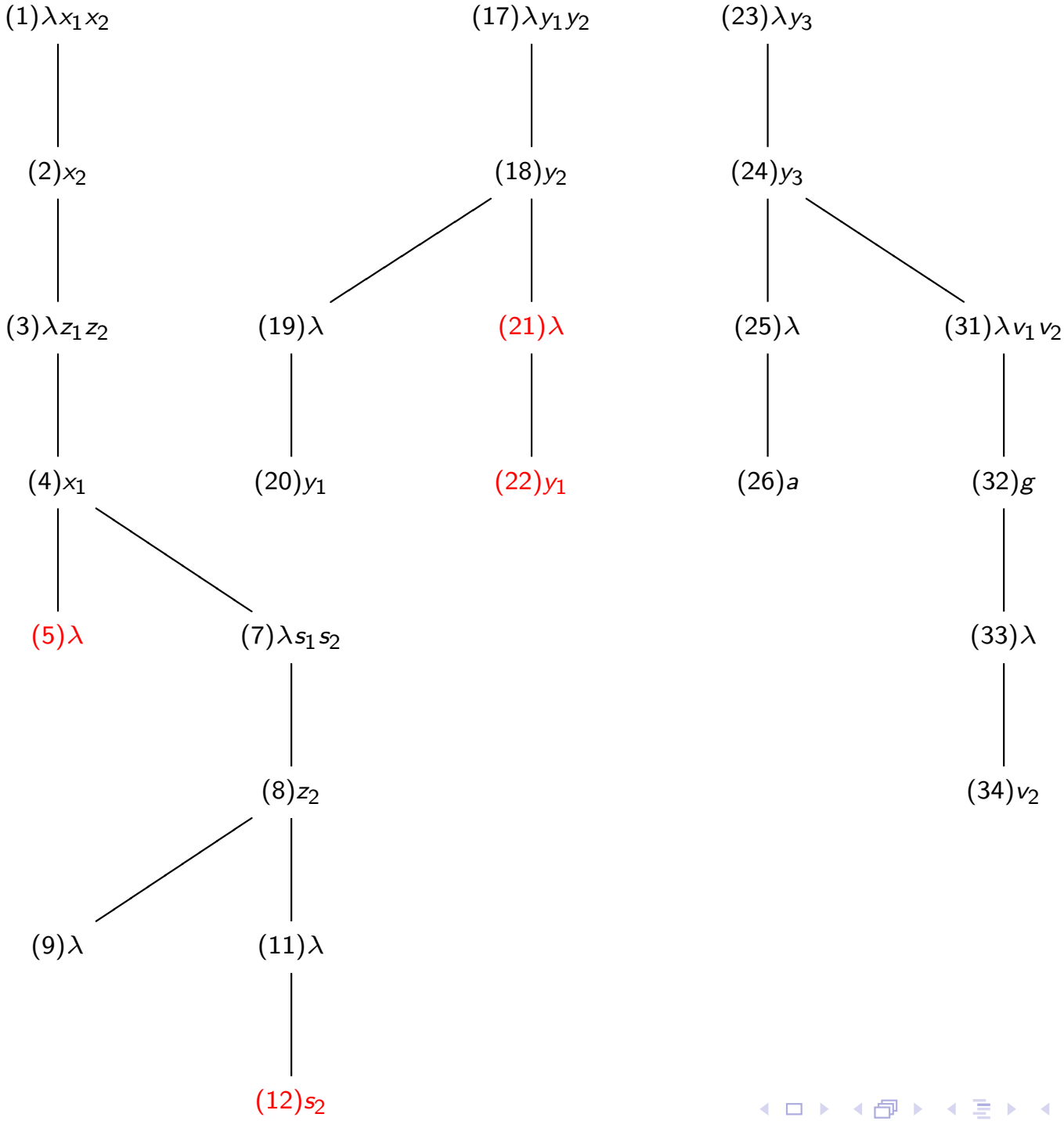
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(22)y1
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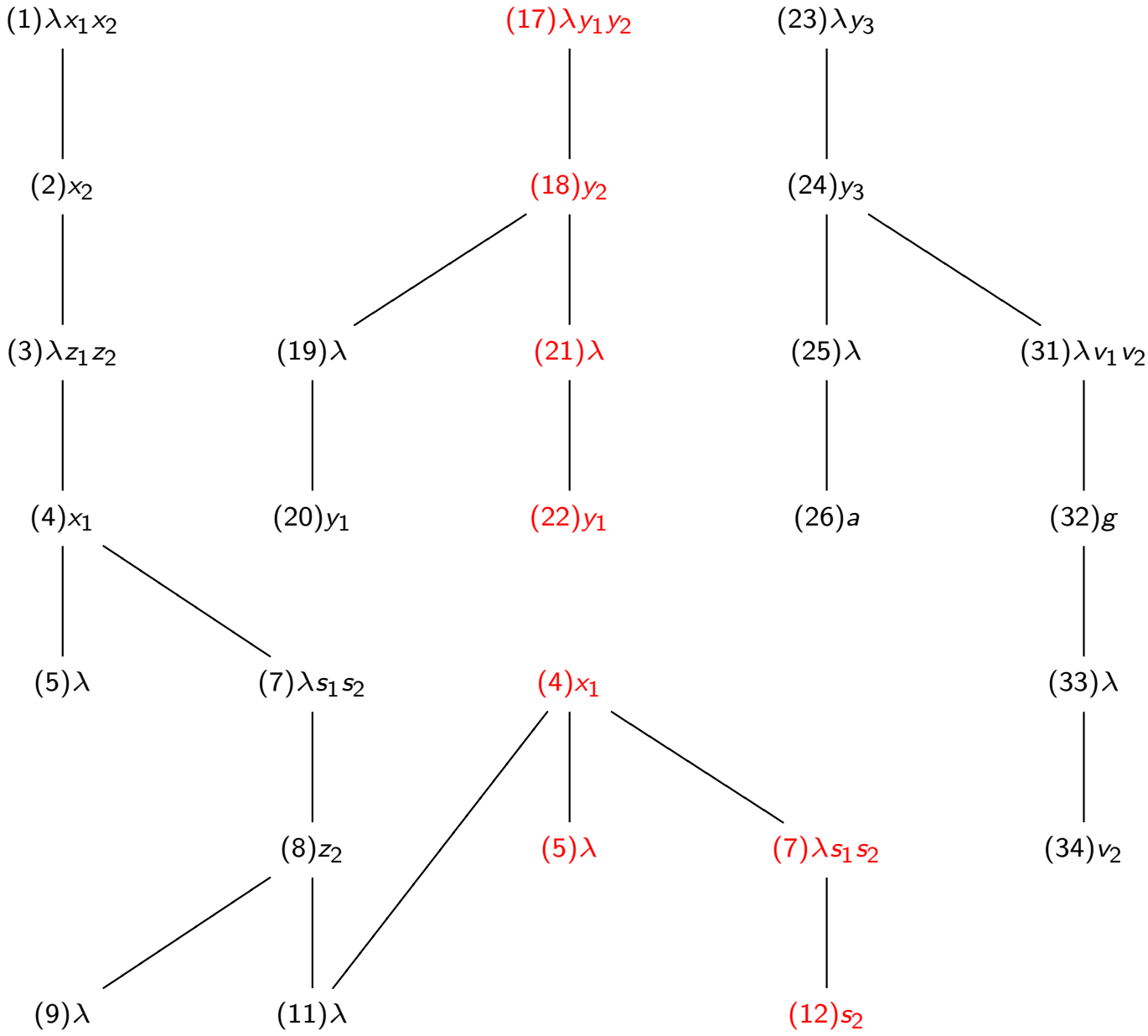
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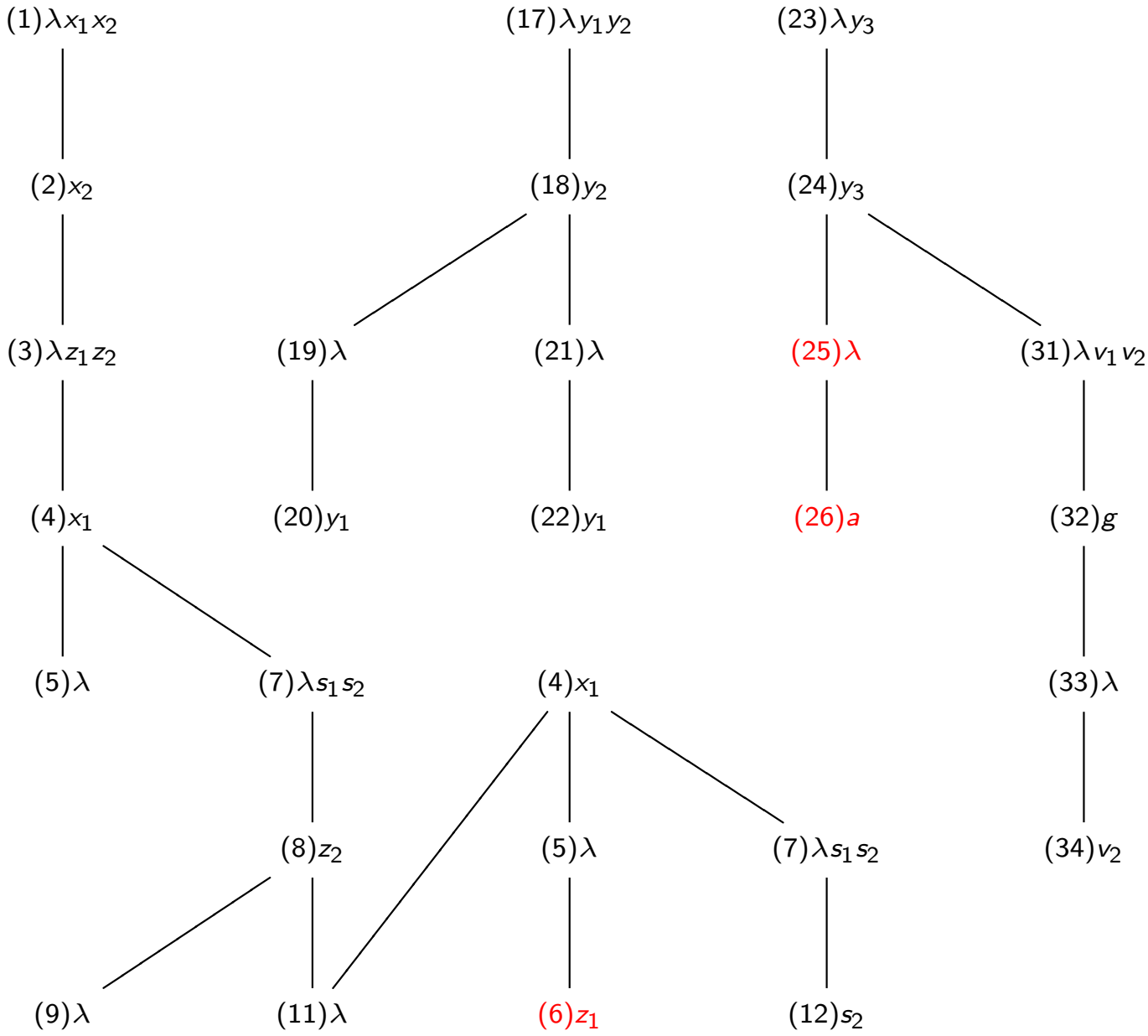
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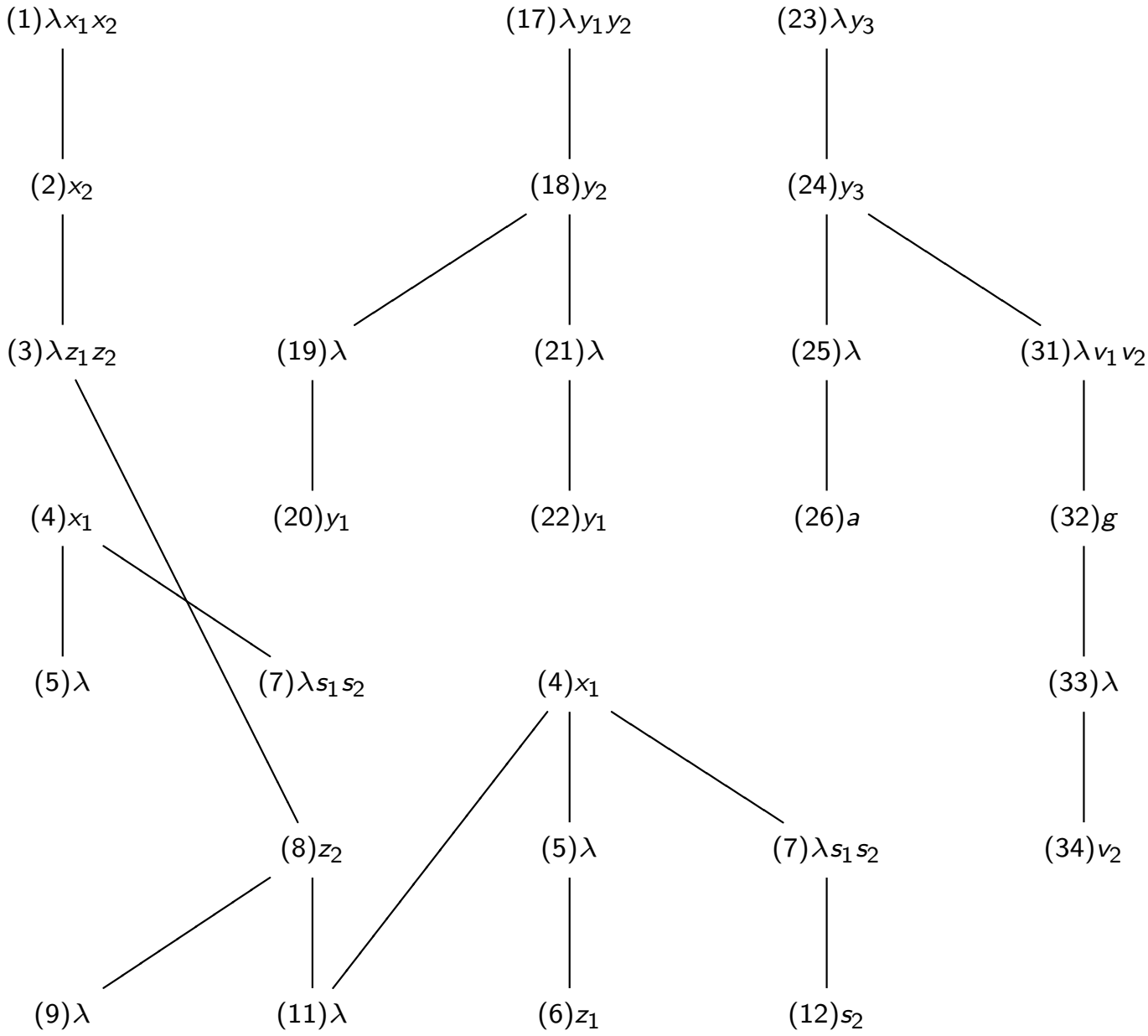
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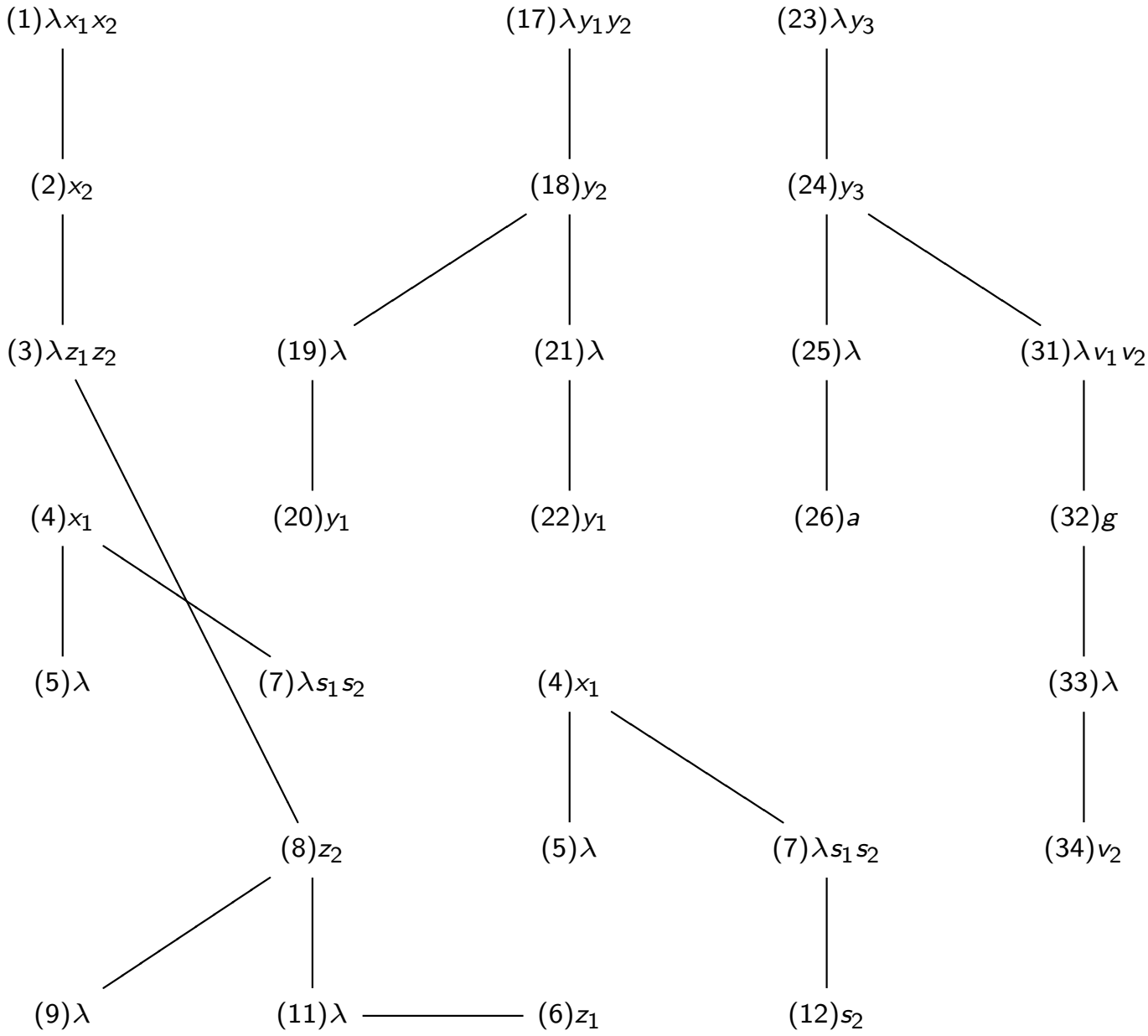
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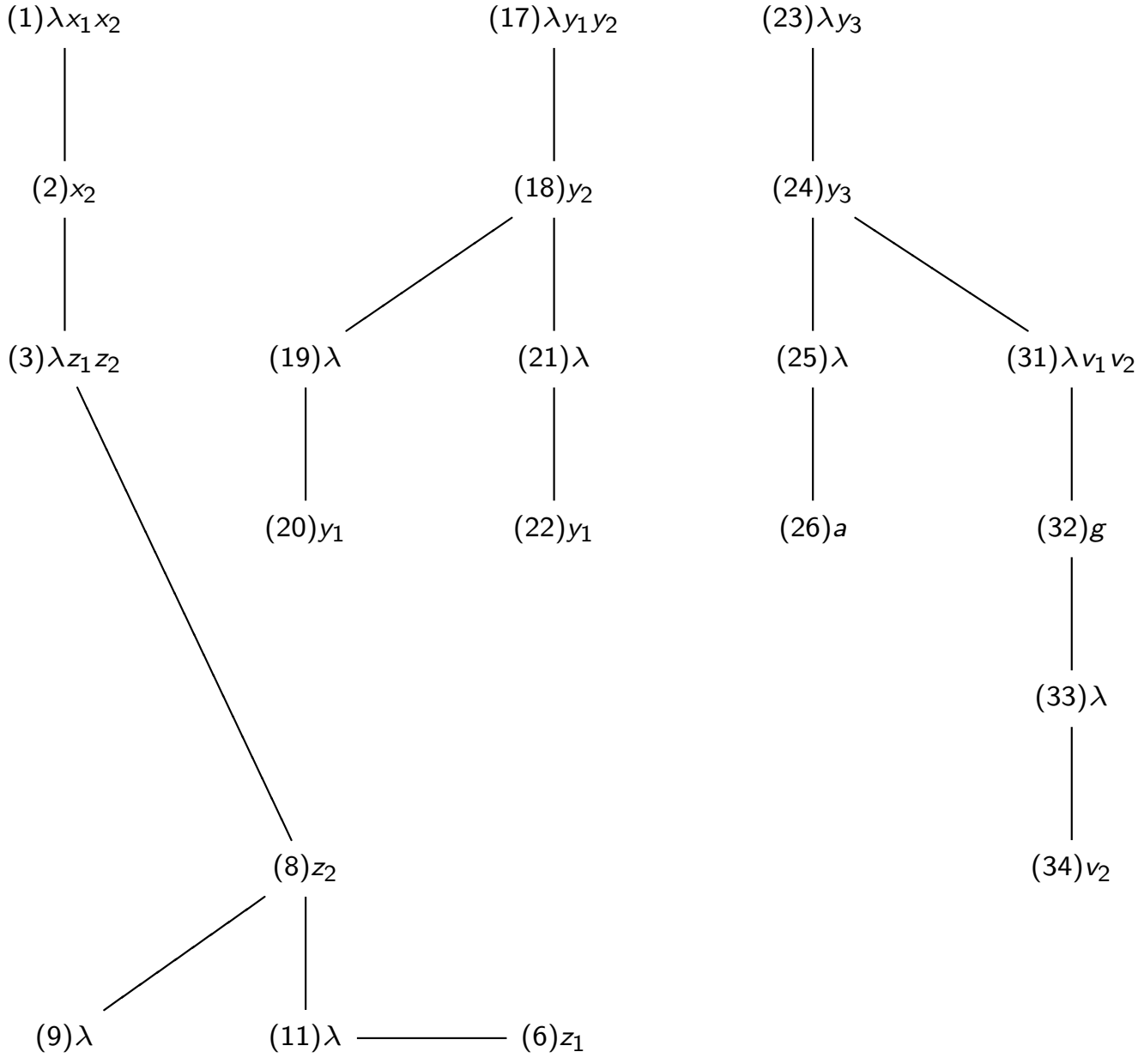
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- Small term property
- Theorem If $x : A$ and A has order $2n + 2$ or $2n + 3$ and arity m then $xw_1 \dots w_k = u$ has a (canonical) solution iff it has a (canonical) solution of depth at most $4n^2 m^{2n-2}|u|$

The Retract Problem

- Type A is a retract of B, $A \trianglelefteq B$, if there are $t : A \to B$ and $s : B \to A$ such that $s(tx) =_{\beta\eta} x$

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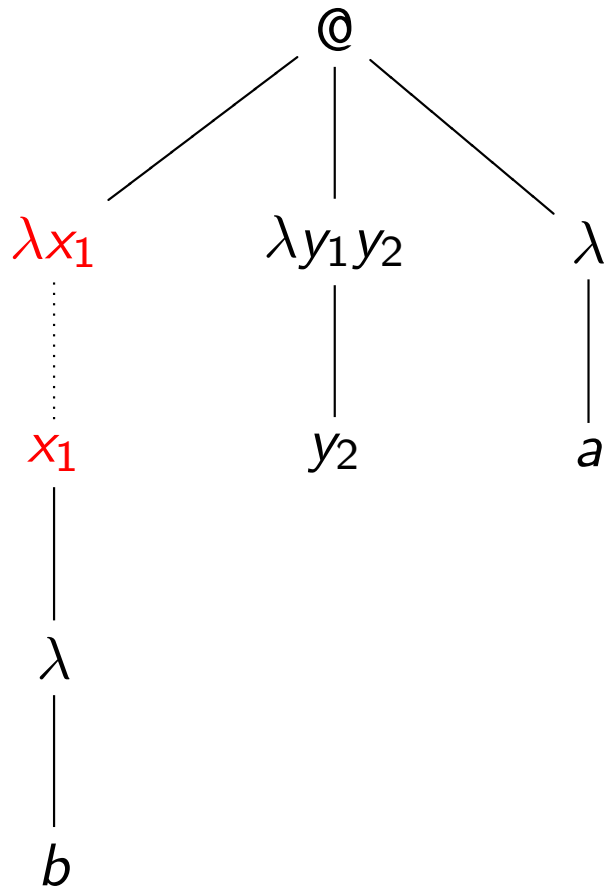
- Work in progress: using games, there is a proof system for the general case

Games only work for long normal forms

- $x : ((\mathbf{0}, \mathbf{0}, \mathbf{0}), \mathbf{0}, \mathbf{0})$ β-interpolation problem $x(\lambda y_1 y_2 . y_2)a = a$



- $\lambda x_1 . x_1 b(\lambda y_1 y_2 . y_2)a \rightarrow_\beta^* a$
- No natural game?