Logic Representation in LF

Report on work in progress*

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Abstract

The purpose of a logical framework such as LF is to provide a language for defining logical systems suitable for use in a logic-independent proof development environment. In previous work we have developed a theory of representation of logics in a logical framework and considered the behaviour of structured theory presentations under representation. That work was based on the simplifying assumption that logics are characterized as families of consequence relations on "closed" sentences. In this report we extend the notion of logical system to account for open formula, and study its basic properties. Following standard practice, we distinguish two types of logical system of open formulae that differ in the treatment of free variables, and show how they may be induced from a logical system of closed sentences. The technical notions of a logic presentation and a uniform encoding of a logical system in LF are generalized to the present setting.

1 Introduction

The Logical Framework (LF) [HHP87] is a language for defining formal systems. The language is a three-level typed λ-calculus with Π-types, closely related to the AUTOMATH type theories [dB80, vD80]. A formal system is specified by giving an LF signature, a finite list of constant declarations that specifies the syntax, judgement forms, and inference rules of the system. All of the syntactic apparatus of a formal system, including proofs, are represented as LF terms. The LF type system is sufficiently expressive to capture the uniformities of a large class of logical systems of interest to computer science, including notions of schematic rules and proofs, derived rules of inference, and higher-order judgement forms expressing consequence and generality. Throughout this paper we assume a reasonably good acquaintance with the concepts and formalism of LF as presented in [HHP87].

In [HST89] we have studied a notion of representation of a logical system in LF. A logical system (or *logic*) is formalized in [HST89] as a family of consequence relations between *sentences* of the logical system uniformly defined over *signatures* of the system. To represent such an object logic $\mathcal{L}$ in LF, a uniform presentation of $\mathcal{L}$-signatures as extensions of an LF signature is required. Then, for each signature, $\mathcal{L}$-sentences over this signature are mapped to closed LF types of a specified form in such a way that this yields a full and faithful embedding of the consequence relation $\vdash^{\mathcal{L}}$ in the consequence relation $\vdash^{\mathcal{LF}}$ of LF. (The consequence relation of LF is given by considering inhabitation assertions, as in NuPRL [Con86].) By focusing on the embedding of logical systems, LF may be viewed as a "universal metalogic" in which all inferential activity is to be conducted: object logics exists (for the purposes of implementation) only insofar as they are encodable in LF. In

*Most definitions in this paper are rather tentative and may change at further stages of our work on these problems.

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[HST89] we have studied in detail the issues concerned in lifting inferential activity in object logics to LF via their representations. In particular we studied the problems of inference in theories presented in a structured way, much as in [SB83].

In our earlier work on logic representation [HST89] we focused on the notion of a logical system as a family of *simple consequence relations* [Avr87] satisfying certain natural closure conditions. For the sake of simplicity we considered only logics of "closed" sentences (referred to in this paper as *ground logical systems*), taking no explicit account of the behaviour of variables in a logic. Although it is difficult to say in general what are "closed" sentences, this assumption is perhaps best explained by noting that in our notion of representation, the sentences of a logical system are naturally (compositionally) encoded in LF as closed types. The purpose of this work is to remove this simplifying assumption by considering a notion of logical system that includes an explicit treatment of open formulae, and to consider the representation of such systems in a logical framework. It should be stressed that we are still making the simplifying assumption that logics are presented as consequence relations. In future work we intend to consider not just consequence, but also proofs.

Open formulae are not just a fancy feature we wish to add to our framework for laughs. First of all, an important motivation for our work is to adequately model logical concepts as described and used in mathematical logic, and open formulae certainly occur there. More specifically and perhaps even more importantly, open formulae are necessary to adequately study standard finite presentations of some common mathematical theories. For example, the usual presentation of Peano arithmetic includes the following axiom schema (induction schema):

$$P(0) \wedge \forall n.(P(n) \supset P(succ(n))) \supset \forall n.P(n)$$

This is schematic in P, which should not be interpreted as ranging over all closed sentences "with holes for n" only. For instance, the associativity of $+$, $\forall k, n, m.\, k + (n + m) = (k + n) + m$, is *not derivable* from such closed instances of the induction schema. It may be proved, however, "by induction on m" using an instance of the schema with k and n *free*.

This paper is organized as follows. In Section 2 we recall from [HST89] the definition of a ground logical system as a family of consequence relations indexed by signatures that satisfies a certain uniformity condition with respect to change of signature. This resembles the formalization of a logical system as an *institution* from [GB84]; the crucial difference is that institutions present a model-theoretic view of logical systems while our formulation is centered directly on the notion of a consequence relation. (See also [FS88], which is based on the notion of a closure operation, and [Mes89], which encompasses both model-theoretic and proof-theoretic points of view.) The sorts of consequence relations that we consider are motivated by the strictures of encoding in LF, and thus are limited to one-sided consequence relations that are closed under weakening, permutation, contraction, and cut, and which satisfy compactness. Generalizing the methodology of [HHP87], we introduce the notion of a *representation* of one logical system in another, taking account of variability in signatures.

In Section 3 we generalize these ideas and introduce the concept of a logical system of open formulae. Roughly, to each signature of the logical system is associated a category of contexts and substitutions (*cf.* [Car86]), and then to each signature and context over this signature is associated a set of formulae over this signature and context. Every formula is always considered with an explicit indication of its context. Hence, the consequence relation associated with each signature of a logical system of open formulae is defined on pairs consisting of a context (over this signature) and a formula built in this context. Of course, this consequence relation is required to satisfy the uniformity condition induced by the morphism structure of the category of signatures. In mathematical logic two types of consequence relations on open formulae are usually considered. Validity-type consequence relation rely on an implicit universal quantification of the free variables in each open formula. Truth-type consequence relations universally quantify free variables "globally" in the whole consequence statement. We characterize the two types of logical systems of open formulae by imposing appropriate structural conditions on their consequence relations.

In Section 4 we show how open formulas may be introduced to ground logical systems. This follows the view in abstract model theory that variables are uninterpreted constants (*cf.* [Bar74]), explored in a similar way in the theory of institutions in [Tar86], [ST88]. We try to justify the formal construction of Section 4 using a model-theoretic view provided via the theory of institutions in Section 5.

In Section 6 we introduce the metalogic of interest, LF, and view it as a logical system of open formulae in two different ways, guided by the validity and truth interpretations of LF contexts. We then define the notion of a logic presentation. A logic presentation is essentially an LF signature equipped with an indication of which LF contexts encode contexts of the object logic and which LF types encode the judgements of the object logic. Such a presentation induces again two logical systems: one of validity type, the other of truth type. A uniform encoding of an object logic in LF is a representation of the object logic in a logic presented in LF satisfying certain additional conditions ensuring adequacy of the encoding of the syntax. We also indicate that all the methodological suggestions on constructing logical systems in a structured way via structuring their presentations as suggested in [HST89] for ground logical systems carry over to this more general framework as well.

Finally, in Section 7 we suggest directions for future research. We stress once more that the current paper is just a report on work very much in progress. Thus, most of these suggestions are in fact research obligations to round off the technical ideas presented here.

2 Consequence relations and ground logical systems

In this section we recall the basic definitions used in [HST89] to capture the concept of a ground logical system and of a representation of one ground logical system in another. We start with some categorial preliminaries.

By a *category with inclusions* we mean any category $\mathcal{K}$ with a "wide" preorder subcategory of morphisms, which will be referred to as *inclusions*, such that the identity map on each object $A \in |\mathcal{K}|$ is the (unique) inclusion of A into itself. Inclusions are designated by $\iota : A \hookrightarrow B$. When convenient we will write the target object B as B^ι, and sometimes even identify the inclusion with its target (when the source is clear from the context). For any two objects $A, B \in |\mathcal{K}|$, we say that A *is included in* B, written $A \hookrightarrow B$, if there is an inclusion $\iota : A \hookrightarrow B$. In many particular cases that we study, morphisms are functions of some kind; in such cases we will normally assume without explicit mention that the inclusions are inclusions in the usual sense.

We will also assume that each category with inclusions $\mathcal{K}$ *has canonical pushouts along inclusions*, *i.e.*, whenever $f : A \to A'$ and $\iota : A \hookrightarrow A''$ are morphisms of $\mathcal{K}$, the pushout of f and ι exists, and, moreover, the morphism opposite the inclusion in the pushout diagram is itself an inclusion:

$$\begin{tikzcd} A' \arrow[r, hook, "\iota^\star"] & f^\star A'' \\ A \arrow[u, "f"] \arrow[r, hook, "\iota"'] & A'' \arrow[u, "p(f, A'')"'] \end{tikzcd}$$

We require a canonical choice of $p(f, A'')$ (and $f^\star A''$) which is functorial in f, *i.e.*, $p(f;f', A'') = p(f, A''); p(f', f^\star A'')$ (dually to contextual categories, *cf.* [Car86]).

For any two morphisms $f : A \to B$ and $f' : A' \to B'$ in a category with inclusions $\mathcal{K}$, we say that f' *is an extension of* f if there are inclusions $\iota_A : A \hookrightarrow A'$ and $\iota_B : B \hookrightarrow B'$ such that $\iota_A;f' = f;\iota_B$. A family of morphisms $f_i : A_i \to B_i$, $i = 1, \ldots, n$, is *compatible* if for all $\{ i_1, \ldots, i_k \} \subseteq \{ 1, \ldots, n \}$, for all objects A such that for $l = 1, \ldots, k$, $A \hookrightarrow A_{i_l}$, there is a morphism $f : A \to B$ (for some B) such that for $l = 1, \ldots, k$, f_{i_l} is an extension of f.

Throughout this paper, all functors between categories with inclusions will be assumed to preserve inclusions. The category of all categories with inclusions and inclusion-preserving functors will be denoted by **ICat**.

For any category with inclusions $\mathcal{K}$, *the category of functors into* $\mathcal{K}$, $\mathbf{Func}_{\mathbf{ICat}}(\mathcal{K})$, is defined as follows (*cf.* [TBG] where a similar category of diagrams in a given category is defined via an indexed category using the Grothendieck construction):

objects: are pairs $\langle \mathcal{I}, F\rangle$ consisting of a category with inclusions $\mathcal{I}$ and a functor $F : \mathcal{I} \to \mathcal{K}$.

morphisms: from $\langle \mathcal{I}, F\rangle$ to $\langle \mathcal{J}, G\rangle$ are pairs $\mu = \langle \mu^1, \mu^2\rangle$ where $\mu^1 : \mathcal{I} \to \mathcal{J}$ is a functor and $\mu^2 : F \dot{\to} \mu^1;G$ is a natural transformation of functors in $\mathcal{I} \to \mathcal{K}$.

composition: is defined by[1]:

$$\langle \mu^1_1, \mu^2_1\rangle;\langle \mu^1_2, \mu^2_2\rangle = \langle \mu^1_1;\mu^1_2, \mu^2_1;(\mu^1_1;\mu^2_2)\rangle.$$

Our treatment of logical systems centers on consequence relations (see [Avr87] for a survey). We take a consequence relation to be a binary relation between finite subsets and elements of a set of "sentences" satisfying three conditions to be given below. We use φ and ψ to range over sentences, Φ to range over arbitrary sets of sentences, and Δ to range over finite sets of sentences. We write Δ, Δ' for union, and write φ, Δ for $\{\varphi\}, \Delta$. If $s : \Phi_1 \to \Phi_2$ is a function, then the extension of s to subsets of Φ_1 is denoted by s as well. Function application will often be denoted by juxtaposition, *e.g.*, $s\varphi$ stands for $s(\varphi)$.

Definition 2.1 *A* consequence relation (CR) *is a pair* $\langle S, \vdash\rangle$ *where* S *is a set of* sentences *and* $\vdash \subseteq Fin(S) \times S$ *is a binary relation such that*

1. *(Reflexivity)* $\varphi \vdash \varphi$;
2. *(Transitivity) If* $\Delta \vdash \varphi$ *and* $\varphi, \Delta' \vdash \psi$, *then* $\Delta, \Delta' \vdash \psi$.
3. *(Weakening) If* $\Delta \vdash \psi$, *then* $\varphi, \Delta \vdash \psi$.

If $S' \subseteq S$, *then* $\langle S, \vdash\rangle\lceil S'$ *is defined to be the consequence relation* $\langle S', \vdash \cap (Fin(S') \times S')\rangle$.

The choice of conditions on consequence relations is motivated by our intention to consider encodings of logical systems in LF (in a sense to be made precise below.)

Definition 2.2 *A* morphism of consequence relations (CR morphism) $s : \langle S_1, \vdash_1\rangle \to \langle S_2, \vdash_2\rangle$ *is a function* $s : S_1 \to S_2$ *(the* translation of sentences*) such that if* $\Delta \vdash_1 \varphi$, *then* $s\Delta \vdash_2 s\varphi$. *The CR morphism* s *is* conservative *if* $\Delta \vdash_1 \varphi$ *whenever* $s\Delta \vdash_2 s\varphi$. **CR** *is the category with inclusions whose objects are consequence relations and whose morphisms are CR morphisms. Identity, composition and inclusions are inherited from the category of sets. By* $|\_| : \mathbf{CR} \to \mathbf{Set}$ *we denote the functor which maps each consequence relation to its underlying set of sentences.*

CLEAR-like techniques for structuring theory presentations are based on the separation between the language of a theory and the set of axioms that generates it [BG80]. We therefore consider a logical system to be a family of consequence relations indexed by a collection of *signatures* which determine the language of a theory. Moreover, it is important for the development that consequence be preserved under variation in signature (for example, renaming constants or replacing constants by terms over another signature). This leads to the following definition:

[1] We use ";" to denote not only composition in a category (*e.g.*, the usual composition of functions and functors), written in diagrammatic order, but also both vertical composition of natural transformations and the composition of a natural transformation with a functor so that $(\mu^2_1;(\mu^1_1;\mu^2_2))_A = (\mu^2_1)_A;(\mu^2_2)_{\mu^1_1(A)}$.

Definition 2.3 *A* ground logical system, *or* ground logic, *is a functor* $\mathcal{G} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{CR}$ *where* $\mathbf{Sig}^{\mathcal{G}}$ *is a category with inclusions and* $\mathcal{G}$ *is an inclusion-preserving functor.*[2].

The category $\mathbf{Sig}^{\mathcal{G}}$ is called the *category of signatures* of $\mathcal{G}$, with objects denoted by Σ and morphisms by $\sigma : \Sigma_1 \to \Sigma_2$. A signature morphism $\sigma : \Sigma_1 \to \Sigma_2$ is to be thought of as specifying a "relative interpretation" of the language defined by Σ_1 into the language defined by Σ_2. Writing $\mathcal{G}(\Sigma) = (|\mathcal{G}|_\Sigma, \vdash^{\mathcal{G}}_\Sigma)$, the definition of logical system implies that if $\sigma : \Sigma_1 \to \Sigma_2$ and $\Delta \vdash^{\mathcal{G}}_{\Sigma_1} \varphi$, then $\mathcal{G}(\sigma)(\Delta) \vdash^{\mathcal{G}}_{\Sigma_2} \mathcal{G}(\sigma)(\varphi)$. The function $|\mathcal{G}|(\sigma)$ underlying the CR morphism is called the *translation function* induced by σ (we write $|\mathcal{G}|$ for the composition $\mathcal{G};|\_| : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{Set}$). To simplify notation, we write $\sigma(\varphi)$ for $\mathcal{G}(\sigma)(\varphi)$ and $\sigma(\Delta)$ for $\mathcal{G}(\sigma)(\Delta)$ when no confusion is likely.

Definition 2.4 *The category of ground logical systems,* **GLog**, *is defined as* $\mathbf{Func}_{\mathbf{ICat}}(\mathbf{CR})$. *Hence, a* morphism of ground logics $\gamma : \mathcal{G} \to \mathcal{G}'$ *is a pair* $(\gamma^{Sig}, \gamma^{CR})$ *where* $\gamma^{Sig} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{Sig}^{\mathcal{G}'}$ *is a functor and* $\gamma^{CR} : \mathcal{G} \dot{\to} \gamma^{Sig};\mathcal{G}' : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{CR}$ *is a natural transformation.*

A morphism of ground logics is to be thought of as an "encoding" of one logical system in another in such a way that consequence is preserved. Let $\gamma : \mathcal{G} \to \mathcal{G}'$ be a morphism of ground logics. To simplify notation, we write $\gamma(\Sigma)$ for $\gamma^{Sig}(\Sigma)$, and $\gamma(\varphi)$ for $\gamma^{CR}_\Sigma(\varphi)$ (for appropriate choice of Σ).

Definition 2.5 *A ground logic morphism* $\gamma : \mathcal{G} \to \mathcal{G}'$ *is a* representation *if* γ^{Sig} *is an embedding and each* γ^{CR}_Σ *is conservative. A representation is* surjective *if each* γ^{CR}_Σ *is surjective as a function on the underlying sets.*

We refer to [HST89] for simple examples of ground logics and their representations.

3 Logical systems of open formulae

In the previous section we studied ground logical systems, where the logical sentences considered are closed (intuitively, built entirely out of the symbols given in the signature). These may perhaps be best characterized by referring to model theory: the truth of a closed sentence is unambigously determined by an interpretation of the symbols in the signature (*i.e.*, a model over this signature). In many logical systems studied in mathematical logic, however, logical formulae may additionally contain "free variables". To determine the truth of such a formula, an interpretation must be provided not only for the symbols in the signature, but also for the free variables (*i.e.*, not only a model, but also a "valuation" of the free variables in the model must be given). The free variables of an open formula, usually together with information on their typing, form the "context" in which the formula is built. We will avoid the sloppiness of leaving implicit the context in which an open formula is built, and always consider formulae over a given signature together with an explicitly indicated context. Moreover, where for ground logical systems we have assumed that signatures form a category and that signature morphisms induce translation of sentences, here we deal as well with a category of signatures, where signature morphisms induce translations of both contexts and formulae. Furthermore, the contexts over each signature are assumed to form a category (with morphisms which may be thought of as substitutions of terms for variables) and context morphisms then induce a translation of formulae. As usual, the translations induced are assumed to be mutually consistent.

Thus, the formulae of a logical system of open formulae are given by a functor

$$\mathcal{F} : \mathbf{Sig} \to \mathbf{Func}_{\mathbf{ICat}}(\mathbf{Set}).$$

The functor $\mathcal{F}$, for each signature $\Sigma \in |\mathbf{Sig}|$, yields a functor $\mathcal{F}(\Sigma) : \mathbf{Ctxt}_\Sigma \to \mathbf{Set}$, where $\mathbf{Ctxt}_\Sigma$ is a category of Σ-*contexts* with inclusions. Then, for any Σ-context $\Gamma \in |\mathbf{Ctxt}_\Sigma|$, the set $\mathcal{F}(\Sigma)(\Gamma)$ is a

[2]Of course this definition captures only one aspect of what is usually meant by the informal notion of "logical system."

set of Σ-*formulae in context* Γ. As mentioned before, we will always consider open formulae together with the context they are built in, and so we in fact will be dealing with the following set of open Σ-formulae:

$$Form_{\mathcal{F}}(\Sigma) = \{ \langle \Gamma, \varphi \rangle \mid \Gamma \in |\mathbf{Ctxt}_\Sigma|, \varphi \in \mathcal{F}(\Sigma)(\Gamma) \}.$$

The functor $\mathcal{F}$ also determines translations of contexts and formulae as mentioned above: for any signature morphism $\sigma : \Sigma \to \Sigma'$, $\mathcal{F}(\sigma)$ is a morphism in $\mathbf{Func_{ICat}}(\mathbf{Set})$ from $\mathcal{F}(\Sigma) : \mathbf{Ctxt}_\Sigma \to \mathbf{Set}$ to $\mathcal{F}(\Sigma') : \mathbf{Ctxt}_{\Sigma'} \to \mathbf{Set}$. By definition (*cf.* Sec. 2), we thus have a functor $\mathcal{F}(\sigma)^1 : \mathbf{Ctxt}_\Sigma \to \mathbf{Ctxt}_{\Sigma'}$ and a natural transformation $\mathcal{F}(\sigma)^2 : \mathcal{F}(\Sigma) \dot{\to} \mathcal{F}(\sigma)^1;\mathcal{F}(\Sigma')$, and hence for each Σ-context Γ, a function $\mathcal{F}(\sigma)^2_\Gamma : \mathcal{F}(\Sigma)(\Gamma) \to \mathcal{F}(\Sigma')(\mathcal{F}(\sigma)^1(\Gamma))$. When no confusion is likely, we will write $\sigma(\Gamma)$ for $\mathcal{F}(\sigma)^1(\Gamma)$ (for any $\Gamma \in |\mathbf{Ctxt}_\Sigma|$), $\sigma_\Gamma(\varphi)$ for $\mathcal{F}(\sigma)^2_\Gamma(\varphi)$ (for any $\Gamma \in |\mathbf{Ctxt}_\Sigma|$ and $\varphi \in \mathcal{F}(\Sigma)(\Gamma)$) and $\gamma(\varphi)$ for $\mathcal{F}(\Sigma)(\gamma)(\varphi)$ (for any Σ-context morphism $\gamma : \Gamma \to \Gamma'$ and $\varphi \in \mathcal{F}(\Sigma)(\Gamma)$).

This in turn induces a natural extension of the map $\Sigma \mapsto Form_{\mathcal{F}}(\Sigma)$ to a functor

$$Form_{\mathcal{F}} : \mathbf{Sig} \to \mathbf{Set}$$

where for any signature morphism $\sigma : \Sigma \to \Sigma'$, for any $\langle \Gamma, \varphi \rangle \in Form_{\mathcal{F}}(\Sigma)$, $Form_{\mathcal{F}}(\sigma)(\langle \Gamma, \varphi \rangle) = \langle \sigma(\Gamma), \sigma_\Gamma(\varphi) \rangle$.

Definition 3.1 *A* logical system of open formulae *(or, a logic of open formulae, or simply a* logic*) $\mathcal{L}$ consists of*

- *a functor $\mathcal{F}_\mathcal{L} : \mathbf{Sig}^\mathcal{L} \to \mathbf{Func_{ICat}}(\mathbf{Set})$, called the* formula functor *of $\mathcal{L}$, and*
- *a ground logical system $\mathcal{C}_\mathcal{L} : \mathbf{Sig}^\mathcal{L} \to \mathbf{CR}$, called the* consequence functor *of $\mathcal{L}$,*

such that the underlying sentence functor $|\mathcal{C}_\mathcal{L}| : \mathbf{Sig}^\mathcal{L} \to \mathbf{Set}$ of the consequence functor of $\mathcal{L}$ coincides with the functor $Form_{\mathcal{F}_\mathcal{L}} : \mathbf{Sig}^\mathcal{L} \to \mathbf{Set}$ as determined by the formula functor of $\mathcal{L}$.

Example 3.2 As a simple example of a logic of open formulae we present first-order logic. Since this is very standard and well known, we will omit many standard definitions and refer to the reader's intuition.

A *first-order signature* is a set of operation and predicate names with indicated arities (≥ 0). A first-order signature morphism maps operation names to terms of the same arity and renames predicate names preserving their arities. This defines the category $\mathbf{Sig}^{\mathcal{FO}}$ of first-order logic signatures.

For any first-order signature Σ, Σ-*contexts* are finite sets of variables, and so the category $\mathbf{Ctxt}^{\mathcal{FO}}_\Sigma$ of Σ-contexts has finite sets (of variables) as contexts and substitutions of terms with variables Y for variables X as morphisms from X to Y. Any first-order signature morphism $\sigma : \Sigma \to \Sigma'$ determines an obvious functor from $\mathbf{Ctxt}^{\mathcal{FO}}_\Sigma$ to $\mathbf{Ctxt}^{\mathcal{FO}}_{\Sigma'}$, which is the identity on objects (sets of variables).

Finally, for any first-order signature Σ and Σ-context X, the set of *first-order formulae* is defined in the usual way; these are first-order formulae over Σ with all free variables in X. Any context morphism (substitution) determines the usual translation of first-order formulae, as does any signature morphism.

All this defines the formula functor $\mathcal{F}_{\mathcal{FO}} : \mathbf{Sig}^{\mathcal{FO}} \to \mathbf{Func_{ICat}}(\mathbf{Set})$. The consequence relation of first-order logic may be defined model-theoretically as follows.

For any first-order signature Σ, a *first-order* Σ-*structure* A consists of a non-empty carrier set $|A|$ and an interpretation of operation names in Σ as functions on $|A|$, and of the predicate names in Σ as relations on $|A|$, of the arity indicated in Σ. Let $\mathbf{Str}^{\mathcal{FO}}(\Sigma)$ be the collection of all first-order Σ-structures (this forms a category with Σ-homomorphisms as morphisms).

Consider now a Σ-structure $A \in \mathbf{Str}^{\mathcal{FO}}(\Sigma)$, a set of variables X and a *valuation* $v : X \to |A|$. For any Σ-formula with free variables in X, $\varphi \in \mathcal{F}_{\mathcal{FO}}(\Sigma)(X)$, *the satisfaction of* φ *in* A *under* v, written $A[v] \models^{\mathcal{FO}}_\Sigma \varphi$, is defined in the usual way.

This standard notion of satisfaction may be used to determine consequence relations over the set of first-order formulae in two different ways. One possibility is to consider free variables as always implicitly universally quantified. This leads to the following family of *validity* consequence relations: for each first-order signature Σ, sets X_i of variables and open first-order formulae $\varphi_i \in \mathcal{F}_{\mathcal{FO}}(\Sigma)(X_i)$, $i = 0, \dots, n$,

$\{\langle X_i, \varphi_i\rangle\}_{i=1}^n \vdash_\Sigma^{\mathcal{FO}^v} \langle X_0, \varphi_0\rangle$ if and only if for all $A \in \mathbf{Str}^{\mathcal{FO}}(\Sigma)$, $A[v_0] \models_\Sigma^{\mathcal{FO}} \varphi_0$ for all valuations $v_0 : X_0 \to |A|$ whenever $A[v_i] \models_\Sigma^{\mathcal{FO}} \varphi_i$ for $i = 1, \dots, n$, for all valuations $v_i : X_i \to |A|$.

This yields the logical system $\mathcal{FO}^v$ of open formulae of first-order logic under the validity interpretation. From the point of view of proof theory, the usual Hilbert-type presentations of first-order logic present the same validity consequence relation.

Another possible view is to consider open formulae as truly open, and hence to identify occurrences of the same variable in different formulae. This leads to the following family of *truth* consequence relations: for each first-order signature Σ, sets X_i of variables and open first-order formulae $\varphi_i \in \mathcal{F}_{\mathcal{FO}}(\Sigma)(X_i)$, $i = 0, \dots, n$,

$\{\langle X_i, \varphi_i\rangle\}_{i=1}^n \vdash_\Sigma^{\mathcal{FO}^t} \langle X_0, \varphi_0\rangle$ if and only if for all $A \in \mathbf{Str}^{\mathcal{FO}}(\Sigma)$, for all valuations $v_i : X_i \to |A|$, $i = 0, \dots, n$, such that for all $\{i_1, \dots, i_k\} \subseteq \{0, \dots, n\}$ the v_{i_l} coincide on $X = \bigcap_{l=1}^k X_{i_l}$, $A[v_0] \models_\Sigma^{\mathcal{FO}} \varphi_0$ whenever $A[v_i] \models_\Sigma^{\mathcal{FO}} \varphi_i$ for $i = 1, \dots, n$.

This yields the logical system $\mathcal{FO}^t$ of open formulae of first-order logic under the truth interpretation. Natural-deduction-style presentations of first-order logic present the truth consequence relation via the notion of derivation under hypotheses. □

The above example illustrates two different views of the rôle of free variables in open formulae. The first option is to assume that free variables in an open formula are "local" to the formula, and so open formulae are always implicitly universally closed. This corresponds to so-called *validity* consequence relations, determined by a model-theoretic satisfaction relation according to the scheme:

$\Phi \vdash^v \varphi$ if and only if in every model, if Φ holds under every valuation then φ holds under every valuation as well.

A characteristic structural property of such consequence relations is that its conclusion (the formula on the right) may be instantiated, and its premises (formulae on the left) may be generalized.

Definition 3.3 *A logic of open formulae $\mathcal{L}$ is* of validity type *if its consequence relations admit instantiation on the right, i.e., for any signature $\Sigma \in |\mathbf{Sig}^{\mathcal{L}}|$ and open Σ-formulae $\Delta \subseteq Form_{\mathcal{F}_{\mathcal{L}}}(\Sigma)$ and $\langle\Gamma, \varphi\rangle \in Form_{\mathcal{F}_{\mathcal{L}}}(\Sigma)$, whenever*

$$\Delta \vdash_\Sigma^{\mathcal{L}} \langle\Gamma, \varphi\rangle$$

then for any Σ-context morphism $\gamma : \Gamma \to \Gamma'$,

$$\Delta \vdash_\Sigma^{\mathcal{L}} \langle\Gamma', \gamma(\varphi)\rangle.$$

Proposition 3.4 *If $\mathcal{L}$ is a logical system of validity type then its consequence relations admit generalisation on the left, i.e., for any signature $\Sigma \in |\mathbf{Sig}^{\mathcal{L}}|$ and open formulae $\langle\Gamma_i, \varphi_i\rangle \in Form_{\mathcal{F}_{\mathcal{L}}}(\Sigma)$, $i = 0, \dots, n$, whenever*

$$\{\langle\Gamma_i, \varphi_i\rangle\}_{i=1}^n \vdash_\Sigma^{\mathcal{L}} \langle\Gamma_0, \varphi_0\rangle$$

then for any Σ-context morphisms $\gamma_i : \Gamma_i' \to \Gamma_i$ and formulae $\varphi_i' \in \mathcal{F}_{\mathcal{L}}(\Sigma)(\Gamma_i')$ such that $\varphi_i = \gamma_i(\varphi_i')$, $i = 1, \dots, n$,

$$\{\langle\Gamma_i', \varphi_i'\rangle\}_{i=1}^n \vdash_\Sigma^{\mathcal{L}} \langle\Gamma_0, \varphi_0\rangle.$$

Proof Using instantiation on the right, we get $\langle\Gamma_i', \varphi_i'\rangle \vdash_\Sigma^{\mathcal{L}} \langle\Gamma_i, \varphi_i\rangle$ for $i = 1, \dots, n$. The conclusion then follows by the transitivity of the consequence relation $\vdash_\Sigma^{\mathcal{L}}$. □

Proposition 3.5 *If $\mathcal{L}$ is a logical system of validity type then its consequence relations admit renamings, i.e., for any signature $\Sigma \in |\mathbf{Sig}^{\mathcal{L}}|$ and open formulae $\langle \Gamma_i, \varphi_i \rangle \in Form_{\mathcal{F}_{\mathcal{L}}}(\Sigma)$, $i = 0, \ldots, n$, whenever*

$$\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}} \langle \Gamma_0, \varphi_0 \rangle$$

then for any Σ-context isomorphisms $r_i : \Gamma_i \to \Gamma'_i$, $i = 0, \ldots, n$,

$$\{ \langle \Gamma'_i, r_i(\varphi_i) \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}} \langle \Gamma'_0, r_0(\varphi_0) \rangle.$$

Proof Use instantiation on the right w.r.t. $r_0 : \Gamma_0 \to \Gamma'_0$ and generalisation on the left w.r.t. $r_i^{-1} : \Gamma'_i \to \Gamma_i$. □

Proposition 3.6 *If $\mathcal{L}$ is a logical system of validity type then its consequence relations admit elimination of dummy variables on the left and introduction of dummy variables on the right, i.e., for any signature $\Sigma \in |\mathbf{Sig}^{\mathcal{L}}|$ and open formulae $\langle \Gamma_i, \varphi_i \rangle \in Form_{\mathcal{F}_{\mathcal{L}}}(\Sigma)$, $i = 0, \ldots, n$, whenever*

$$\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}} \langle \Gamma_0, \varphi_0 \rangle$$

then for any Σ-contexts $\Gamma'_i \hookrightarrow \Gamma_i$ such that $\varphi_i \in \mathcal{F}_{\mathcal{L}}(\Sigma)(\Gamma'_i)$, $i = 1, \ldots, n$, and $\Gamma_0 \hookrightarrow \Gamma'_0$,

$$\{ \langle \Gamma'_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}} \langle \Gamma'_0, \varphi_0 \rangle.$$

Proof Use instantiation on the right and generalisation on the left w.r.t. the context inclusions. □

It is worth emphasizing that validity consequence relations need not admit introduction of dummy variables on the left, nor elimination of dummy variables on the right.

A different view of free variables in open formulae is that they denote an arbitrary but (in some sense) fixed value throughout all the formulae they occur in. Consequently, occurrences of the same variable on both sides of the consequence relation are assumed to denote the same value. This corresponds to so-called *truth* consequence relations, determined by a model-theoretic satisfaction relation according to the scheme:

$\Phi \vdash^{v} \varphi$ if and only if in every model under every valuation, if Φ holds then φ holds as well.

A characteristic structural property of such consequence relations is that any variable may be instantiated at the same time on both sides of the consequence relation.

Definition 3.7 *A logic of open formulae $\mathcal{L}$ is* of truth-type *if its consequence relations admit global instantiation, i.e., for any signature $\Sigma \in |\mathbf{Sig}^{\mathcal{L}}|$ and open formulae $\langle \Gamma_i, \varphi_i \rangle \in Form_{\mathcal{F}_{\mathcal{L}}}(\Sigma)$, $i = 0, \ldots, n$, whenever*

$$\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}} \langle \Gamma_0, \varphi_0 \rangle$$

then for any compatible family of Σ-context morphisms $\gamma_i : \Gamma_i \to \Gamma'_i$, $i = 0, \ldots, n$,

$$\{ \langle \Gamma'_i, \gamma_i(\varphi_i) \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}} \langle \Gamma'_0, \gamma_i(\varphi_0) \rangle.$$

Proposition 3.8 *If $\mathcal{L}$ is a logical system of the truth type then its consequence relations admit introduction of dummy variables on the left and on the right, i.e., for any signature $\Sigma \in |\mathbf{Sig}^{\mathcal{L}}|$ and open formulae $\langle \Gamma_i, \varphi_i \rangle \in Form_{\mathcal{F}_{\mathcal{L}}}(\Sigma)$, $i = 0, \ldots, n$, whenever*

$$\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}} \langle \Gamma_0, \varphi_0 \rangle$$

then for any Σ-contexts Γ'_i such that $\Gamma_i \hookrightarrow \Gamma'_i$ for $i = 0, \ldots, n$,

$$\{ \langle \Gamma'_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}} \langle \Gamma'_0, \varphi_0 \rangle.$$

Proof This is a particular instance of global instantiation; just notice that any family of inclusions is compatible. □

It is important to realize that truth consequence relations in general do not admit elimination of dummy variables, neither on the left nor on the right.

Turning now to morphisms of logical systems, let us note that formula functors, as functors into the category $\mathbf{Func_{ICat}}(\mathbf{Set})$, come naturally equipped with a notion of morphism. Given two functors $\mathcal{F} : \mathbf{Sig} \to \mathbf{Func_{ICat}}(\mathbf{Set})$ and $\mathcal{F}' : \mathbf{Sig}' \to \mathbf{Func_{ICat}}(\mathbf{Set})$, a morphism $\mu : \mathcal{F} \to \mathcal{F}'$ consists of a functor $\mu^1 : \mathbf{Sig} \to \mathbf{Sig}'$ and a natural transformation $\mu^2 : \mathcal{F} \dot{\to} \mu^1;\mathcal{F}'$. The latter, in turn, is a family of morphisms in $\mathbf{Func_{ICat}}(\mathbf{Set})$, consisting for each $\Sigma \in |\mathbf{Sig}|$ of a functor $(\mu^2_\Sigma)^1 : \mathbf{Ctxt}_\Sigma \to \mathbf{Ctxt}'_{\mu^1(\Sigma)}$ and a natural transformation $(\mu^2_\Sigma)^2 : \mathcal{F}(\Sigma) \dot{\to} (\mu^2_\Sigma)^1;\mathcal{F}'(\mu^1(\Sigma))$. Finally, $(\mu^2_\Sigma)^2$ is a family indexed by Σ-contexts Γ of functions $(\mu^2_\Sigma)^2_\Gamma : \mathcal{F}(\Sigma)(\Gamma) \to \mathcal{F}'(\mu^1(\Sigma))((\mu^2_\Sigma)^1(\Gamma))$. When no confusion is likely, for any $\Sigma \in |\mathbf{Sig}|$, $\Gamma \in |\mathbf{Ctxt}_\Sigma|$ and $\varphi \in \mathcal{F}(\Sigma)(\Gamma)$, we write $\mu(\Sigma)$, $\mu_\Sigma(\Gamma)$ and $\mu_{\Sigma,\Gamma}(\varphi)$ for $\mu^1(\Sigma)$, $(\mu^2_\Sigma)^1(\Gamma)$ and $(\mu^2_\Sigma)^2_\Gamma(\varphi)$, respectively. Notice also that any such morphism $\mu : \mathcal{F} \to \mathcal{F}'$ defines a morphism $\hat{\mu} : \mathit{Form}_\mathcal{F} \to \mathit{Form}_{\mathcal{F}'}$, where $\hat{\mu}^1 : \mathbf{Sig} \to \mathbf{Sig}'$ is simply μ^1 and the natural transformation $\hat{\mu}^2 : \mathit{Form}_\mathcal{F} \dot{\to} \mu^1;\mathit{Form}_{\mathcal{F}'}$ between functors in $\mathbf{Sig} \to \mathbf{Set}$ is given by: for each $\Sigma \in |\mathbf{Sig}|$ and $\langle \Gamma, \varphi \rangle \in \mathit{Form}_\mathcal{F}(\Sigma)$, $\hat{\mu}^2_\Sigma(\langle \Gamma, \varphi \rangle) = \langle \mu_\Sigma(\Gamma), \mu_{\Sigma,\Gamma}(\varphi) \rangle$.

Definition 3.9 *Consider two logical systems of open formulae $\mathcal{L}$ and $\mathcal{L}'$. A* logic morphism *$\mu : \mathcal{L} \to \mathcal{L}'$ is a morphism between their formula functors $\mu : \mathcal{F}_\mathcal{L} \to \mathcal{F}_{\mathcal{L}'}$ such that $\hat{\mu} : \mathit{Form}_{\mathcal{F}_\mathcal{L}} \to \mathit{Form}_{\mathcal{F}_{\mathcal{L}'}}$ is a ground logic morphism $\hat{\mu} : \mathcal{C}_\mathcal{L} \to \mathcal{C}_{\mathcal{L}'}$.* **Log** *is the category of logical systems of open formulae and their morphism (it is easy to see that a composition of logic morphisms is a logic morphism).*

Just as for ground logical system, logic morphisms are a bit too crude to model the informal notion of logic representation.

Definition 3.10 *A logic morphism $\mu : \mathcal{L} \to \mathcal{L}'$ is a* representation *if $\hat{\mu} : \mathcal{C}_\mathcal{L} \to \mathcal{C}_{\mathcal{L}'}$ is a ground logic representation, i.e., if for each $\Sigma \in |\mathbf{Sig}^\mathcal{L}|$, $\hat{\mu}^2_\Sigma : \mathit{Form}_{\mathcal{F}_\mathcal{L}}(\Sigma) \to \mathit{Form}_{\mathcal{F}_{\mathcal{L}'}}(\mu^1(\Sigma))$ is a conservative morphism of consequence relations $\hat{\mu}^2_\Sigma : \mathcal{C}_\mathcal{L}(\Sigma) \to \mathcal{C}_\mathcal{L}(\mu^1(\Sigma))$, and all the functors $\mu^2_\Sigma : \mathbf{Ctxt}_\Sigma \to \mathbf{Ctxt}'_{\mu^1(\Sigma)}$ for $\Sigma \in |\mathbf{Sig}^\mathcal{L}|$ are injective. A logic representation $\mu : \mathcal{L} \to \mathcal{L}'$ is* exact *if in addition all the functors $\mu^2_\Sigma : \mathbf{Ctxt}_\Sigma \to \mathbf{Ctxt}'_{\mu^1(\Sigma)}$ for $\Sigma \in |\mathbf{Sig}^\mathcal{L}|$ are surjective and $\hat{\mu} : \mathcal{C}_\mathcal{L} \to \mathcal{C}_{\mathcal{L}'}$ is a* surjective *ground logic representation.*

4 From closed sentences to open formulae

The treatment of variables in abstract model theory [Bar74] suggests that at least in many typical cases the complete structure of a logical system of open formulae is in a sense redundant, since it may be recovered from a corresponding ground logical system of closed sentences. We have used the ideas of [Bar74] to develop a concept simulating open formulae in an arbitrary institution in [Tar86], [ST88]. Here we apply the same technique to construct a logical system of open formulae out of a ground logical system.

In the example of first-order logic sketched in the previous section, for any first-order signature Σ and Σ-context (set of variables X), the open Σ-formulae built in context X are exactly the closed sentences over the signature $\Sigma(X)$ defined as the extension of Σ by the variables in X as new constants (0-ary operation names). Moreover, given any Σ-structure A, a valuation of X in $|A|$ corresponds to an expansion of A to a $\Sigma(X)$-structure which indeed additionally determines an interpretation of the new constants, *i.e.*, a valuation of variables. Then, an open first-order Σ-formula φ with free variables in X holds in a structure A under a valuation $v : X \to |A|$ if and only if φ viewed as a closed $\Sigma(X)$-sentence holds in the expansion of A to a $\Sigma(X)$-structure determined by v. Hence, a consequence relation on open Σ-formulae with variables X may be recovered from the satisfaction relation on closed $\Sigma(X)$-sentences, and (much less directly) from the consequence relation on closed sentences. This idea readily generalizes to an arbitrary ground logical system. Notice that the "global" view of logical systems as uniformly-defined families indexed by a category of signatures is crucial here.

Consider an arbitrary ground logical system $\mathcal{G} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{CR}$, fixed throughout this section.

Given the category $\mathbf{Sig}^{\mathcal{G}}$ of signatures (with inclusions), we define contexts as signature extensions. More formally, for each $\Sigma \in |\mathbf{Sig}^{\mathcal{G}}|$, the category $\mathbf{Ctxt}^{\mathcal{G}}_{\Sigma}$ of Σ-contexts is the full subcategory of the slice category $\Sigma \downarrow \mathbf{Sig}^{\mathcal{G}}$ determined by inclusions $\iota : \Sigma \hookrightarrow \Sigma'$. That is, $\mathbf{Ctxt}^{\mathcal{G}}_{\Sigma}$ has as objects pairs consisting of a signature Σ' and a signature inclusion $\iota : \Sigma \hookrightarrow \Sigma'$; and a Σ-context morphism from $\iota_1 : \Sigma \to \Sigma_1$ to $\iota_2 : \Sigma \to \Sigma_2$ is a signature morphism $\gamma : \Sigma_1 \to \Sigma_2$ such that $\iota_1;\gamma = \iota_2$. As before, we will denote contexts by Γ; the target signature of the extension is then written as Σ^{Γ}, and the inclusion $\Sigma \hookrightarrow \Sigma^{\Gamma}$ is then determined unambigously. This allows us to identify contexts with signature inclusions or even with their target signatures when no confusion is likely. Notice that for any signature Σ, the identity on Σ, which is a signature inclusion, is the "empty context" $\langle\rangle \in |\mathbf{Ctxt}^{\mathcal{G}}_{\Sigma}|$. This is the least object in $|\mathbf{Ctxt}^{\mathcal{G}}_{\Sigma}|$ w.r.t. the inclusion ordering.

The map $\Sigma \mapsto \mathbf{Ctxt}^{\mathcal{G}}_{\Sigma}$ extends to a functor $\mathbf{Ctxt}^{\mathcal{G}} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{ICat}$ using the canonical pushout construction in $\mathbf{Sig}^{\mathcal{G}}$. Namely, for any signature morphism $\sigma : \Sigma \to \Sigma'$, $\mathbf{Ctxt}^{\mathcal{G}}(\sigma) : \mathbf{Ctxt}^{\mathcal{G}}_{\Sigma} \to \mathbf{Ctxt}^{\mathcal{G}}_{\Sigma'}$ is defined on objects as follows: for any $\Gamma \in |\mathbf{Ctxt}^{\mathcal{G}}_{\Sigma}|$, $\mathbf{Ctxt}^{\mathcal{G}}(\sigma)(\Gamma) = \Gamma^{\star}$.

$$
\begin{tikzcd}
\Sigma' \arrow[r, hook, "\Gamma^{\star}"] & \sigma^{\star}\Sigma^{\Gamma} \\
\Sigma \arrow[u, "\sigma"] \arrow[r, hook, "\Gamma"'] & \Sigma^{\Gamma} \arrow[u, "p(\sigma, \Sigma^{\Gamma})"']
\end{tikzcd}
$$

This extends to context morphisms using the pushout property.

The open formulae determined by $\mathcal{G}$ are given by the functor

$$\mathcal{F}^{\mathcal{G}} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{Func}_{\mathbf{ICat}}(\mathbf{Set})$$

defined as follows:

- for each $\Sigma \in |\mathbf{Sig}^{\mathcal{G}}|$, $\mathcal{F}^{\mathcal{G}}(\Sigma) : \mathbf{Ctxt}^{\mathcal{G}}_{\Sigma} \to \mathbf{Set}$ is defined by:
  - for each $\Gamma \in \mathbf{Ctxt}^{\mathcal{G}}_{\Sigma}$, $\mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma) = |\mathcal{G}|(\Sigma^{\Gamma})$
  - for each $\gamma : \Gamma \to \Gamma'$ in $\mathbf{Ctxt}^{\mathcal{G}}_{\Sigma}$, *i.e.*, $\gamma : \Sigma^{\Gamma} \to \Sigma^{\Gamma'}$ in $\mathbf{Sig}^{\mathcal{G}}$, $\mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma) = |\mathcal{G}|(\gamma)$
- for each $\sigma : \Sigma \to \Sigma'$ in $\mathbf{Sig}^{\mathcal{G}}$
  - $\mathcal{F}^{\mathcal{G}}(\sigma)^1 : \mathbf{Ctxt}^{\mathcal{G}}_{\Sigma} \to \mathbf{Ctxt}^{\mathcal{G}}_{\Sigma'}$ is the functor $\mathbf{Ctxt}^{\mathcal{G}}(\sigma)$ as defined above
  - for each $\Gamma \in |\mathbf{Ctxt}^{\mathcal{G}}_{\Sigma}|$, $\mathcal{F}^{\mathcal{G}}(\sigma)^2_{\Gamma} : |\mathcal{G}|(\Sigma^{\Gamma}) \to |\mathcal{G}|(\mathbf{Ctxt}^{\mathcal{G}}(\sigma)(\Gamma))$ is the function $|\mathcal{G}|(p(\sigma, \Sigma^{\Gamma})) : |\mathcal{G}|(\Sigma^{\Gamma}) \to |\mathcal{G}|(\sigma^{\star}\Sigma^{\Gamma})$.

It is easy to see that everything is well-defined here; in particular that all the functoriality and naturality requirements follow.

As before, $\mathcal{F}^{\mathcal{G}} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{Func}_{\mathbf{ICat}}(\mathbf{Set})$ determines open formulae of the logics we derive from $\mathcal{G}$. The functor $Form_{\mathcal{F}^{\mathcal{G}}} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{Set}$ is defined as in Sec. 3. We have yet to define the consequence relations of the system. This may be done in two different ways, depending on whether we consider the validity or the truth interpretation of consequence relations on open formulae.

Definition 4.1 *The* validity logic $\mathcal{L}^v(\mathcal{G})$ of open formulae determined by the ground logic $\mathcal{G} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{CR}$ *consists of the formula functor*

$$\mathcal{F}^{\mathcal{G}} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{Func}_{\mathbf{ICat}}(\mathbf{Set})$$

and the consequence functor

$$C_{\mathcal{L}^v(\mathcal{G})} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{CR}$$

where for each $\Sigma \in |\mathbf{Sig}^{\mathcal{G}}|$, *the consequence relation* $\vdash_{\Sigma}^{\mathcal{L}^v(\mathcal{G})}$ *on* $Form_{\mathcal{F}^{\mathcal{G}}}(\Sigma)$ *is defined as follows: for any* $\Gamma_i \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{G}}|$ *and* $\varphi_i \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma_i)$ $(i = 0, \ldots, n)$,

$\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^v(\mathcal{G})} \langle \Gamma_0, \varphi_0 \rangle$ *if and only if for some* $\{ i_1, \ldots, i_k \} \subseteq \{ 1, \ldots, n \}$ *there exist* Σ*-context morphisms* $\gamma_l : \Gamma_{i_l} \to \Gamma_0$, $l = 1, \ldots, k$, *such that*[3] $\{ \gamma_l(\varphi_{i_l}) \}_{l=1}^{k} \vdash_{\Sigma^{\Gamma_0}}^{\mathcal{G}} \varphi_0$.

In the above definition, $i_1, \ldots, i_k$ are not assumed to be distinct: we may use premises in many different ways.

Proposition 4.2 *$\mathcal{L}^v(\mathcal{G})$ as defined in Def. 4.1 is indeed a logical system of open formulae of validity type.*

Proof

- For each $\Sigma \in |\mathbf{Sig}^{\mathcal{G}}|$, $\vdash_{\Sigma}^{\mathcal{L}^v(\mathcal{G})}$ is indeed a consequence relation:

  Reflexivity and weakening follow directly from the definition. Transitivity follows from the fact that the consequence relations of $\mathcal{G}$ are preserved under translations induced by signature morphisms, which applies to context morphisms in $\mathcal{L}^v(\mathcal{G})$ as well since they are in fact signature morphisms in $\mathcal{G}$, and from the transitivity of the consequence relations of $\mathcal{G}$.

- For each $\sigma : \Sigma \to \Sigma'$, $\vdash_{\Sigma}^{\mathcal{L}^v(\mathcal{G})}$ is preserved under the translation of formulae induced by σ.

  This again follows from the fact that the consequence relations of $\mathcal{G}$ are preserved under translations induced by signature morphisms. Here are some details: let $\Gamma_i \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{G}}|$, $\varphi_i \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma_i)$, $i = 0, \ldots, n$, be such that $\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^v(\mathcal{G})} \langle \Gamma_0, \varphi_0 \rangle$. We have to prove that

  $$\{ \langle \sigma^\star \Sigma^{\Gamma_i}, \mathcal{G}(p(\sigma, \Sigma^{\Gamma_i}))(\varphi_i) \rangle \}_{i=1}^{n} \vdash_{\Sigma'}^{\mathcal{L}^v(\mathcal{G})} \langle \sigma^\star \Sigma^{\Gamma_0}, \mathcal{G}(p(\sigma, \Sigma^{\Gamma_0}))(\varphi_0) \rangle.$$

  By definition, there are $\{ i_1, \ldots, i_k \} \subseteq \{ 1, \ldots, n \}$ and Σ-context morphisms $\gamma_l : \Gamma_{i_l} \to \Gamma_0$, $l = 1, \ldots, k$, such that $\{ \mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma_l)(\varphi_{i_l}) \}_{l=1}^{k} \vdash_{\Sigma^{\Gamma_0}}^{\mathcal{G}} \varphi_0$.

  Since $\vdash_{\Sigma^{\Gamma_0}}^{\mathcal{G}}$ is preserved by translation of formulae induced by $p(\sigma, \Sigma^{\Gamma_0}) : \Sigma^{\Gamma_0} \to \sigma^\star \Sigma^{\Gamma_0}$, it follows that

  $$\{ \mathcal{G}(p(\sigma, \Sigma^{\Gamma_0}))(\mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma_l)(\varphi_{i_l})) \}_{l=1}^{k} \vdash_{\sigma^\star \Sigma^{\Gamma_0}}^{\mathcal{G}} \mathcal{G}(p(\sigma, \Sigma^{\Gamma_0}))(\varphi_0).$$

  For $l = 1, \ldots, k$, by the pushout property of $\sigma^\star \Sigma^{\Gamma_{i_l}}$, there is a Σ'-context morphism $\gamma'_l : \sigma^\star \Sigma^{\Gamma_{i_l}} \to \sigma^\star \Sigma^{\Gamma_0}$ such that

  $$p(\sigma, \Sigma^{\Gamma_{i_l}}); \gamma'_l = \gamma_l; p(\sigma, \Sigma^{\Gamma_0}).$$

  Then, for $l = 1, \ldots, k$,

  $$\begin{aligned} \mathcal{G}(p(\sigma, \Sigma^{\Gamma_0}))(\mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma_l)(\varphi_{i_l})) &= \mathcal{G}(\gamma_l; p(\sigma, \Sigma^{\Gamma_0}))(\varphi_{i_l}) \\ &= \mathcal{G}(p(\sigma, \Sigma^{\Gamma_{i_l}}); \gamma'_l)(\varphi_{i_l}) \\ &= \mathcal{F}^{\mathcal{G}}(\Sigma')(\gamma'_l)(\mathcal{G}(p(\sigma, \Sigma^{\Gamma_{i_l}})(\varphi_{i_l})). \end{aligned}$$

  Thus:

  $$\{ \mathcal{F}^{\mathcal{G}}(\Sigma')(\gamma'_l)(\mathcal{G}(p(\sigma, \Sigma^{\Gamma_{i_l}}))(\varphi_{i_l})) \}_{l=1}^{k} \vdash_{\sigma^\star \Sigma^{\Gamma_0}}^{\mathcal{G}} \mathcal{G}(p(\sigma, \Sigma^{\Gamma_0}))(\varphi_0)$$

  which completes the proof.

[3] We are using freely the notational conventions introduced in Sec. 3. In particular, $\gamma_l(\varphi_{i_l})$ is really $\mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma_l)(\varphi_{i_l})$.

- For each $\Sigma \in |\mathbf{Sig}^{\mathcal{G}}|$, $\vdash_{\Sigma}^{\mathcal{L}^v(\mathcal{G})}$ admits instantiation on the right:

  This again follows from the fact that the consequence relations of $\mathcal{G}$ are preserved under translations induced by signature morphisms.

  Let $\Gamma_i \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{G}}|$, $\varphi_i \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma_i)$, $i = 0, \ldots, n$, be such that $\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^v(\mathcal{G})} \langle \Gamma_0, \varphi_0 \rangle$. Consider a Σ-context morphism $\gamma : \Gamma_0 \to \Gamma_0'$. By definition, there are $\{ i_1, \ldots, i_k \} \subseteq \{ 1, \ldots, n \}$ and Σ-context morphisms $\gamma_l : \Gamma_{i_l} \to \Gamma_0$, $l = 1, \ldots, k$, such that

$$\{ \mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma_l)(\varphi_{i_l}) \}_{l=1}^{k} \vdash_{\Sigma^{\Gamma_0}}^{\mathcal{G}} \varphi_0.$$

  Then also

$$\{ \mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma_l;\gamma)(\varphi_{i_l}) \}_{l=1}^{k} \vdash_{\Sigma^{\Gamma_0'}}^{\mathcal{G}} \mathcal{G}(\gamma)(\varphi_0)$$

  from which it follows that

$$\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^v(\mathcal{G})} \langle \Gamma_0', \mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma)(\varphi_0) \rangle.$$

□

As a special case of the closure condition embodied in the above definition we have the following "rule of universal closure" and "rule of universal elimination."

Corollary 4.3 *Let $\mathcal{G} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{CR}$ be any ground logic, let $\Sigma \in |\mathbf{Sig}^{\mathcal{G}}|$ be any signature of $\mathcal{G}$, and let $\Gamma \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{G}}|$ be any Σ-context of $\mathcal{G}$.*

1. *For any formulae $\varphi_i \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\langle\rangle)$, $i = 1, \ldots, n$, and $\psi \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma)$, if $\{ \varphi_i \}_{i=1}^{n} \vdash_{\Sigma^{\Gamma}}^{\mathcal{G}} \psi$ then $\{ \langle \langle\rangle, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^v(\mathcal{G})} \langle \Gamma, \psi \rangle$.*

2. *For any formula $\varphi \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma)$ and any context morphism $\gamma : \Gamma \to \langle\rangle$, $\langle \Gamma, \varphi \rangle \vdash_{\Sigma}^{\mathcal{L}^v(\mathcal{G})} \langle \langle\rangle, \gamma(\varphi) \rangle$.*

Definition 4.4 *The* truth logic $\mathcal{L}^t(\mathcal{G})$ of open formulae determined by the ground logic $\mathcal{G} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{CR}$ *consists of the formula functor*

$$\mathcal{F}^{\mathcal{G}} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{Func}_{\mathbf{ICat}}(\mathbf{Set})$$

and the consequence functor

$$\mathcal{C}_{\mathcal{L}^t(\mathcal{G})} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{CR}$$

where for each $\Sigma \in |\mathbf{Sig}^{\mathcal{G}}|$, the consequence relation $\vdash_{\Sigma}^{\mathcal{L}^t(\mathcal{G})}$ on $Form_{\mathcal{F}^{\mathcal{G}}}(\Sigma)$ is defined as the transitive closure[4] *of the relation defined as follows: for any $\Gamma_i \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{G}}|$ and $\varphi_i \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma_i)$ $(i = 0, \ldots, n)$,*

$\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^t(\mathcal{G})} \langle \Gamma_0, \varphi_0 \rangle$ if and only if for some $\{ i_1, \ldots, i_k \} \subseteq \{ 1, \ldots, n \}$ there is a Σ-context $\Gamma \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{G}}|$ such that $\Gamma \hookrightarrow \Gamma_{i_l}$ for $l = 1, \ldots, k$ and $\Gamma \hookrightarrow \Gamma_0$, and for some formulae $\psi_l \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma)$, $l = 0, \ldots, k$, such that

- $\varphi_{i_l} \vdash_{\Sigma^{\Gamma_{i_l}}}^{\mathcal{G}} \psi_l$ *for* $l = 1, \ldots, k$
- $\{ \psi_l \}_{l=1}^{k} \vdash_{\Sigma^{\Gamma}}^{\mathcal{G}} \psi_0$
- $\psi_0 \vdash_{\Sigma^{\Gamma_0}}^{\mathcal{G}} \varphi_0$

If all the contexts are the same, say $\Gamma_i = \Gamma$ for $i = 0, \ldots, n$, then the condition stated in the definition amounts to the requirement that

$$\{ \varphi_i \}_{i=1}^{n} \vdash_{\Sigma^{\Gamma}}^{\mathcal{G}} \varphi_0.$$

Moreover, the relation defined in such a way is a consequence relation between formulas with a fixed context.

[4] *i.e.*, the least relation containing the relation defined below and satisfying the transitivity condition of Def. 2.1.

Proposition 4.5 *$\mathcal{L}^t(\mathcal{G})$ as defined in Def. 4.4 is indeed a logical system of open formulae of truth type.*

Proof

- For each $\Sigma \in |\mathbf{Sig}^{\mathcal{G}}|$, $\vdash_{\Sigma}^{\mathcal{L}^t(\mathcal{G})}$ is indeed a consequence relation:

  Transitivity follows directly from the definition. Reflexivity and weakening obviously hold for the generating relation, and it is easy to see that they are preserved by the transitive closure.

- For each $\sigma : \Sigma \to \Sigma'$, $\vdash_{\Sigma}^{\mathcal{L}^t(\mathcal{G})}$ is preserved under the translation of formulae induced by σ.

  It is enough to show that the generating relation is preserved. Consider: $\Gamma_i \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{G}}|$, $\varphi_i \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma_i)$, $i = 0, \ldots, n$, such that $\{\langle \Gamma_i, \varphi_i \rangle\}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^t(\mathcal{G})} \langle \Gamma_0, \varphi_0 \rangle$ and such that for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ there is a Σ-context $\Gamma \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{G}}|$ such that $\Gamma \hookrightarrow \Gamma_{i_l}$ for $l = 1, \ldots, k$ and $\Gamma \hookrightarrow \Gamma_0$, and there exist formulae $\psi_l \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma)$, $l = 0, \ldots, k$, such that

  - $\varphi_{i_l} \vdash_{\Sigma^{\Gamma_{i_l}}}^{\mathcal{G}} \psi_l$ for $l = 1, \ldots, k$
  - $\{\psi_l\}_{l=1}^{k} \vdash_{\Sigma^{\Gamma}}^{\mathcal{G}} \psi_0$
  - $\psi_0 \vdash_{\Sigma^{\Gamma_0}}^{\mathcal{G}} \varphi_0$

  Then $\sigma^\star\Gamma$ is a Σ'-context such that $\sigma^\star\Gamma \hookrightarrow \sigma^\star\Gamma_{i_l}$ for $l = 1, \ldots, k$ and $\sigma^\star\Gamma \hookrightarrow \sigma^\star\Gamma_0$. Moreover,

  - $\mathcal{G}(p(\sigma, \Sigma^{\Gamma_{i_l}}))(\varphi_{i_l}) \vdash_{\sigma^\star\Sigma^{\Gamma_{i_l}}}^{\mathcal{G}} \mathcal{G}(p(\sigma, \Sigma^{\Gamma}))(\psi_l)$ for $l = 1, \ldots, k$
  - $\{\mathcal{G}(p(\sigma, \Sigma^{\Gamma}))(\psi_l)\}_{l=1}^{k} \vdash_{\sigma^\star\Sigma^{\Gamma}}^{\mathcal{G}} \mathcal{G}(p(\sigma, \Sigma^{\Gamma}))(\psi_0)$
  - $\mathcal{G}(p(\sigma, \Sigma^{\Gamma}))(\psi_0) \vdash_{\sigma^\star\Sigma^{\Gamma_0}}^{\mathcal{G}} \mathcal{G}(p(\sigma, \Sigma^{\Gamma_0}))(\varphi_0)$

  Thus indeed:

  $$\{\langle \sigma^\star\Sigma^{\Gamma_i}, \mathcal{G}(p(\sigma, \Sigma^{\Gamma_i}))(\varphi_i)\rangle\}_{i=1}^{n} \vdash_{\Sigma'}^{\mathcal{L}^t(\mathcal{G})} \langle \sigma^\star\Sigma^{\Gamma_0}, \mathcal{G}(p(\sigma, \Sigma^{\Gamma_0}))(\varphi_0)\rangle$$

- For each $\Sigma \in |\mathbf{Sig}^{\mathcal{G}}|$, $\vdash_{\Sigma}^{\mathcal{L}^t(\mathcal{G})}$ admits global instantiation.

  Again, it is enough to prove this for the generating relation. Consider: $\Gamma_i \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{G}}|$, $\varphi_i \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma_i)$, $i = 0, \ldots, n$, such that $\{\langle \Gamma_i, \varphi_i \rangle\}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^t(\mathcal{G})} \langle \Gamma_0, \varphi_0 \rangle$ and such that for some set $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ there is a Σ-context $\Gamma \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{G}}|$ such that $\Gamma \hookrightarrow \Gamma_{i_l}$ for $l = 1, \ldots, k$ and $\Gamma \hookrightarrow \Gamma_0$, and for some formulae $\psi_l \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma)$, $l = 0, \ldots, k$, such that

  - $\varphi_{i_l} \vdash_{\Sigma^{\Gamma_{i_l}}}^{\mathcal{G}} \psi_l$ for $l = 1, \ldots, k$
  - $\{\psi_l\}_{l=1}^{k} \vdash_{\Sigma^{\Gamma}}^{\mathcal{G}} \psi_0$
  - $\psi_0 \vdash_{\Sigma^{\Gamma_0}}^{\mathcal{G}} \varphi_0$

  Let then $\gamma_i : \Gamma_i \to \Gamma'_i$, $i = 0, \ldots, n$ be a compatible family of Σ-context morphisms. By definition, there is a Σ-context morphism $\gamma : \Gamma \to \Gamma'$ such that for $i = 0, \ldots, n$, γ_i is an extension of γ. In particular this implies that for $i = 0, \ldots, n$, $\Gamma' \hookrightarrow \Gamma_i$. Since the consequence relations of $\mathcal{G}$ are preserved under translations induced by signature (and hence Σ-context) morphisms, we have:

  - $\mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma_{i_l})(\varphi_{i_l}) \vdash_{\Sigma^{\Gamma'_{i_l}}}^{\mathcal{G}} \mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma)(\psi_{i_l})$ for $l = 1, \ldots, k$
  - $\{\mathcal{F}^{\mathcal{G}}(\Sigma)(\dot{\gamma})(\psi_{i_l})\}_{l=1}^{k} \vdash_{\Sigma^{\Gamma'}}^{\mathcal{G}} \mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma)(\psi_0)$
  - $\mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma)(\psi_0) \vdash_{\Sigma^{\Gamma'_0}}^{\mathcal{G}} \mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma_0)(\varphi_0)$

Thus, as required

$$\{\langle \Gamma_i', \mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma_i)(\varphi_i)\rangle\}_{i=1}^n \vdash_\Sigma^{\mathcal{L}^t(\mathcal{G})} \langle \Gamma_0', \mathcal{F}^{\mathcal{G}}(\Sigma)(\gamma)(\varphi_0)\rangle.$$

□

The simplest case (and the most typical for considerations on truth consequence relations) is when the context of open formulae used in a deduction sequence is fixed. The closure property embodied in the condition in Def. 4.4 then amounts to the following:

Proposition 4.6 *For any ground logic $\mathcal{G} : \mathbf{Sig}^{\mathcal{G}} \to \mathbf{CR}$, any signature $\Sigma \in |\mathbf{Sig}^{\mathcal{G}}|$, Σ-context $\Gamma \in |\mathbf{Ctxt}_\Sigma^{\mathcal{G}}|$ and formulae $\varphi_i \in \mathcal{F}^{\mathcal{G}}(\Sigma)(\Gamma)$, $i = 0, \dots, n$, if $\{\varphi_i\}_{i=1}^n \vdash_{\Sigma^\Gamma}^{\mathcal{G}} \varphi_0$ then $\{\langle\Gamma, \varphi_i\rangle\}_{i=1}^n \vdash_\Sigma^{\mathcal{L}^t(\mathcal{G})} \langle \Gamma, \varphi_0\rangle$.*

5 Logical systems and institutions

One justification for the definitions we gave in Section 4 may be sought in model theory. The theory of *institutions* (*cf.* [GB84]) provides a framework to study this issue at a sufficiently abstract level.

Definition 5.1 *An* institution $\mathcal{I}$ *consists of:*

- *a category $\mathbf{Sig}^{\mathcal{I}}$ (of signatures);*
- *a functor $\mathbf{Sen}^{\mathcal{I}}:\mathbf{Sig}^{\mathcal{I}} \to \mathbf{Set}$ ($\mathbf{Sen}^{\mathcal{I}}$ gives for any signature Σ the set $\mathbf{Sen}^{\mathcal{I}}(\Sigma)$ of Σ-sentences and for any signature morphism $\sigma:\Sigma \to \Sigma'$ the function $\mathbf{Sen}^{\mathcal{I}}(\sigma):\mathbf{Sen}^{\mathcal{I}}(\Sigma) \to \mathbf{Sen}^{\mathcal{I}}(\Sigma')$ translating Σ-sentences to Σ'-sentences);*
- *a functor $\mathbf{Mod}^{\mathcal{I}}:\mathbf{Sig}^{\mathcal{I}} \to \mathbf{Cat}^{op}$ (where $\mathbf{Cat}$ is the category of all categories; $\mathbf{Mod}^{\mathcal{I}}$ gives for any signature Σ the category $\mathbf{Mod}^{\mathcal{I}}(\Sigma)$ of Σ-models and for any signature morphism $\sigma:\Sigma \to \Sigma'$ the σ-reduct functor $\mathbf{Mod}^{\mathcal{I}}(\sigma):\mathbf{Mod}^{\mathcal{I}}(\Sigma') \to \mathbf{Mod}^{\mathcal{I}}(\Sigma)$ translating Σ'-models to Σ-models); and*
- *a satisfaction relation $\models_{\mathcal{I},\Sigma} \subseteq |\mathbf{Mod}^{\mathcal{I}}(\Sigma)| \times \mathbf{Sen}^{\mathcal{I}}(\Sigma)$ for each signature Σ.*

such that for any signature morphism $\sigma:\Sigma \to \Sigma'$ the translations $\mathbf{Mod}^{\mathcal{I}}(\sigma)$ of models and $\mathbf{Sen}^{\mathcal{I}}(\sigma)$ of sentences preserve the satisfaction relation, i.e. for any $\varphi \in \mathbf{Sen}^{\mathcal{I}}(\Sigma)$ and $M' \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma')|$,

$$M' \models_{\mathcal{I},\Sigma'} \mathbf{Sen}^{\mathcal{I}}(\sigma)(\varphi) \iff \mathbf{Mod}^{\mathcal{I}}(\sigma)(M') \models_{\mathcal{I},\Sigma} \varphi \qquad \textit{(Satisfaction condition)}$$

In the following we will assume in addition that the institutions we consider have categories of signatures *with inclusions* and that the sentence functors $\mathbf{Sen}^{\mathcal{I}}$ preserve inclusions. For any signature morphism $\sigma : \Sigma \to \Sigma'$, the function $\mathbf{Sen}^{\mathcal{I}}(\sigma)$ will be written simply as σ and the reduct functor $\mathbf{Mod}^{\mathcal{I}}(\sigma)$ as $\_|_\sigma$. Moreover, for any signature inclusion $\iota : \Sigma \hookrightarrow \Sigma'$, the reduct functor $\_|_\iota$ will be written as $\_|_\Sigma$.

Definition 5.2 *An institution $\mathcal{I}$ determines a ground logical system*

$$\mathcal{G}(\mathcal{I}) : \mathbf{Sig}^{\mathcal{I}} \to \mathbf{CR}$$

where $|\mathcal{G}(\mathcal{I})| : \mathbf{Sig}^{\mathcal{I}} \to \mathbf{Set}$ is just $\mathbf{Sen}^{\mathcal{I}} : \mathbf{Sig}^{\mathcal{I}} \to \mathbf{Set}$ and for each $\Sigma \in |\mathbf{Sig}^{\mathcal{I}}|$, for $\varphi_i \in \mathbf{Sen}^{\mathcal{I}}(\Sigma)$, $i = 0, \dots, n$,

$\{\varphi_i\}_{i=1}^n \vdash_\Sigma^{\mathcal{G}(\mathcal{I})} \varphi_0$ if and only if for all models $M \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma)|$, $M \models_{\mathcal{I},\Sigma} \varphi_0$ whenever $M \models_{\mathcal{I},\Sigma} \varphi_i$ for $i = 1, \dots, n$.

Proposition 5.3 *For any insitution $\mathcal{I}$, the logical system $\mathcal{G}(\mathcal{I})$ given by Definition 5.2 is indeed a ground logical system.*

Definition 5.4 *Let $\mathcal{I}$ be an institution. By a ground logic* sound for $\mathcal{I}$ *we mean any ground logic $\mathcal{G} : \mathbf{Sig}^{\mathcal{I}} \to \mathbf{CR}$ such that $|\mathcal{G}| = \mathbf{Sen}^{\mathcal{I}}$ and for each $\Sigma \in |\mathbf{Sig}^{\mathcal{I}}|$ and $\varphi_i \in \mathbf{Sen}^{\mathcal{I}}(\Sigma)$, $i = 0, \ldots, n$, if $\{ \varphi_i \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{G}} \varphi_0$ then $\{ \varphi_i \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{G}(\mathcal{I})} \varphi_0$. If the opposite implication holds as well, we say that $\mathcal{G}$ is* complete for $\mathcal{I}$.

The sentence part of any institution $\mathcal{I}$ may be used to determine open formulae in this institution in exactly the same way as in Section 4 for ground logics, giving a functor $\mathcal{F}_{\mathbf{Sen}^{\mathcal{I}}} : \mathbf{Sig}^{\mathcal{I}} \to \mathbf{Func}_{\mathbf{ICat}}(\mathbf{Set})$. To use the model-theoretic satisfaction relation of $\mathcal{I}$ to determine consequence relations on open formulae, we need an "institutional" version of the notion of a valuation. This may be introduced in a rather straightforward way: for any signature $\Sigma \in |\mathbf{Sig}^{\mathcal{I}}|$, Σ-context Γ (a signature extension), and model $M \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma)|$, a *valuation* of context Γ in the model M is any expansion of M to a Σ^{Γ}-model, *i.e.*, a model $M' \in \mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma})$ such that $M'|_{\Sigma} = M$.

Definition 5.5 *Let $\mathcal{I}$ be an arbitrary institution. The* validity logic $\mathcal{L}^{v}(\mathcal{I})$ of open formulae determined by $\mathcal{I}$ *consists of the formula functor $\mathcal{F}_{\mathbf{Sen}^{\mathcal{I}}} : \mathbf{Sig}^{\mathcal{I}} \to \mathbf{Func}_{\mathbf{ICat}}(\mathbf{Set})$ with the consequence relations on open formulas defined as follows: for each signature $\Sigma \in |\mathbf{Sig}^{\mathcal{I}}|$, Σ-contexts Γ_i and formulae $\varphi_i \in \mathcal{F}_{\mathbf{Sen}^{\mathcal{I}}}(\Sigma)(\Gamma_i)$ for $i = 0, \ldots, n$*

$\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^{v}(\mathcal{I})} \langle \Gamma_0, \varphi_0 \rangle$ if and only if for each model $M \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma)|$, $M_0 \models_{\Sigma^{\Gamma_0}} \varphi_0$ for all $M_0 \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma})|$ such that $M_0|_{\Sigma} = M$ whenever for all $i = 1, \ldots, n$, $M_i \models_{\Sigma^{\Gamma_0}} \varphi_i$ for all $M_i \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma})|$ such that $M_i|_{\Sigma} = M$.

Proposition 5.6 *For any institution $\mathcal{I}$, the logical system $\mathcal{L}^{v}(\mathcal{I})$ as defined in Def. 5.5 is indeed a logical system of open formulae of validity type.*

Definition 5.7 *Let $\mathcal{I}$ be an institution and $\mathcal{L}$ be a logic of open formulae. We say that $\mathcal{L}$* is sound for $\mathcal{I}$ under the validity interpretation *if*

- *$\mathcal{L}$ is of validity type*
- *$\mathcal{F}_{\mathcal{L}}$ is $\mathcal{F}_{\mathbf{Sen}^{\mathcal{I}}}$*
- *For all signatures $\Sigma \in |\mathbf{Sig}^{\mathcal{I}}|$, Σ-contexts Γ_i and formulae $\varphi_i \in \mathcal{F}_{\mathbf{Sen}^{\mathcal{I}}}(\Sigma)(\Gamma_i)$, $i = 0, \ldots, n$, if $\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}} \langle \Gamma_0, \varphi_0 \rangle$ then $\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^{v}(\mathcal{I})} \langle \Gamma_0, \varphi_0 \rangle$.*

If the implication opposite to the one in the last condition holds as well, we say that $\mathcal{L}$ is complete for $\mathcal{I}$ under the validity interpretation.

Proposition 5.8 *If a ground logical system $\mathcal{G}$ is sound for an institution $\mathcal{I}$, then $\mathcal{L}^{v}(\mathcal{G})$ is sound for $\mathcal{I}$ under the validity interpretation.*

Proof Consider any $\Sigma \in |\mathbf{Sig}^{\mathcal{I}}|$, Σ-contexts Γ_i and formulae $\varphi_i \in \mathcal{F}_{\mathbf{Sen}^{\mathcal{I}}}(\Sigma)(\Gamma_i)$, $i = 0, \ldots, n$, such that $\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^{v}(\mathcal{G})} \langle \Gamma_0, \varphi_0 \rangle$. Then, for some $\{ i_1, \ldots, i_k \} \subseteq \{ 1, \ldots, n \}$ there exist Σ-context morphisms $\gamma_l : \Gamma_{i_l} \to \Gamma_0$, $l = 1, \ldots, k$, such that $\{ \gamma_l(\varphi_{i_l}) \}_{l=1}^{k} \vdash_{\Sigma^{\Gamma_0}}^{\mathcal{G}} \varphi_0$. The soundness of $\mathcal{G}$ implies that $\{ \mathcal{G}(\gamma_l)(\varphi_{i_l}) \}_{l=1}^{k} \vdash_{\Sigma^{\Gamma_0}}^{\mathcal{G}(\mathcal{I})} \varphi_0$, *i.e.*, for every $M_0 \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma_0})|$, $M_0 \models_{\mathcal{I},\Sigma^{\Gamma_0}} \varphi_0$ whenever $M_0 \models_{\mathcal{I},\Sigma^{\Gamma_0}} \mathbf{Sen}^{\mathcal{I}}(\gamma_l)(\varphi_{i_l})$ for all $l = 1, \ldots, k$.

Consider now $M \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma)|$ such that for all $i = 1, \ldots, n$, for all $M_i \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma_i})|$ such that $M_i|_{\Sigma} = M$, $M_i \models_{\mathcal{I},\Sigma^{\Gamma_i}} \varphi_i$. Let then $M_0 \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma_0})|$ be such that $M_0|_{\Sigma} = M$. Then for $l = 1, \ldots, k$, since $\mathbf{Mod}^{\mathcal{I}}(\gamma_l)(M_0)|_{\Sigma} = M_0|_{\Sigma} = M$, the satisfaction condition implies that $M_0 \models_{\mathcal{I},\Sigma^{\Gamma_0}} \mathbf{Sen}^{\mathcal{I}}(\gamma_l)(\varphi_{i_l})$. Hence, $M_0 \models_{\mathcal{I},\Sigma^{\Gamma_0}} \varphi_0$ as well, and we conclude

$$\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^{n} \vdash_{\Sigma}^{\mathcal{L}^{v}(\mathcal{I})} \langle \Gamma_0, \varphi_0 \rangle$$

□

Completeness, as usual, is much more difficult. In general, $\mathcal{L}^v(\mathcal{G})$ need not be complete for $\mathcal{I}$ under the validity interpretation even if $\mathcal{G}$ is complete for $\mathcal{I}$. We are working on natural conditions on the institution $\mathcal{I}$ that would ensure this to be the case.

We can, however, ensure so-called *weak completeness*: if $\mathcal{G}$ is weakly complete for $\mathcal{I}$, *i.e.* all theorems (consequences of the empty set of premises) of $\mathcal{I}$ are theorems of $\mathcal{G}$, then $\mathcal{L}^v(\mathcal{G})$ is weakly complete for $\mathcal{I}$ under the validity interpretation.

To introduce the logic of open formulae of an institution with the truth interpretation, we need one more technical concept.

Definition 5.9 *Let $\mathcal{I}$ be an institution. Consider $\Sigma \in |\mathbf{Sig}^{\mathcal{I}}|$ and Σ-contexts Γ_i, $i = 0, \ldots, n$. We say that a family of models $\{ M_i \}_{i=0}^n$, $M_i \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma_i})|$ for $i = 0, \ldots, n$, is* compatible *if for all $j, k \in \{ 0, \ldots, n \}$, for all Σ-contexts Γ such that $\Gamma \hookrightarrow \Gamma_j$ and $\Gamma \hookrightarrow \Gamma_k$, $M_j|_{\Sigma^\Gamma} = M_k|_{\Sigma^\Gamma}$.*

Definition 5.10 *Let $\mathcal{I}$ be an arbitrary institution. The* truth logic $\mathcal{L}^t(\mathcal{I})$ of open formulae *determined by $\mathcal{I}$ consists of the formula functor $\mathcal{F}_{\mathbf{Sen}^{\mathcal{I}}} : \mathbf{Sig}^{\mathcal{I}} \to \mathbf{Func}_{\mathbf{ICat}}(\mathbf{Set})$ with the consequence relations on open formulas defined as follows: for each signature $\Sigma \in |\mathbf{Sig}^{\mathcal{I}}|$, Σ-contexts Γ_i and formulae $\varphi_i \in \mathcal{F}_{\mathbf{Sen}^{\mathcal{I}}}(\Sigma)(\Gamma_i)$ for $i = 0, \ldots, n$*

$\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^n \vdash_\Sigma^{\mathcal{L}^t(\mathcal{I})} \langle \Gamma_0, \varphi_0 \rangle$ *if and only if (for each model $M \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma)|$) for every compatible family of models $\{ M_i \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma_i})| \}_{i=0}^n$ (such that $M_i|_\Sigma = M$ for any $i = 0, \ldots, n$) $M_0 \models_{\Sigma^{\Gamma_0}} \varphi_0$ whenever $M_i \models_{\Sigma^{\Gamma_0}} \varphi_i$ for all $i = 1, \ldots, n$.*

OOPS! Unfortunately, in general this is not well-defined, since the relations $\vdash_\Sigma^{\mathcal{L}^t(\mathcal{I})}$ as specified above need not be transitive. The (lack of) existence of valuations for intermediate contexts causes the problem.

Definition 5.11 *An institution $\mathcal{I}$ is* regular *if for any signature Σ, family of Σ-contexts Γ_i, $i = 0, \ldots, n$, and compatible family $\{ M_i \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma_i})| \}_{i=1}^n$ there exists a model $M_0 \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma_0})|$ such that the family $\{ M_i \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma_i})| \}_{i=0}^n$ is compatible.*

In the most typical situations (or more abstractly, under some additional assumptions on the inclusion ordering of the collection of signatures) the requirement of regularity is equivalent to the condition that all reduct functors induced by signature inclusions are surjective. For example, the usual institution of first-order logic *where the structures are assumed to have non-empty carriers* satisfies this requirement. There are, however, numerous natural institutions which are not regular.

Proposition 5.12 *If $\mathcal{I}$ is a regular institution then $\mathcal{L}^t(\mathcal{I})$ as defined in Def. 5.10 is indeed a logical system of open formulae of truth type.*

Proof The only problem is to prove the transitivity of $\vdash_\Sigma^{\mathcal{L}^t(\mathcal{I})}$, which follows in a rather straightforward way when regularity is assumed. □

Definition 5.13 *Let $\mathcal{I}$ be an institution and $\mathcal{L}$ be a logic of open formulae. We say that $\mathcal{L}$* is sound for $\mathcal{I}$ under the truth interpretation *if*

- *$\mathcal{L}$ is of truth type*
- *$\mathcal{F}_\mathcal{L}$ is $\mathcal{F}_{\mathbf{Sen}^{\mathcal{I}}}$*
- *For all signatures $\Sigma \in |\mathbf{Sig}^{\mathcal{I}}|$, Σ-contexts Γ_i and formulae $\varphi_i \in \mathcal{F}_{\mathbf{Sen}^{\mathcal{I}}}(\Sigma)(\Gamma_i)$, $i = 0, \ldots, n$, if $\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^n \vdash_\Sigma^{\mathcal{L}} \langle \Gamma_0, \varphi_0 \rangle$ then $\{ \langle \Gamma_i, \varphi_i \rangle \}_{i=1}^n \vdash_\Sigma^{\mathcal{L}^t(\mathcal{I})} \langle \Gamma_0, \varphi_0 \rangle$.*

If the implication opposite to the one in the last condition holds as well, we say that $\mathcal{L}$ is complete for $\mathcal{I}$ under the truth interpretation.

Proposition 5.14 *If a ground logical system $\mathcal{G}$ is sound for a regular institution $\mathcal{I}$ then $\mathcal{L}^t(\mathcal{G})$ is sound for $\mathcal{I}$ under the truth interpretation.*

Proof Consider any $\Sigma \in |\mathbf{Sig}^{\mathcal{I}}|$, Σ-contexts Γ_i and formulae $\varphi_i \in \mathcal{F}_{\mathbf{Sen}^{\mathcal{I}}}(\Sigma)(\Gamma_i)$, $i = 0, \ldots, n$, such that $\{\langle\Gamma_i, \varphi_i\rangle\}_{i=1}^n \vdash_\Sigma^{\mathcal{L}^t(\mathcal{G})} \langle\Gamma_0, \varphi_0\rangle$. It is enough to prove the soundness for the generating relations of $\vdash_\Sigma^{\mathcal{L}^t(\mathcal{G})}$. So, we can assume that for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ there is a Σ-context Γ such that $\Gamma \hookrightarrow \Gamma_{i_l}$ for $l = 1, \ldots, k$ and $\Gamma \hookrightarrow \Gamma_0$, and there exist formulae $\psi_l \in \mathbf{Sen}^{\mathcal{I}}(\Sigma^\Gamma)$, $l = 0, \ldots, k$, such that

- $\varphi_{i_l} \vdash^{\mathcal{G}}_{\Sigma^{\Gamma_{i_l}}} \psi_l$ for $l = 1, \ldots, k$
- $\{\psi_l\}_{l=1}^k \vdash^{\mathcal{G}}_{\Sigma^\Gamma} \psi_0$
- $\psi_0 \vdash^{\mathcal{G}}_{\Sigma^{\Gamma_0}} \varphi_0$

Since $\mathcal{G}$ is sound for $\mathcal{I}$, we have

- $\varphi_{i_l} \vdash^{\mathcal{G}(\mathcal{I})}_{\Sigma^{\Gamma_{i_l}}} \psi_l$ for $l = 1, \ldots, k$
- $\{\psi_l\}_{l=1}^k \vdash^{\mathcal{G}(\mathcal{I})}_{\Sigma^\Gamma} \psi_0$
- $\psi_0 \vdash^{\mathcal{G}(\mathcal{I})}_{\Sigma^{\Gamma_0}} \varphi_0$

Consider now any compatible family of models $\{M_i \in |\mathbf{Mod}^{\mathcal{I}}(\Sigma^{\Gamma_i})|\}_{i=0}^n$ such that for $i = 1, \ldots, n$, $M_i \models_{\mathcal{I},\Sigma^{\Gamma_i}} \varphi_i$. Then for $l = 1, \ldots, k$, $M_{i_l} \models_{\mathcal{I},\Sigma^{\Gamma_{i_l}}} \psi_l$ and hence by the satisfaction condition $M_{i_l}|_{\Sigma^\Gamma} \models_{\mathcal{I},\Sigma^\Gamma} \psi_l$. This implies (by the compatibility of the family $\{M_i\}_{i=0}^n$) that $M_0|_{\Sigma^\Gamma} \models_{\mathcal{I},\Sigma^\Gamma} \psi_0$ and hence, again by the satisfaction condition, $M_0 \models_{\mathcal{I},\Sigma^{\Gamma_0}} \psi_0$. Thus, $M_0 \models_{\mathcal{I},\Sigma^{\Gamma_0}} \varphi_0$, which proves $\{\langle\Gamma_i, \varphi_i\rangle\}_{i=1}^n \vdash_\Sigma^{\mathcal{L}^t(\mathcal{I})} \langle\Gamma_0, \varphi_0\rangle$ □

In general, completeness is not preserved in the case of the truth interpretation either. However, if $\mathcal{G}$ is complete for $\mathcal{I}$, then the logic $\mathcal{L}^t(\mathcal{G})$ is complete for $\mathcal{I}$ under the truth interpretation relative to the restriction of the consequence relations to formulae with the same context.

6 Logical systems and LF

In order to discuss representations of logical systems in LF, we first recall from [HST89] the logical system associated with the LF type theory. The basic form of assertion in this logic is that a closed type is inhabited. We then remove the restriction to closed types and define two logical systems of open formulae determined by the LF type theory, one of validity type, the other of truth type.

Definition 6.1 *An* LF signature morphism $\sigma : \Sigma \to \Sigma'$ *is a function σ mapping constants to closed terms such that if $c{:}A$ ($c{:}K$) occurs in Σ, then $\vdash_{\Sigma'} \sigma(c) : \sigma^\sharp A$ ($\vdash_{\Sigma'} \sigma(c) : \sigma^\sharp K$). (The function $\sigma^\sharp$ is the natural extension of σ to LF terms.) $\mathbf{Sig}^{\mathcal{LF}}$ is the category with inclusions of LF signatures and LF-signature morphisms, with composition defined in the obvious way.*

Proposition 6.2 *If $\sigma : \Sigma \to \Sigma'$ and $\vdash_\Sigma \alpha$, then $\vdash_{\Sigma'} \sigma^\sharp\alpha$ for each assertion α of the LF type system.*

Definition 6.3 *Let Σ be an LF signature. $\mathcal{GLF}(\Sigma)$ is the pair $\langle \mathrm{Types}_\Sigma, \vdash_\Sigma^{\mathcal{LF}}\rangle$ where $\mathrm{Types}_\Sigma = \{A \mid \vdash_\Sigma A : \mathsf{Type}\}$ and*

$$A_1, \ldots, A_n \vdash_\Sigma^{\mathcal{LF}} A \quad \textit{iff} \quad x_1{:}A_1, \ldots, x_n{:}A_n \vdash_\Sigma M : A$$

for some M and any pairwise distinct variables $x_1, \ldots, x_n$.

This consequence relation has a straightforward Gentzen-style axiomatization similar to that used in NuPRL [Con86].

Definition 6.4 *The* ground logic of LF *is a functor* $\mathcal{GLF} : \mathbf{Sig}^{\mathcal{LF}} \to \mathbf{CR}$ *which is the extension of the map* $\Sigma \mapsto \mathcal{GLF}(\Sigma)$ *defined by taking* $\mathcal{GLF}(\sigma)$, *for* $\sigma : \Sigma \to \Sigma'$, *to be* $\sigma^\sharp \restriction \mathrm{Types}_\Sigma$, *the restriction of* $\sigma^\sharp$ *to closed* Σ*-types.*

Logical systems of open formulae are determined by the LF type theory in much the same style. We have already defined the category of signatures. Contexts are just LF contexts with substitutions of terms for variables as morphisms:

Definition 6.5 *For any LF signature* $\Sigma \in |\mathbf{Sig}^{\mathcal{LF}}|$ *and any* Σ*-contexts* Γ *and* Γ', *an* LF context morphism $\gamma : \Gamma \to \Gamma'$ *is a function* γ *mapping variables to terms such that for each declaration* $x : A$ *in* Γ, $\Gamma' \vdash_\Sigma \gamma(x){:}\gamma^\sharp A$. *(The function* $\gamma^\sharp$ *is the obvious extension of* γ *to LF terms.) The category* $\mathbf{Ctxt}_\Sigma^{\mathcal{LF}}$ *of LF* Σ*-contexts and their morphisms is defined in a straightforward way.*

Open formulae of the systems we define are just LF types:

Definition 6.6 *For any LF signature* $\Sigma \in |\mathbf{Sig}^{\mathcal{LF}}|$ *and LF context* $\Gamma \in |\mathbf{Ctxt}_\Sigma^{\mathcal{LF}}|$, Σ*-formulae in context* Γ *are LF types formed over signature* Σ *in context* Γ,

$$\mathcal{F}_{\mathcal{LF}}(\Sigma)(\Gamma) = \{\, A \mid \Gamma \vdash_\Sigma A{:}\mathsf{Type}\,\}$$

Definition 6.7 *The* $\mathcal{LF}$ *formula functor* $\mathcal{F}_{\mathcal{LF}} : \mathbf{Sig}^{\mathcal{LF}} \to \mathbf{Func}_{\mathbf{ICat}}(\mathbf{Set})$ *is the obvious extension of the mappings defined in Definitions 6.5 and 6.6 to a functor: for any signature morphisms* σ, $\mathcal{F}_{\mathcal{LF}}(\sigma)$ *is essentially given by the natural translation of LF terms induced by* σ; *for any signature* Σ *and a* Σ*-context morphism* γ, $\mathcal{F}_{\mathcal{LF}}(\Sigma)(\gamma)$ *is again the natural translation of LF terms induced by* γ.

Proposition 6.8 *There is an obvious, componentwise inclusion morphism*

$$S_{\mathcal{LF}} : \mathcal{F}_{\mathcal{LF}} \to \mathcal{F}_{\mathcal{GLF}}$$

where $\mathcal{F}_{\mathcal{GLF}}$ *is the formula functor determined by the ground logical system of LF,* $\mathcal{GLF}$, *as in Section 4. The inclusions are proper at the context level: LF contexts used in* $\mathcal{F}_{\mathcal{LF}}$ *are extensions of signatures by object constants (constants of a type), which excludes extensions by type constants (constants of a kind).*

Definition 6.9 *The* validity LF logic $\mathcal{LF}^v$ *is the logic of open formulae with the formula functor* $\mathcal{F}_{\mathcal{LF}} : \mathbf{Sig}^{\mathcal{LF}} \to \mathbf{Func}_{\mathbf{ICat}}(\mathbf{Set})$ *and, for each LF signature* $\Sigma \in |\mathbf{Sig}^{\mathcal{LF}}|$, *with the consequnce relation on* $Form_{\mathcal{F}_{\mathcal{LF}}}(\Sigma)$ *given as follows: for any* Σ *contexts* Γ_i *and* A_i *such that* $\Gamma_i \vdash_\Sigma A_i{:}\mathsf{Type}$, *for* $i = 0, \ldots, n$,

$$\{\, \langle \Gamma_i, A_i \rangle \,\}_{i=1}^{n} \vdash_\Sigma^{\mathcal{LF}^v} \langle \Gamma_0, A_0 \rangle$$

if and only if

$$x_1{:}\Pi\Gamma_1.A_1, \ldots, x_n{:}\Pi\Gamma_n.A_n \vdash_\Sigma M{:}\Pi\Gamma_0.A_0$$

for some mutually distinct variables $x_1, \ldots, x_n$ *and term* M. *(We use the informal notation* $\Pi\Gamma_i.A_i$ *for the type obtained by* Π*-closure of* A_i *w.r.t. the variables in the context* Γ_i.*)*

Proposition 6.10 $\mathcal{LF}^v$ *as defined in Def. 6.9 is indeed a logical system of open formulae of the validity type.*

Definition 6.11 *The* truth LF logic $\mathcal{LF}^t$ *is the logic of open formulae with the formula functor* $\mathcal{F}_{\mathcal{LF}} : \mathbf{Sig}^{\mathcal{LF}} \to \mathbf{Func}_{\mathbf{ICat}}(\mathbf{Set})$ *and, for each LF signature* $\Sigma \in |\mathbf{Sig}^{\mathcal{LF}}|$, *with the consequnce relation on* $Form_{\mathcal{F}_{\mathcal{LF}}}(\Sigma)$ *defined as the transitive closure of the relation given as follows: for any* Σ*-contexts* Γ_i *and types* A_i *such that* $\Gamma_i \vdash_\Sigma A_i{:}\mathsf{Type}$ *for* $i = 0, \ldots, n$,

$\{\, \langle \Gamma_i, A_i \rangle \,\}_{i=1}^{n} \vdash_\Sigma^{\mathcal{LF}^t} \langle \Gamma_0, A_0 \rangle$ *if and only if for some* $\{\, i_1, \ldots, i_k \,\} \subseteq \{\, 1, \ldots, n \,\}$, *there exists a* Σ*-context* Γ *and types* A'_l, $l = 1, \ldots, k$, *such that for* $l = 1, \ldots, k$, $\Gamma \vdash_\Sigma A'_l{:}\mathsf{Type}$ *and moreover*

- $\Gamma_{i_l}, x{:}A_{i_l} \vdash_\Sigma M_l{:}A'_l$ *for* $l = 1, \ldots, k$, *for some variable* x *and term* M_l
- $\Gamma, x_1{:}A'_1, \ldots, x_k{:}A'_k \vdash_\Sigma M{:}A'_0$ *for some mutually distinct variables* $x_1, \ldots, x_k$ *and term* M
- $\Gamma_0, x{:}A'_0 \vdash_\Sigma M_0{:}A_0$

Remark 6.12 *If all contexts in Def. 6.11 are the same, say* $\Gamma_i = \Gamma$ *for* $i = 0, \ldots, n$, *then the condition on the generating relation given in the definition is equivalent to the requirement that for some mutually distinct variables* $x_1, \ldots, x_n$ *and term* N,

$$\Gamma, x_1{:}A_1, \ldots, x_n{:}A_n \vdash_\Sigma N{:}A_0.$$

Moreover, the relation defined in this way is already transitive on the formulae built in the same context.

Proposition 6.13 $\mathcal{LF}^t$ *as defined in Def. 6.11 is indeed a logical system of open formulae of truth type.*

Proposition 6.14 *The inclusion morphism* $S_{\mathcal{LF}} : \mathcal{F}_{\mathcal{LF}} \to \mathcal{F}_{\mathcal{GLF}}$ *is a logic morphism*

$$S_{\mathcal{LF}} : \mathcal{LF}^t \to \mathcal{L}^t(\mathcal{GLF})$$

A reasonable question at this point is whether $S_{\mathcal{LF}}$ is a logic representation between $\mathcal{LF}^t$ and $\mathcal{L}^t(\mathcal{GLF})$ (and/or between $\mathcal{LF}^v$ and $\mathcal{L}^v(\mathcal{GLF})$). Unfortunately, this is not the case in general; in most cases it seems that $\mathcal{L}^t(\mathcal{GLF})$ (resp. $\mathcal{L}^v(\mathcal{GLF})$) is somewhat too weak. This topic needs further study.

For the purposes of encoding a logical system $\mathcal{L}$, we are interested in "specializations" of the logical system determined by the LF type theory obtained by fixing a "base" signature $\Sigma_{\mathcal{L}}$ specifying the syntax, assertions, and rules of $\mathcal{L}$ [HHP87]. The signatures of $\mathcal{L}$ are then represented as extensions to $\Sigma_{\mathcal{L}}$, and signature morphisms are represented as LF signature morphisms on these extensions leaving $\Sigma_{\mathcal{L}}$ fixed.

Definition 6.15 *Let* Σ *be an LF signature. The* category of extensions *of* Σ, *written* $\mathbf{Sig}^{\mathcal{LF}}_\Sigma$, *is the full subcategory of the slice category* $\Sigma{\downarrow}\mathbf{Sig}^{\mathcal{LF}}$ *determined by the inclusions* $\iota : \Sigma \hookrightarrow \Sigma'$.

Every LF signature induces logical systems based on that signature as follows:

Definition 6.16 *Let* Σ *be an LF signature.*

- *The* validity logical system presented by Σ, $\mathcal{LF}^v_\Sigma$, *is the restriction of* $\mathcal{LF}^v$ *to* $\mathbf{Sig}^{\mathcal{LF}}_\Sigma$.
- *The* truth logical system presented by Σ, $\mathcal{LF}^t_\Sigma$, *is the restriction of* $\mathcal{LF}^t$ *to* $\mathbf{Sig}^{\mathcal{LF}}_\Sigma$.

An encoding of a logical system $\mathcal{L}$ in LF consists not only of an LF signature $\Sigma_{\mathcal{L}}$, but also of an "internal type family" distinguishing the basic judgements of $\mathcal{L}$ in the encoding [HST89]. For example, in the encoding of first-order logic given in [HHP87], the constant *true* of kind $o \to \mathsf{Type}$ represents the basic judgement form of first-order logic. The significance of *true* for the encoding becomes apparent in the statement of the adequacy theorem: terms of type $true(\hat{\varphi})$ in a context with variables x_i of type $true(\hat{\varphi}_i)$ represent proofs of φ from the φ_i's. Similarly, we also indicate which LF contexts are used to represent $\mathcal{L}$-contexts. In the encoding of first-order logic given in [HHP87], first-order contexts are represented by LF contexts with variables of type ι (a distinguished type of individuals). This methodology is formalized in our setting as follows.

Definition 6.17 *An* internal type family of Σ *is a term F such that* $\vdash_\Sigma F : K$ *for some kind K. (Note that if* $\vdash_\Sigma K$*, then K has normal form* $\Pi x_1{:}A_1.\ldots.\Pi x_k{:}A_k.\mathsf{Type}$ *for some* $x_1, \ldots, x_k$ *and* $A_1, \ldots, A_k$*.) The* range *of an internal type family F of Σ in a Σ-context Γ is defined to be the set*

$$\mathrm{Rng}^{\Gamma}_{\Sigma}(F) = \{ F\, M_1 \ldots M_k \mid \Gamma \vdash_\Sigma F\, M_1 \ldots M_k : \mathsf{Type} \},$$

(where terms are identified up to $\beta\eta$-conversion.) If $\mathcal{J}$ is a set of internal type families of Σ, then

$$\mathrm{Rng}^{\Gamma}_{\Sigma}(\mathcal{J}) = \bigcup_{F \in \mathcal{J}} \mathrm{Rng}^{\Gamma}_{\Sigma}(F).$$

Definition 6.18 *A* logic presentation *is a triple $\langle \Sigma, \mathcal{T}, \mathcal{J} \rangle$ where Σ is an LF signature and $\mathcal{T}$ and $\mathcal{J}$ are finite sets of internal type families of Σ.*

Definition 6.19 *Let $\langle \Sigma, \mathcal{T}, \mathcal{J} \rangle$ be a logic presentation.*

The validity (truth, respectively) logical system presented by $\langle \Sigma, \mathcal{T}, \mathcal{J} \rangle$, $\mathcal{P}^v(\Sigma, \mathcal{T}, \mathcal{J})$ *($\mathcal{P}^t(\Sigma, \mathcal{T}, \mathcal{J})$, respectively) is the restriction of $\mathcal{LF}^v_\Sigma$ ($\mathcal{LF}^t_\Sigma$, respectively):*

- *to signatures and signature morphisms in* $\mathbf{Sig}^{\mathcal{LF}}_{\Sigma}$,
- *for each signature $\Sigma' \in |\mathbf{Sig}^{\mathcal{LF}}_{\Sigma}|$, to Σ'-contexts of the form $\langle x_1{:}A_1, \ldots, x_n{:}A_n \rangle$ where $A_i \in \mathrm{Rng}^{\langle x_1:A_1,\ldots,x_{i-1}:A_{i-1}\rangle}_{\Sigma'}(\mathcal{T})$ for $i = 1, \ldots, n$,*
- *for each signature $\Sigma' \in |\mathbf{Sig}^{\mathcal{LF}}_{\Sigma}|$ and Σ'-context Γ (satisfying the above requirement), to the formulae that are types in $\mathrm{Rng}^{\Gamma}_{\Sigma}(\mathcal{J})$.*

Definition 6.20 *A logical system is* uniformly validity-encodable *(*uniformly truth-encodable*, respectively) in LF if there exists a logic presentation $\langle \Sigma_\mathcal{L}, \mathcal{T}_\mathcal{L}, \mathcal{J}_\mathcal{L} \rangle$ and a surjective exact representation $\rho_\mathcal{L} : \mathcal{L} \to \mathcal{P}^v(\Sigma_\mathcal{L}, \mathcal{J}_\mathcal{L})$ ($\rho_\mathcal{L} : \mathcal{L} \to \mathcal{P}^t(\Sigma_\mathcal{L}, \mathcal{J}_\mathcal{L})$, respectively). The tuple $\langle \Sigma_\mathcal{L}, \mathcal{J}_\mathcal{L}, \rho_\mathcal{L} \rangle$ with an indication of the type of the system $\mathcal{L}$ is called a* uniform encoding *of $\mathcal{L}$ in LF.*

The word "uniform" reflects the fact that we require a "natural" (or "compositional") encoding of the entire family of consequence relations of $\mathcal{L}$ in LF, rather than a signature-by-signature encoding as is suggested by the account in [HHP87]. The requirement of exactness ensures that $\mathcal{T}$ accurately describes the images of $\mathcal{L}$-contexts in LF. The requirement of surjectivity ensures that $\mathcal{J}$ accurately describes the images of $\mathcal{L}$-sentences in LF. For example, in the encoding of first-order logic in [HHP87], only proofs of true(M) in contexts with variables of type ι (in addition to those labelling assumptions) are considered, for otherwise a complete correspondence with first-order logic cannot in general be expected.

All the methodological consequences of the notions presented above, as described in [HST89] for ground logical systems, carry over to the present framework of logical systems of open formulae, their presentations and encodings in LF. We refer to that paper, as well as to [HHP87] and [AHM87] for examples of logic encodings in LF that may be readily adapted to the framework we have introduced here.

The following technicalities indicate that the ideas on presenting logics in a structured way using the pushout construction suggested in [HST89] carry over as well.

Definition 6.21 *A* logic presentation morphism $\sigma : \langle \Sigma, \mathcal{T}, \mathcal{J} \rangle \to \langle \Sigma', \mathcal{T}'\mathcal{J}' \rangle$ *is a signature morphism $\sigma : \Sigma \to \Sigma'$ in $\mathbf{Sig}^{\mathcal{LF}}$ such that for every $F \in \mathcal{T}$ ($F \in \mathcal{J}$, respectively) with*

$$\vdash_\Sigma F : \Pi x_1{:}A_1.\ldots.x_k{:}A_k.\mathsf{Type},$$

there exists $F' \in \mathcal{T}'$ ($F' \in \mathcal{J}'$, respectively) such that

$$\sigma^\sharp F =_{\beta\eta} \lambda x_1{:}\sigma^\sharp A_1.\ldots.x_k{:}\sigma^\sharp A_k.F'(M_1, \ldots, M_n)$$

for some $M_1, \ldots, M_n$ of suitable type. **LogPres** *is the category of logic presentations and logic presentation morphisms.*

Proposition 6.22 *The assignments $\langle \Sigma, \mathcal{T}, \mathcal{J} \rangle \mapsto \mathcal{P}^v(\Sigma, \mathcal{T}, \mathcal{J})$ and $\langle \Sigma, \mathcal{T}, \mathcal{J} \rangle \mapsto \mathcal{P}^t(\Sigma, \mathcal{T}, \mathcal{J})$ extends to functors $\mathcal{P}^v$: **LogPres** $\to$ **Log** and $\mathcal{P}^t$: **LogPres** $\to$ **Log**, respectively.*

Sketch of construction Consider a presentation morphism $\sigma : \langle \Sigma_1, \mathcal{T}_1, \mathcal{J}_1 \rangle \to \langle \Sigma_2, \mathcal{T}_2, \mathcal{J}_2 \rangle$. The logic morphism $\mathcal{P}^{v(t)}(\sigma) : \mathcal{P}^{v(t)}(\Sigma_1, \mathcal{T}_2, \mathcal{J}_1) \to \mathcal{P}^{v(t)}(\Sigma_2, \mathcal{T}_2, \mathcal{J}_2)$ may be defined as follows:

- $\mathcal{P}^{v(t)}(\sigma)^{Sig} : \mathbf{Sig}^{\mathcal{LF}}_{\Sigma_1} \to \mathbf{Sig}^{\mathcal{LF}}_{\Sigma_2}$ is defined on objects using the canonical pushout construction: $\mathcal{P}^{v(t)}(\sigma)^{Sig}(\iota_1 : \Sigma_1 \hookrightarrow \Sigma_1') = (\iota_2 : \Sigma_2 \hookrightarrow \Sigma_2')$ where $\Sigma_2' = \sigma^\star\Sigma_1'$ and $\iota_2 : \Sigma_2 \hookrightarrow \sigma^\star\Sigma_1'$ is the inclusion morphism to the pushout of σ and ι_1 in $\mathbf{Sig}^{\mathcal{LF}}$. This extends to morphisms using the co-universal property of pushouts.
- $\sigma' : \Sigma_1' \to \Sigma_2'$ in the construction above induces the translation $(\sigma')^\sharp$ of LF terms over Σ_1' to LF terms over Σ_2' and of Σ_1'-contexts to Σ_2'-contexts. Moreover, for any Σ_1'-context Γ_1', $(\sigma')^\sharp : \mathrm{Rng}^{\Gamma_1'}_{\Sigma_1'}(\mathcal{T}_1) \to \mathrm{Rng}^{(\sigma')^\sharp(\Gamma_1')}_{\Sigma_2'}(\mathcal{T}_2)$ and similarly $(\sigma')^\sharp : \mathrm{Rng}^{\Gamma_1'}_{\Sigma_1'}(\mathcal{J}_1) \to \mathrm{Rng}^{(\sigma')^\sharp(\Gamma_1')}_{\Sigma_2'}(\mathcal{J}_2)$. (This uses the fact that σ is a logic presentation morphism.) It is easy to see that this translation preserves consequence relations as required.

We propose to use colimits in the category of logic presentations to build logics in a structured fashion. Although the category of logic presentations is not finitely co-complete, it may be shown that a diagram in **LogPres** has a colimit iff its projection to $\mathbf{Sig}^{\mathcal{LF}}$ has a colimit. The most pertinent case is that of pushouts along inclusions:

Proposition 6.23 **LogPres** *is a category with inclusions, where a logic presentation morphism $\iota : \langle \Sigma, \mathcal{T}, \mathcal{J} \rangle \hookrightarrow \langle \Sigma', \mathcal{T}', \mathcal{J}' \rangle$ is an* inclusion *if $\iota : \Sigma \hookrightarrow \Sigma'$ is an inclusion, $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{J} \subseteq \mathcal{J}'$. In particular,* **LogPres** *has pushouts along inclusions.*

A **LogPres** inclusion can be seen as a parameterized logic presentation where the pushout of this morphism with a "fitting" morphism amounts to instantiation, by analogy with parameterized structured theory presentations. Small examples of this are given in [HST89] for ground logical systems, and may be generalized to the framework of logics of open formulae we present here. Let us just stress here once again that the category **LogPres** of logic presentations, not the category **Log** of logics, seems appropriate for "putting logics together"

7 Directions for further research

The paper presents only a sketch of some of the technicalities necessary to adequately grasp the notion of a logical system of open formulae and of a representation of such systems in a universal logical framework like LF.

An obvious technical gap in the presentation flow of this paper may be found in Section 5 where we try to connect a formal construction on logics with model theory as given by the theory of institutions. Clearly, the issue of (in)completeness of the construction needs more study. A less evident but equally important problem is how to understand (introduce?) a notion of logic encoding in LF via model theory of the encoded logic on one hand and of LF on the other. It seems to us at the moment that there may be some intrinsic difficulties there, as the model theory and proof theory offer inherently different views of logical systems.

A closely related point is to study situations in which a validity-type logical system $\mathcal{L}$ may be viewed as $\mathcal{L}^v(\mathcal{G})$ for the associated ground logical system $\mathcal{G}$ obtained by restricting $\mathcal{L}$ to closed sentences (and similarly for truth-type logical systems). It seems that in most cases the two constructions given in given in Section 4 do not yield the original logical system, but rather a somewhat weaker logic of the appropriate type. In particular, the validity logic of LF cannot be characterized as the validity-type extension of the ground LF logic to open types (due to the presence of binding operators).

Problems with general truth-type logical systems as presented here (the natural truth-type "consequence relations" are not transitive, *cf.* Sections 4 and 5) indicate that perhaps we should adopt a different formalisation, where a "truth context" is fixed throughout a deduction process, rather than being attached to individual formulae (as in this paper, and in the most straightforward view of first-order logic with open formulae, where a reasonable effect like the transitivity of the consequence relation is only due to the implicit assumption that structure carriers are never empty). Consequently, in truth-type logics consequence relations would be defined separately not only for each signature but also for each context over any signature as well. This would also allow us to combine the two types of logical systems (validity and truth) by considering a notion of a logic where for each signature Σ and for each context Γ over Σ, we would have a validity-type consequence relation on formulae built in the (truth) context Γ extended by a (validity) context which is explicitly indicated in the formula.

Part of the motivation for studying open formulae was to enable an adequate treatment of axiom schemes. We believe that the framework presented (or its alternative version mentioned in the previous paragraph) provides an appropriate basis for such treatment — but this remains to be investigated in detail. Finally, the issues of structured logic presentations need further study. Although the definitions in this paper provide the possibility of presenting a greater variety of logics than those in [HST89], concrete examples which exploit this increased flexibility have not yet been worked out.

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