A Domain-Theoretic Banach-Alaoglu Theorem

Gordon Plotkin ⋆

School of Informatics, University of Edinburgh, King's Buildings,
Edinburgh EH9 3JZ, UK.

Abstract. We give a domain-theoretic analogue of the classical Banach-Alaoglu theorem, showing that the patch topology on the weak* topology is compact. Various theorems follow concerning the stable compactness of spaces of valuations on a topological space. We conclude with reformulations of the patch topology in terms of polar sets or Minkowski functionals, showing, in particular, that the 'sandwich set' of linear functionals is compact.

1 Introduction

One of Klaus Keimel's many mathematical interests is the interaction between order theory and functional analysis. In recent years this has led to the beginnings of a 'domain-theoretic functional analysis,' which may be considered to be a topic within 'positive analysis' in the sense of Jimmie Lawson [11]. In the latter, 'notions of positivity and order play a key rôle,' as do lower semicontinuity and (so) T_0 spaces. The present paper contributes a domain-theoretic analogue of the classical Banach-Alaoglu theorem for continuous d-cones, that is, domains endowed with a compatible cone structure [20].

We begin with some historical remarks to set the present work in context. There have been quite extensive developments within functional analysis concerning positivity and order. The topics investigated include lattice-ordered vector spaces, also called Riesz Spaces [12], Banach lattices [16], and, more generally, ordered vector spaces and positive operators; there have also been developments where vector spaces were replaced by ordered cones [3]. However in these contexts the topologies considered were always Hausdorff.

In the early 80s Keimel became interested in the work of Boboc, Bucur and Cornea on axiomatic potential theory [2]. A student of his, Matthias Rauch, considered their work from the viewpoint of domain theory [13], showing, among other things, that a special class of their ordered cones, the standard H-cones, can be viewed as continuous lattice-ordered d-cones, with addition and scalar multiplication being Lawson continuous. Next, starting in the late 80s, Keimel worked on ordered cones with Walter Roth, with a monograph appearing in 1992 [8]. 'Convex' quasi-uniform structures on cones arose there, replacing the standard uniform structure on locally convex topological vector spaces; these

⋆ This work was done with the support of EPSRC grant GR/S86372/01 and of a Royal Society-Wolfson Research Merit Award.

quasi-uniform structures subsume order and topology. This may be the first time that non-Hausdorff topologies were considered in functional analysis.

Roth wrote several papers in this area including his 2000 paper [14] on Hahn-Banach-type theorems for locally convex cones. Later, in her 1999 Ph.D. thesis [18], Keimel's student Regina Tix gave a domain-theoretic version of these theorems in the framework of d-cones, where the order is now that of a dcpo (directed complete partial order), and see also [19]. These Hahn-Banach theorems include sandwich-type theorems, separation theorems and extension theorems. Plotkin subsequently gave another separation theorem, which was incorporated, together with other improvements, into a revised version of Tix's thesis [20].

The present paper can be seen as providing another contribution of that kind. The classical Banach-Alaoglu Theorem states that in a topological vector space the polar of a neighbourhood of zero is weak*-compact [15]. We give an analogue for continuous d-cones. We have a certain advantage in that the range of our functionals, the non-negative reals extended with a point at infinity, is Lawson compact. It turns out that, under an appropriate assumption, an entire topology is compact: the patch topology on the weak*-upper topology of the dual space of the cone. It follows that various kinds of polar sets are weak*-compact.

The work on Hahn-Banach-type theorems has found application in theoretical computer science, viz the study of powerdomains. In her thesis Tix considered powerdomains for combinations of ordinary and probabilistic nondeterminism; more precisely she combined each of the three classical powerdomains for nondeterminism (lower, upper and convex) with the powerdomain of all valuations. It was a pleasant surprise that the separation theorems found application in this development and we anticipate that so too will the domain-theoretic Banach-Alaoglu theorem given here.

We take [4] as a standard reference on domain theory and related topology; we refer the reader particularly to the material on stably compact spaces, and also to [6,1] for more recent material on that topic where it is argued that stably compact spaces are the correct T_0 analogue of compact Hausdorff spaces. The needed background on d-cones can be found in Chapter 2 of [20]. We cover it much more briefly here in Section 2 which concerns technical preliminaries. We derive our domain-theoretic Banach-Alaoglu theorem in Section 3 and then discuss some reformulations of the weak*-upper topology and its dual in terms of polar sets and functional bounds in Section 4.

2 Technical preliminaries

We are concerned with semimodules for two (unitary) semirings: $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$, where by a semimodule we mean a module for a semiring, see [5]. The first semiring is that of the non-negative reals with the usual addition and multiplication; the second extends the first with an infinite element and the extensions of the semiring operations with $\infty + x = \infty$, $\infty \cdot 0 = 0$, and $\infty \cdot x = \infty$, if $x \neq 0$. Then a *d-cone* is an $\overline{\mathbb{R}}_+$-semimodule in the category of dcpos, where $\overline{\mathbb{R}}_+$ is endowed with the usual ordering with least element 0, making it a continuous lattice; an *ordered cone* is an $\mathbb{R}_+$-semimodule in the category of posets, endowing $\mathbb{R}_+$ with the usual ordering; and a *topological cone* is an $\mathbb{R}_+$-semimodule in the category of topological spaces, endowing $\mathbb{R}_+$ with the upper topology. Our definition of a d-cone differs inessentially from that in [20] where the infinite element is avoided.

In any cone the action of the semiring induces an action of the multiplicative group $(0,\infty)$, and therefore all such actions are d-cone automorphisms, and so, in particular, automorphisms of the way below relation on d-cones. Further, 0 is always the least element, taking the specialisation ordering in the topological case.

A function between semimodules is *homogeneous* if it preserves the semiring action, *additive* if it preserves semimodule addition, and *linear* if it preserves both. In case the semimodule is preordered, such a function f is *subadditive* if $f(x+y) \leq f(x)+f(y)$ always holds, *superadditive* if $f(x+y) \geq f(x)+f(y)$ always holds, *sublinear* if it is homogeneous and subadditive, and *superlinear* if it is homogeneous and superadditive. We may sometimes mention the semiring at hand if it is not clear which we mean, writing, for example, '$\mathbb{R}_+$-homogeneous.'

A *functional* on a set X is simply a function on X with range $\overline{\mathbb{R}}_+$. Given a collection $\mathcal{F}$ of such functionals on a set X and a topology on $\overline{\mathbb{R}}_+$, the corresponding *weak** topology on $\mathcal{F}$ is the weakest topology making all point evaluation functions $\mathrm{ev}_x : f \mapsto f(x)$ continuous: so we speak of the *weak***-upper*, or *weak***-Scott*, the *weak***-lower* and the *weak***-Lawson* topologies on $\mathcal{F}$. The weak*-upper topology has as a subbasis the sets:

$$W_{x,r} =_{\mathrm{def}} \{f \in \mathcal{F} \mid f(x) > r\}$$

where $r \in (0,\infty)$; the weak*-lower topology has as a subbasis the sets:

$$L_{x,r} =_{\mathrm{def}} \{f \in \mathcal{F} \mid f(x) < r\}$$

where $r \in (0,\infty)$; and the weak*-Lawson topology is the join of the other two weak* topologies. The weak*-lower topology is always a separating dual topology for the weak*-upper topology (see [20], Definition VI-6.17), and its specialisation ordering is the pointwise one.

We will be particularly interested in C^* the collection of all linear functionals on a cone C, taking these to be continuous or monotone as appropriate to the kind of cone considered. One can endow C^* with a cone structure, when it is called the *dual* cone: the operations are defined pointwise, and then, taking the pointwise order, we have notions of dual cone for the dcpo and poset case, and, taking the weak*-upper topology, gives one for topological cones.

Two examples of d-cones are $\mathcal{L}(X)$, the collection of continuous functionals on a topological space X, taking the upper topology on $\overline{\mathbb{R}}_+$, with the pointwise ordering and $\mathcal{V}(X)$ the collection of continuous valuations on X, again with the pointwise ordering. Their properties are treated in detail in Chapter 2 of [20]; we note here a 'Riesz Representation Theorem,' that $\Lambda : \mathcal{V}(X) \cong \mathcal{L}(X)^*$ is a d-cone isomorphism, where $\Lambda_\nu = f \mapsto \int f d\nu$.

Both d-cones and topological cones whose underlying topology is T_0 yield ordered cones, taking the the underlying order and the specialisation order, respectively. A continuous d-cone, i.e., one whose underlying dcpo is continuous, yields a topological cone, taking the Scott topology: the point is that addition is then continuous in the product topology.

3 The Banach-Alaoglu Theorem

We begin with a Banach-Alaoglu theorem for ordered cones. The proof follows the general lines of the usual proof of the standard Banach-Alaoglu Theorem, e.g., see [15], embedding the dual space in a compact one of functionals and then showing the set one wishes to prove compact to be closed in the induced topology.

Lemma 1. *Let τ and τ_d be separating dual topologies and subtopologies of a compact Hausdorff topology $\overline{\tau}$. Then τ is stably compact, τ_d is its co-compact topology and $\overline{\tau}$ is its patch topology.*

Proof. The join of the two separating topologies is Hausdorff and so equal to $\overline{\tau}$; we can then apply Theorem VI-6.18 of [4] to obtain the desired conclusions.

Theorem 1. *Let C be an ordered cone. Then the weak*-upper topology on C^* is stably compact, and its dual is the weak*-lower topology.*

Proof. By Lemma 1 it is enough to prove that the weak*-Lawson topology on C is compact. The weak*-Lawson topology on the collection $\overline{\mathbb{R}}_+^{|C|}$ of all functionals is the $|C|$-fold power of the Lawson topology on $\overline{\mathbb{R}}_+$, and so compact by the Tychonoff theorem. The weak*-Lawson topology on C^* is evidently the subspace topology induced by the weak*-Lawson topology on $\overline{\mathbb{R}}_+^{|C|}$, and so compact if we can show it is a closed subset of $\overline{\mathbb{R}}_+^{|C|}$ in that topology.

To that end we show, successively, that the subsets of monotone, homogeneous and additive functionals are closed. The subset of monotone functionals can be written in the form:

$$\bigcap_{x \leq y} \langle \mathrm{ev}_x, \mathrm{ev}_y \rangle^{-1}(\leq)$$

and is therefore closed as the order relation on $\overline{\mathbb{R}}_+$ is closed in the product Lawson topology on $\overline{\mathbb{R}}_+^2$ and the point evaluation functionals are continuous with respect to the weak*-Lawson topology.

The subset of homogeneous functionals can be written in the form:

$$\bigcap_{\lambda, x} \langle \mathrm{ev}_{\lambda \cdot x}, (\lambda \cdot -) \circ \mathrm{ev}_x \rangle^{-1}(=)$$

and is therefore closed as the equality relation is closed and multiplication is continuous in the Lawson topology.

Finally, the subset of additive functionals can be written in the form:

$$\bigcap_{x,y} \langle \mathrm{ev}_x, \mathrm{ev}_y, \mathrm{ev}_{x+y} \rangle^{-1}(+)$$

and is therefore closed as addition is continuous in the Lawson topology and so its graph is a closed subset of $\overline{\mathbb{R}}_+^3$.

This theorem does not extend to d-cones. Consider the d-cone $C = \mathcal{V}(\Omega)$ where Ω consists of the natural numbers, with the usual ordering, extended with a point at infinity: then the weak*-Lawson topology on C^* is not compact (and so, too, C^* is not a closed subset of $\overline{\mathbb{R}}_+^{|C|}$). For the set $\{F \in C^* \mid F(\eta_\infty) \geq 1\}$ is weak*-Lawson closed and covered by the increasing sequence $W_{\eta_n,0}$ of weak*-upper open sets, but by no member of it, as $\int f_n \, d-$ is in $\{F \in C^* \mid F(\eta_\infty) \geq 1\}$, but not in $W_{\eta_n,0}$, where $f_n(m) = 0$ if $m \leq n$ and $= 1$, otherwise (η_x is the point valuation at x).

To proceed further we consider the relation between the continuous functionals on a dcpo P and the monotone functionals on it, which we write as $\mathcal{M}(P)$. There is an evident inclusion:

$$\phi : \mathcal{L}(P) \hookrightarrow \mathcal{M}(P)$$

As both $\mathcal{L}(P)$ and $\mathcal{M}(P)$ are complete lattices and the inclusion preserves all sups, ϕ has a right adjoint $\psi : \mathcal{M}(P) \to \mathcal{L}(P)$, which assigns to any monotone functional its *(Scott continuous) lower envelope*, being the greatest continuous function below it; note here that ψ is a retraction with ϕ the corresponding section, so that $\langle \phi, \psi \rangle$ is an embedding-projection pair. In case P is continuous, the lower envelope is given by a standard formula:

$$\psi(f)(x) = \bigvee_{a \ll x} f(a)$$

The idea of using arguments involving both closed subsets and lower envelopes to prove stable compactness appears first in a paper of Jung [6]: the application there was to show the stable compactness of spaces of valuations. We show below that, as may be expected, results of that kind follow from the domain-theoretic Banach-Alaoglu theorem.

Proposition 1. *Let C be a continuous d-cone. Then, for any $f \in \mathcal{M}(P)$, $\psi(f)$ is $\mathbb{R}_+$-homogeneous if f is, subadditive if f is, and, assuming $\ll$ additive on C, superadditive if f is.*

Proof. For the preservation of $\mathbb{R}_+$-homogeneity, $\psi(f)$ is clearly strict if f is and taking $r \in (0, \infty)$ we have:

$$\begin{aligned} \psi(f)(r \cdot x) &= \textstyle\bigvee_{a \ll r \cdot x} f(a) \\ &= \textstyle\bigvee_{b \ll x} f(r \cdot b) \quad (r \cdot - \text{ acts automorphically}) \\ &= r \cdot (\psi(f)(x)) \quad (\text{by the homogeneity of } f \\ &\qquad\qquad\qquad\qquad \text{and the continuity of the action}) \end{aligned}$$

For the preservation of subadditivity we calculate:

$$\begin{array}{rll} \psi(f)(x+y) & = \bigvee_{c \ll x+y} f(c) & \\ & \leq \bigvee_{a \ll x, b \ll y} f(a+b) & \text{(by the continuity of } + \\ & & \quad \text{and the monotonicity of } f) \\ & \leq \bigvee_{a \ll x, b \ll y} f(a) + f(b) & \text{(by the subadditivity of } f) \\ & = \psi(f)(x) + \psi(f)(y) & \end{array}$$

And for the preservation of superadditivity we calculate:

$$\begin{array}{rll} \psi(f)(x+y) & = \bigvee_{c \ll x+y} f(x+y) & \\ & \geq \bigvee_{a \ll x, b \ll y} f(a+b) & \text{(by the additivity of } \ll) \\ & \geq \bigvee_{a \ll x, b \ll y} f(a) + f(b) & \text{(by the superadditivity of } f) \\ & = \psi(f)(x) + \psi(f)(y) & \end{array}$$

Let us remark that the preservation of homogeneity and subadditivity was already shown by Tix, see, e.g., [19].

We also need a different topology from the weak*-Lawson topology. Given a collection $\mathcal{F}$ of continuous functionals on a topological space X, define the *open-lower* topology on $\mathcal{F}$ to have as subbasis all sets of the form:

$$L_{U,r} = \{f \in \mathcal{F} \mid \exists x \in U.\, f(x) < r\}$$

for U open and $r \in (0, \infty)$.

Lemma 2. *Let $\mathcal{F}$ be a collection of continuous functionals on a domain P. Then the open-lower topology is a separating dual topology for the weak*-upper topology.*

Proof. First suppose that $f \leq g$ in the weak*-Scott specialisation ordering, which is the same as the pointwise one. Then if $g \in L_{U,r}$ we also clearly have that $f \in L_{U,r}$. Conversely, suppose we have $f \not\leq g$ in the pointwise ordering. Then there is an x and $r \in (0, \infty)$ such that $g(x) < r < f(x)$. So, as f is continuous there is an $a \ll x$ such that $f(a) > r$, and it follows that $f \in W_{a,r}$ and $g \in L_{a\Uparrow,r}$; note that the sets $W_{a,r}$ and $L_{a\Uparrow,r}$ are disjoint. It follows that $g \not\leq f$ in the open-lower specialisation ordering, as otherwise we would have that $f \in L_{a\Uparrow,r} \cap W_{a,r}$. So the topologies are dual and separating, as required.

We now have everything needed for the domain-theoretic analogue of the Banach-Alaoglu theorem:

Theorem 2. *Let C be a continuous d-cone with an additive way-below relation. Then the weak*-upper topology on C^* is stably compact, and its co-compact topology is the open-lower topology.*

Proof. We know from Proposition 1 that ψ cuts down to a function from $(C_m)^*$ to C^*, and ϕ evidently cuts down to a function in the opposite direction. Both these functions are continuous with respect to the weak*-upper topology. This is obvious for ϕ, and for ψ we calculate:

$$\begin{array}{rcl}\psi^{-1}(\{f \in C^* \mid f(x) > r\}) & = & \{f \in (C_m)^* \mid \bigvee_{b \ll x} f(b) > r\} \\ & = & \bigcup_{b \ll x}\{f \in (C_m)^* \mid f(b) > r\}\end{array}$$

So as C^* is a weak*-upper retract of $(C_m)^*$ and as, by Theorem 1, that topology is stably compact, the weak*-upper topology on C^* is also stably compact as retracts of stably compact spaces are stably compact [10,6].

If the $L_{U,r}$ are co-compact, it follows, using Lemma 1 (and Lemma 2) that the open-lower topology is the co-compact topology for the stably compact weak*-upper topology. So we show that all sets of the form $\{f \in C^* \mid \forall x \in U.\, f(x) \geq r\}$ with $r \in (0, \infty)$ are weak*-upper compact, and that follows from the equation:

$$\psi(\{g \in (C_m)^* \mid \forall x \in U.\, g(x) \geq r\}) = \{f \in C^* \mid \forall x \in U.\, f(x) \geq r\}$$

as ψ is weak*-upper continuous and $\{g \in (C_m)^* \mid \forall x \in U.\, g(x) \geq r\}$ is weak*-upper compact by Theorem 1, being a weak*-lower closed subset of $(C_m)^*$. To see that this equation holds, first note that, as ψ acts as the identity on continuous functionals, the right hand side is included in the left hand side. Conversely, suppose that g is in the left hand side, and $x \in U$. Take any $s < r$. Then, as U is open, $x \gg$ some $a \in U$, and then we see that $g(a) \geq r > s$ and so that $\psi(g)(x) > s$. It follows that, $\psi(g)(x) \geq r$, as required.

We remark that C^* is always compact in the weak*-upper topology, for any d-cone C; this is just because 0 is its least element. So the force of the conclusion is more the local stable compactness of C^*. The d-cone $\mathcal{V}(\Omega)$ used in the counterexample above satisfies the conditions of the theorem (see [20], Chapter 2.2) and so also provides an example where the open-lower and the weak*-lower topologies disagree: $\{F \in \mathcal{V}(\Omega)^* \mid F(\eta_\infty) \geq 1\}$ is closed in the latter but not the former.

Comparing our domain-theoretic Banach-Alaoglu theorem with the standard one, one may notice the assumption that $\ll$ is additive, and also the difference in the proofs, where we consider projections as well as subsets. It is shown in [20] that the condition on the way-below relation is equivalent to the requirement that addition is *quasi-open*, meaning that $(U + V)\uparrow$ is open whenever U and V are. In the case of topological vector spaces not only is addition an open map, but a stronger condition holds, that each map $x + -$ is open. This entails that any linear functional (to $\mathbb{R}$ or $\mathbb{C}$) bounded on a neighbourhood of 0 is continuous, and so difficulties with continuity do not arise in that setting.

Some theorems of [7,6,20,1] concerning the stable compactness of spaces of valuations on a topological space X follow from Theorem 2. We sometimes slightly weaken the hypothesis on X from stable compactness to local stable compactness or strengthen the conclusion by identifying the co-compact topology. Write $\mathcal{V}_{\leq 1}(X)$ for the collection of subprobability valuations and $\mathcal{V}_1(X)$ for the collection of probability valuations.

Corollary 1. *Let X be a stably locally compact topological space. Then $\mathcal{V}(X)$ is stably compact in the weak*-upper topology with co-compact topology the open-lower topology. The same is true of $\mathcal{V}_{\leq 1}(X)$, and also of $\mathcal{V}_1(X)$ in case X is stably compact.*

Proof. Since X is stably locally compact we have, by Propositions 2.25 and 2.28 of [20], that $\mathcal{L}(X)$ is a continuous d-cone with additive $\ll$, and so, by Theorem 2, that $\mathcal{L}(X)^*$ is weak*-upper stably compact, with co-compact topology the open-lower topology.

The isomorphism, $\Lambda\colon \mathcal{V}(X) \cong \mathcal{L}(X)^*$ induces a corresponding pair of topologies on $\mathcal{V}(X)$. We will show that these include the weak*-upper and co-compact topologies, respectively. Then as, by Lemma 2, those are a separating dual pair of topologies, it follows by Lemma 1 that the weak*-upper topology is indeed stably compact with co-compact topology the open-lower one.

For the inclusion of the weak*-upper topology we need only observe that $\Lambda(\{\nu \in \mathcal{V}(X) \mid \nu(U) > r\}) = \{F \in \mathcal{L}(X)^* \mid F(\chi_U) > r\}$ for any open set U and $r \in (0, \infty)$. For the inclusion of the open-lower topology it suffices to prove that:

$$\Lambda(\{\nu \mid \forall U \in O.\, \nu(U) \geq r\}) = \bigcap_{U \in O, s \in (0,1)} \{F \mid \forall f \in (s\chi_U)\Uparrow.\, F(f) \geq sr\})$$

for O an open set of $\mathcal{O}(X)$ and $r \in (0, \infty)$, since the set on the right is closed in the open-lower topology on $\mathcal{L}(X)^*$. The inclusion from left to right is clear. In the other direction, take a Λ_ν in the set on the right, and a $U \in O$ to prove $\nu(U) \geq r$. Then there is a $U' \ll U$ with $U' \in O$, since O is open. Choose $s \in (0, 1)$. Then, by Lemma 2.26 of [20], $\chi_U \gg s\chi_{U'}$, and so $\nu(U) \geq sr$. But s is an arbitrary element of $(0, 1)$, and so we see that $\nu(U) \geq r$, as required.

The set of subprobability valuations $\{\nu \in \mathcal{V}(X) \mid \nu(X) \geq 1\}$ is weak*-upper closed in $\mathcal{V}(X)$ and, when X is compact, the set $\{\nu \in \mathcal{V}(X) \mid \nu(X) \geq 1\}$ is closed in the open-lower topology on $\mathcal{V}(X)$, and so the set of probability valuations is closed in the patch topology. The rest of the theorem follows from these two observations, using Proposition 2.16 of [6].

When X is locally compact, the open-lower topology on $\mathcal{V}(X)$ has a subbasis of closed sets of the form:

$$\{\nu \in \mathcal{V}(X) \mid \forall U \supset K.\, \nu(U) \geq r\}$$

for K compact and $r \in (0, \infty)$. This form of the co-compact topology was noted, without proof, for $\mathcal{V}_{\leq 1}(X)$ and $\mathcal{V}_1(X)$ in [6]; one can evidently then restrict to $r \in (0, 1)$. The restriction to stably compact X in the last part of the corollary is necessary as a topological space Y is compact if $\mathcal{V}_1(Y)$ is compact in the weak*-upper topology. (To see this, suppose $\mathcal{V}_1(Y)$ so compact and let U_i be a directed covering of Y by open sets. Then $W_i =_{\text{def}} \{\nu \mid \nu(U_i) > 0\}$ is a directed covering of $\mathcal{V}_1(Y)$ by weak*-upper open sets, and so some W_i includes it. But then U_i includes Y, as $x \in U_i$ holds iff $\eta_x \in W_i$ does.)

One can specialise these results to domains following, e.g., [20]. A domain, qua topological space, is stably locally compact iff it is coherent, and stably compact iff its Lawson topology is compact. If P is a domain then both $\mathcal{V}(P)$ and $\mathcal{V}_{\leq 1}(P)$ are—but not, in general, $\mathcal{V}_1(P)$. Lastly, on both $\mathcal{V}(P)$ and $\mathcal{V}_{\leq 1}(P)$ the weak*-upper topology coincides with the Scott topology [9,17]. So we see from the corollary that if P is a coherent domain then both $\mathcal{V}(P)$ and $\mathcal{V}_{\leq 1}(P)$ are Lawson compact.

Lawson has proved a certain converse: for a domain P, if $\mathcal{V}(P)$ is Lawson compact then P is coherent, see [20], Theorem 2.10 (d). The necessity of the additivity condition of Theorem 2 follows. Take any domain P and assume that the weak*-upper topology on $\mathcal{L}(P)^*$ is stably compact. Then, following the proof of the corollary, the weak*-upper topology on $\mathcal{V}(P)$ is also stably compact, and so, by Lawson's result the Scott topology on P is stably locally compact, and it follows, by Proposition 2.28 of [20], that $\mathcal{L}(P)$ has an additive way below relation. So if we take any non-coherent domain P we see that $\mathcal{L}(P)$ is continuous, but that $\ll$ is not additive, by Proposition 2.29 of [20], and then that $\mathcal{L}(P)^*$ is not stably compact.

Finally, we mention a natural question: having Banach-Alaoglu theorems for ordered cones and d-cones, is there also one for topological cones? In this respect note that the conclusion of Theorem 2 relates to the dual of C considered as a topological cone.

4 Polar sets and Minkowski functionals

The weak*-upper topology and its dual can be defined in two other ways: using polar sets and using Minkowski functionals, more precisely, their domain-theoretic analogues.

Definition 1. *Let X be a subset of a d-cone C. Then its* lower polar *is defined to be $X_\circ = \{f \in C^* \mid \forall x \in X.\, f(x) \leq 1\}$, and its* upper polar *is defined to be $X^\circ = \{f \in C^* \mid \forall x \in X.\, f(x) \geq 1\}$.*

Proposition 2. *Let C be a d-cone.*

1. *The weak*-upper topology has as a subbasis of closed sets all lower polars, and also all lower polars of non-empty Scott-closed convex sets.*
2. *The open-lower topology has as subbasis of closed sets all upper polars of open sets (not containing 0), and also, if C is continuous, all upper polars of convex open sets (not containing 0).*

Proof. 1. Regarding the first assertion, every lower polar set is evidently closed in the weak*-upper topology on C^*, and, conversely, $W_{a,r}$ is the complement of $\{r^{-1}\cdot a\}_\circ$. For the second, we can evidently disregard lower polars of empty sets, and the lower polar of a set is easily seen to be the same as the least Scott-closed convex set containing it.

2. Regarding the first assertion, every upper polar set is evidently closed in the topology generated by the $L_{U,r}$, and, conversely, the complement of $L_{U,r}$ is $(r^{-1}\cdot U)^\circ$. We can evidently disregard upper polars of sets containing 0. The

second assertion follows from the fact that when C is continuous it is locally convex in the sense that every neighbourhood contains a convex open one, see [20], Proposition 2.5.

Our main alternative description of the polar topology is in terms of functional bounds; the connection between the two is given using *Minkowski functionals*. For any subset X of a d-cone C, define its *upper* and *lower* Minkowski functionals by:

$$\mu_X(x) = \bigvee\{r \in (0, \infty) \mid x \in r \cdot X\}$$

and

$$\nu_X(x) = \bigwedge\{r \in (0, \infty) \mid x \in r \cdot X\}$$

yielding two monotone functions, $\mu : \mathcal{P}(C) \to \overline{\mathbb{R}}_+$ and $\nu : \mathcal{P}(C) \to (\overline{\mathbb{R}}_+)^{op}$. It would be equivalent to let r range over $\overline{\mathbb{R}}_+$, but we find the above form of the definition more convenient. Our Minkowski functionals are defined by an obvious analogy with the standard ones; they, and some of their properties, are also implicit in the proof of Tix's Separation Theorem, see, e.g., [20], Theorem 3.4.

In the other direction, given any functional g on C we define:

$$\mathrm{S}(g) = \{x \in C \mid g(x) > 1\}$$

and

$$\mathrm{L}(g) = \{x \in C \mid g(x) \leq 1\}$$

Both μ and ν are complete lattice homomorphisms and so left adjoints. The next lemma provides relevant information on these two adjunctions; we do not distinguish notationally between functions and their various restrictions and corestrictions.

Lemma 3. *Let C be a d-cone. Then*

1. *μ cuts down to an isomorphism of the complete lattice of the Scott open subsets of C not containing 0 and that of the homogeneous continuous functionals on C, with the pointwise ordering; it has inverse S. This cuts down, in its turn, to an isomorphism of the complete lattice of convex open subsets of C not containing 0 and that of the superlinear continuous functionals on C. Further, for any homogeneous continuous functional g and open set U not containing 0 we have that $g \geq \mu_U$ iff $g \in U^\circ$.*
2. *ν cuts down to an adjunction between the complete lattice of the subsets of C containing 0 and that of the (opposite of) the $\mathbb{R}_+$-homogeneous functionals on C; it has right adjoint L. They, in turn, cut down to an adjunction between the complete lattice of the non-empty convex down-closed subsets of C and that of the (opposite of) the $\mathbb{R}_+$-sublinear monotone functionals on C. Further, for any $\mathbb{R}_+$-homogeneous functional g and non-empty set X we have that $g \leq \nu_X$ iff $g \in X_\circ$.*

Proof. 1. That μ sends (convex) open sets not containing 0 to (superadditive) homogeneous continuous functionals is a straightforward verification; the corresponding properties of S are immediate. Next, μ is monotone and S evidently is too, and we prove that they are inverses. To see that $\mu_{\mathrm{S}(g)} = g$ for any homogeneous continuous function g, note that $x \in r \cdot \mathrm{S}(g)$ iff $g(x) > r$, for any $x \in C$ and $r \in (0, 1)$. To see that $\mathrm{S}(\mu_U) = U$, for any open set U not containing 0, note that, for any $x \in C$:

$$\begin{array}{rll} x \in \mathrm{S}(\mu_U) & \text{iff} & \mu_U(x) > 1 \\ & \text{iff} & \exists r \in (1, \infty).\, x \in r \cdot U \\ & \text{iff} & x \in U \qquad\qquad \text{(as } U \text{ is open)} \end{array}$$

All these equivalences are obvious except the last. The 'only if' holds as U is open and therefore upper closed; the 'if' holds as for any $x \in C$ we have that $x = \bigvee r_n \cdot x$ where r_n is any increasing sequence of positive reals tending to 1. So if $x \in U$ then for some r_n, $r_n \cdot x \in U$ and so $x \in r_n^{-1} \cdot U$.

By the isomorphism, for any positively homogeneous continuous functional g and open set U, we have $g \geq \mu_U$ iff $\mathrm{S}(g) \supset U$ and we now show that the latter is equivalent to $g \in U^{\circ}$. Only the implication from right to left is in question, so suppose that $g(z) \geq 1$ for all $z \in U$, and choose $x \in U$. As U is open we have $r_n \cdot x \in U$ for some r_n (the r_n are as before) and so $g(r_n \cdot x) \geq 1$, which implies that $g(x) > 1$, as required.

2. That ν sends (convex down-closed) subsets of C not containing 0 to $\mathbb{R}_+$-homogeneous (subadditive monotone) functionals is a straightforward verification; the corresponding properties of L are immediate. To see that the right adjoint is S, we calculate:

$$\begin{array}{rll} g \leq \nu_X & \text{iff} & \forall x.\, g(x) \leq \bigwedge\{r \in (0, \infty) \mid x \in r \cdot X\} \\ & \text{iff} & \forall x.\, \forall r \in (0, \infty).\, x \in r \cdot X \supset g(x) \leq r \\ & \text{iff} & \forall x.\, x \in X \supset g(x) \leq 1 \qquad\qquad \text{(as } g \text{ is } \mathbb{R}_+\text{-homogeneous)} \\ & \text{iff} & X \subset \mathrm{L}(g) \end{array}$$

The final assertion follows from the adjunction and the fact that $X \subset \mathrm{L}(g)$ iff $g \in X_{\circ}$.

We remark that in part 2, L is actually a closure operation: one easily verifies that $\nu_{L(g)} = g$ for any $\mathbb{R}_+$-homogeneous functional g.

We next consider a 'homogenising' operation. For any functional f on a set X define:

$$H_u(f) = \bigvee_{r \in (0, \infty)} r^{-1} \cdot f(r \cdot x)$$

Lemma 4. *Let f be a strict continuous functional on a d-cone C. Then $H_u(f)$ is the least homogeneous continuous functional above it.*

Proof. It is evident that $H_u(f)$ is continuous and below any homogeneous functional above f. To see that it is homogeneous, note that it is strict and that for any $s \in (0, \infty)$ and $x \in C$:

$$\begin{aligned} H_u(f)(s \cdot x) &= \bigvee_{r \in (0,\infty)} r^{-1} \cdot f(r \cdot (s \cdot x)) \\ &= s \cdot \bigvee_{r \in (0,\infty)} (rs)^{-1} \cdot f(rs \cdot x) \\ &= s \cdot \bigvee_{t \in (0,\infty)} t^{-1} \cdot f(t \cdot x) \\ &= s \cdot H_u(f)(x) \end{aligned}$$

It is interesting to note that the Minkowski functionals can be understood as homogenised characteristic functions, since $\mu(X) = H_u(\chi_X)$ and $\nu(X) = H_l(\chi_X)$, where H_l is defined analogously to H_u, but taking infs instead of sups.

We can now reformulate the weak*-upper and the open-lower topologies in terms of functional bounds:

Proposition 3. *Let C be a d-cone.*

1. *The weak*-upper topology has subbases of closed sets of each the following forms: all sets of the form $\{g \in C^* \mid g \leq h\}$ with h any functional; all sets of that form with h an $\mathbb{R}_+$-sublinear monotone functional; and, if C is continuous, all sets of that form with h a sublinear continuous functional.*
2. *The open-lower topology has subbases of closed sets of each of the following forms: all sets of the form $\{g \in C^* \mid f \leq g\}$ with f a strict continuous functional; all sets of that form with f a homogeneous continuous functional; and, if C is continuous, all sets of that form with f a superlinear continuous functional.*

Proof.

1. That the sets of the form $\{g \in C^* \mid g \leq h\}$ form a subbasis of closed sets for the weak*-upper topology is evident. By Proposition 2.1 all lower polars of non-empty down-closed convex subsets X also form a subbasis, and by Lemma 3.2, these can be written in the form $\{g \in C^* \mid g \leq \nu_X\}$. So, as ν_X is $\mathbb{R}_+$-sublinear and monotone, we can restrict the subbasis to the required form. Finally, if C is continuous, then such lower polars can also be written as $\{g \in C^* \mid g \leq \psi(\nu_X)\}$ and, by Proposition 1, $\psi(\nu_X)$ is sublinear and continuous.
2. We know from Proposition 2.2 that the open-lower topology has a subbasis consisting of sets of the form U° with U an open subset of C not containing 0. Lemma 3.1 tells us that $U^\circ = \{g \in C^* \mid \mu_U \leq g\}$ and that μ_U is homogeneous and continuous; further applying Lemma 3.1, we obtain that $\{g \in C^* \mid f \leq g\} = \{g \in C^* \mid \mu_{\mathrm{S}(f)} \leq g\} = \mathrm{S}(f)^\circ$, for any homogeneous continuous functional f. We therefore conclude that the open-lower topology has a subbasis consisting of all sets of the form $\{g \in C^* \mid f \leq g\}$ with f a homogeneous continuous functional. Similar reasoning shows that we can restrict to superlinear continuous functionals f in the case that C is continuous.
   Finally, for any strict continuous functional f, by Lemma 4 we have that $H_u(f)$ is homogeneous and continuous and also that $f \leq g$ iff $H_u(f) \leq g$, for any $g \in C^*$. So the sets of the form $\{g \in C^* \mid f \leq g\}$ with f homogeneous and continuous also form a subbasis.

This proposition evidently allows quite a number of equivalent formulations of the weak*-upper topology and its dual. We note the following immediate consequence of Theorem 2 and the proposition:

Corollary 2. *Let C be a continuous d-cone with an additive way-below relation, and suppose that f is a continuous superlinear functional on C and h is an $\mathbb{R}_+$-sublinear functional on C. Then the 'sandwich set' of functionals:*

$$\{g \in C^* \mid f \le g \le h\}$$

is compact in the patch topology (on the weak*-upper topology).

This complements the Sandwich Theorem, see, e.g., [20], Theorem 3.2, which says—though without the assumption that way-below is additive—that the sandwich set is non-empty.

Acknowledgements

I would like to thank Klaus Keimel for many illuminating discussions.

References

1. M. Alvarez-Manilla, A. Jung and K. Keimel, The probabilistic powerdomain for stably compact spaces, *TCS*, Vol. 328, Iss. 3, pp. 221–244, Elsevier, 2004.
2. N. Boboc, G. Bucur and A. Cornea, *Order and convexity in potential theory: H-cones*, Lecture Notes in Mathematics, Vol. 853, Berlin: Springer-Verlag, 1981.
3. B. Fuchssteiner and W. Lusky, *Convex Cones*, Amsterdam: North-Holland , 1981.
4. G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *Continuous Lattices and Domains*, Encyclopedia of Mathematics and its Applications, Vol. 93, Cambridge University Press, 2003.
5. J. S. Golan, *Semirings and Their Applications*, Springer-Verlag, 1999.
6. A. Jung, Stably compact spaces and the probabilistic powerspace construction, *Domain-theoretic Methods in Probabilistic Processes* (eds. J. Desharnais and P. Panangaden), ENTCS, Vol. 87, pp. 5–20, Elsevier, 2004.
7. A. Jung and R. Tix, The troublesome probabilistic powerdomain, *Proc. Third Workshop on Computation and Approximation* (eds. A. Edalat, A. Jung, K. Keimel and M. Kwiatkowska), ENTCS, Vol. 13, pp. 70–91, Elsevier, 1998.
8. K. Keimel and W. Roth, *Ordered Cones and Approximation*, Lecture Notes in Mathematics, Vol. 1517, 140 pp., Springer Verlag, 1992.
9. O. Kirch, *Bereiche und Bewertungen*, Master's thesis, Technische Universität Darmstadt, 1993.
10. J. D. Lawson, The versatile continuous order, *Proc. 3rd. MFPS* (eds. M. Main, A. Melton, M. Mislove, and D. Schmidt), LNCS, Vol. 298, pp. 134–160, Springer Verlag, 1988.
11. J. D. Lawson, Domains, integration and 'positive analysis,' *Mathematical Structures in Computer Science*, Vol. 14, No. 6, pp. 815–832, Cambridge University Press, 2004.
12. W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces, I*, Amsterdam: North Holland, 1971.

13. M. Rauch, Stetige Verbände in der axiomatischen Potentialtheorie, in *Continuous Lattices and Related Topics* (ed. R. E. Hoffman), Mathematische Arbeitspapiere, Vol. 27, pp. 260–308, Universität Bremen, 1982.
14. W. Roth, Hahn-Banach type theorems for locally convex cones, *Journal of the Australian Mathematical Society*, Vol. 68, No. 1, pp. 104–125, 2000.
15. W. Rudin, *Functional Analysis* (2nd. edition), International Series in Pure and Applied Mathematics, McGraw-Hill, 1991.
16. H. H. Schaefer, *Banach Lattices and Positive Operators*, Berlin: Springer-Verlag, 1975.
17. R. Tix, *Stetige Bewertungen auf Topologischen Räumen*, Master's thesis, Technische Universität Darmstadt, 1995.
18. R. Tix, *Continuous D-cones: Convexity and Powerdomain Constructions*, Ph.D. thesis, Technische Universität Darmstadt, Aachen: Shaker Verlag, 1999.
19. R. Tix, Some results on Hahn-Banach theorems for continuous d-cones, *TCS*, Vol. 264, Iss. 2, pp. 205–218, Elsevier, 2001.
20. R. Tix, K. Keimel and G. Plotkin, *Semantic Domains for Combining Probability and Non-Determinism*, ENTCS, Vol. 129, pp. 1–104, Amsterdam: Elsevier, 2005.