Suppose that we run belief propagation on the HMM in Figure 1, with the following schedule: First all of the observed nodes $o_t$ send a message $m_{o_t}(s_t)$ to their parent:

$$m_{o_t}(s_t) = p(o_t|s_t),$$  \hspace{1cm} (1)

and then we send messages $m_{t-1,t}(j)$ forward, starting with $s_0$, and finally we send messages $m_{t+1,t}(i)$ backward, starting with $s_T$. These messages are depicted in Figure 1, and given by:

$$m_{t-1,t}(j) = \sum_i p(s_t = j|s_{t-1} = i)m_{t-2,t-1}(i)m_{o_{t-1}}(i)$$  \hspace{1cm} (2)

$$m_{t+1,t}(i) = \sum_j p(s_{t+1} = j|s_t = i)m_{t+2,t+1}(j)m_{o_{t+1}}(j),$$  \hspace{1cm} (3)

where the summations over $i$ and $j$ are over the possible labels of the HMM.

The purpose of this note is simply to say: forward-backward is equivalent to belief propagation using this schedule, because in Figure 1 each $\alpha$ is the product of the red messages and $\beta$ is the blue message.

Figure 1: A sample HMM, marked with the messages sent by belief propagation. The forward-backward value for $\alpha_t$ is the product of the red messages, and $\beta_t$ is the blue message.
This can be proven by induction. For \( t = 0 \), we allow by convention \( m_{-1,0}(j) = p(s_0 = j) \), so that

\[
\alpha_0(j) = p(s_0 = j)p(o_0|s_0 = j) = m_{s_{-1}}(s_0)m_{o_0}(j). \tag{4}
\]

For \( t > 0 \), assume \( \alpha_{t-1}(i) = m_{t-1,t}(i)m_{o_{t-1}}(i) \) for all \( i \). Then

\[
\alpha_t(j) = p(o_t|s_t = j) \sum_i p(s_t = j|s_{t-1} = i)\alpha_{t-1}(i) \tag{5}
\]

\[
= m_{o_t}(j) \sum_i p(s_t = j|s_{t-1} = i)m_{t-1,t}(i) \tag{6}
\]

\[
= m_{o_t}(j)m_{t-1,t}(j), \tag{7}
\]

which completes the proof.

Similarly, it can be shown by induction that \( \beta_t(i) = m_{t+1,t}(i) \). We assume that \( m_{T+1,T} \) is uniformly 1, so that for all \( i \)

\[
\beta_T(i) = p(o_T|s_T = i) = m_{o_T}(i)m_{T+1,T}(i). \tag{8}
\]

Inductively, assume that \( \beta_{t+1}(i) = m_{t+2,t+1}(i) \). Then

\[
\beta_t(i) = \sum_j p(s_{t+1} = j|s_t = i)p(o_{t+1}|s_{t+1} = j)\beta_{t+1}(j) \tag{9}
\]

\[
= \sum_j p(s_{t+1} = j|s_t = i)m_{o_{t+1}}(j)m_{t+2,t+1}(j) \tag{10}
\]

\[
= m_{t+1,t}(i), \tag{11}
\]

which completes the proof.