Chapter 4 Solvability of the general problem

We are now in a position to consider questions of the solvability of the general problem raised at the end of chapter 2, which we restate:

Given a notion of niceness $\prec$ and f and e, it is required to find the sets of clauses, H, satisfying:

1) $H \leq H_0$ (Th)

2) $\forall H \wedge Irr \wedge Th \wedge \bigwedge_{i=1}^{n} (e_i \wedge f_i)$ is consistent.

3) H is minimal with respect to $\prec$ amongst those clauses satisfying 1) and 2).

Under the assumption that every finite set of clauses has a l.g.g. relative to Th, we can locate the set of solutions when $\prec$ is $\prec_{cpg}$. Let $\inf_{Th}$ H be a l.g.g. of H relative to Th. Note that any tautology is a l.g.g. of $\emptyset$ relative to any Th. When Th = $\emptyset$, we require that $\inf_{\emptyset} H = \inf H$, if H does not contain a tautology, where inf is the function inf defined in section 3.3.2 in order that the notation be consistent.

There is a lattice which contains all the possible solutions. Let $\mathcal{J}_{Th}(H)$ where H is a finite set of clauses be the set of equivalence classes of the set $\{\inf_{Th} H' \mid H' \subseteq H\}$, where the equivalence relation is equivalence relative to Th. The equivalence class of $\inf_{Th} H'$ is denoted by $[\inf_{Th} H']$. $\mathcal{J}_{Th}(H)$ is ordered by:

$[\inf_{Th} H_1] \sqsubseteq [\inf_{Th} H_2]$ iff $\inf_{Th} H_1 \leq \inf_{Th} H_2 (Th)$.

This is evidently a good definition. $\mathcal{J}_{Th}(H)$ is actually a lattice with infs given by:

$$[\inf_{Th} H_1] \sqcap [\inf_{Th} H_2] = [\inf_{Th}(H_1 \cup H_2)]$$

This is a well-defined operation. If $H_1 \subseteq H$ and $H_2 \subseteq H$ then $[\inf_{Th}(H_1 \cup H_2)]$ is in $\mathcal{J}_{Th}(H)$. Suppose $\inf_{Th} H_1 \sim \inf_{Th} H_1' \ (Th)$ and $\inf_{Th} H_2 \sim \inf_{Th} H_2' \ (Th)$. Then,

$$\begin{aligned} \inf_{Th}(H_1 \cup H_2) &\sim \inf_{Th}\{\inf_{Th} H_1, \inf_{Th} H_2\} \ (Th) \\ &\sim \inf_{Th}\{\inf_{Th} H_1', \inf_{Th} H_2'\} \ (Th) \\ &\sim \inf_{Th}(H_1' \cup H_2') \ (Th). \end{aligned}$$

Therefore $\sqcap$ is well-defined.

As $\inf_{Th}(H_1 \cup H_2) \leq \inf_{Th} H_i \ (Th)$ for $i=1,2$, $\sqcap$ does give a lower bound. Suppose $\inf_{Th} H_3 \leq \inf_{Th} H_i \ (Th)$ for $i=1,2$. Then

$$\begin{aligned} \inf_{Th} H_3 &\leq \inf_{Th}\{\inf_{Th} H_1, \inf_{Th} H_2\} \ (Th) \\ &\sim \inf_{Th}(H_1 \cup H_2) \ (Th). \end{aligned}$$

Therefore $[\inf_{Th} H_3] \sqsubseteq [\inf_{Th} H_1] \sqcap [\inf_{Th} H_2]$, and $\sqcap$ gives the greatest lower bound. As $\mathcal{J}_{Th}(H)$ is finite, arbitrary subsets have infs, given by:

$$\bigsqcap\{[\inf_{Th} H_i] \mid i=1,n\} = [\inf_{Th} H_1] \sqcap ([\inf_{Th} H_2] \ldots ([\inf_{Th} H_{n-1}] \sqcap [\inf_{Th} H_{n-1}]) \ldots),$$

$$\bigsqcap\{[\inf_{Th}H_1]\} = [\inf_{Th}H_1],$$
$$\bigsqcap\emptyset = [\{\{P(x), \overline{P(x)}\}\}].$$

Consequently there is a sup operation defined by:

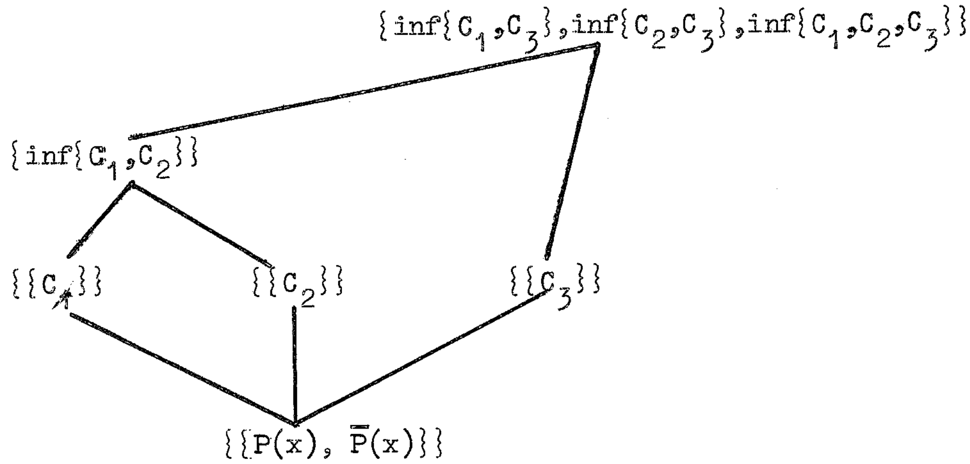
$$[\inf_{Th}H_1] \sqcup [\inf_{Th}H_2] = \bigsqcap\{[\inf_{Th}H_3] \in \mathcal{I}_{Th}(H) \mid [\inf_{Th}H_i] \sqsubseteq [\inf_{Th}H_3], \text{ for } i=1,2\}.$$

Thus, equipped with $\sqsubseteq$, $\sqcap$ and $\sqcup$, $\mathcal{I}_{Th}(H)$ is a finite lattice.

<u>Example</u> Suppose $H = \{C_1, C_2, C_3\}$ where $C_1 = \{P(f()), Q(f())\}$,
$C_2 = \{P(g()), Q(g())\}$,
$C_3 = \{P(h())\}$.

Then $\inf\{C_1, C_2\} \sim \{P(x), Q(x)\}$,
$\inf\{C_1, C_3\} \sim \inf\{C_2, C_3\} \sim \inf\{C_1, C_2, C_3\} = \{P(x)\}$.

The lattice $\mathcal{I}_{\emptyset}(H)$ is displayed in figure 1



<u>Figure 1</u>

We can now show that the solutions are located in $\mathcal{J}_{Th}(H_0)$ when $\preceq$ is $\preceq_{cpg}$.

The set of clauses in H_0 that an arbitrary clause C explains is defined to be:

$$\text{Explainset}(C) = \{C' \in H_0 \mid C \leq C' \ (Th)\}.$$

Notice that Power(C) = cardinality of Explainset(C).

We also set Complexity(H) = $||H||$, the cardinality of H.

<u>Relative equivalence</u> between sets of clauses is defined by: $H \sim H' \ (Th)$ iff $H \leq H' \ (Th)$ and $H' \leq H \ (Th)$.

A set, H, of clauses is <u>reduced</u>, relative to Th, iff $H' \subseteq H$ and $H' \sim H$ implies that $H' = H$.

If $H \sim H' \ (Th)$ and both H and H' are reduced then there is a unique bijection $\theta : H' \Rightarrow H$ such that $\theta(C') \sim C'$ for every C' in H'. This may be proved in a way analogous to the corresponding part of the proof of the corresponding part of the statement of theorem 3.3.1.2. One can easily extend the analogy to the rest of the theorem if relative generalisation is a recursive relation.

<u>Theorem 1</u> H is a solution when $\preceq$ is $\preceq_{cpg}$ iff it is reduced relative to Th and is equivalent, relative to Th, to an H' satisfying:

1) $H' \subseteq \{\inf_{Th} H_1 \mid H_1 \subseteq H_0\}$.

2) $H' \leq H_0 \ (Th)$.

3) $\forall H' \wedge Th \wedge Irr \wedge \bigwedge_{i=1}^{n}(e_i \wedge f_i)$ is consistent.

4) H' is minimal w.r.t. $\preceq_{cp}$ amongst those sets of clauses satisfying 1), 2) and 3).

Further, any H' satisfying these conditions is a solution.

Proof Let H be a solution. If it is not reduced relative to Th then let H'' be reduced and equivalent, relative to Th, to H. Then Complexity(H'') < Complexity(H), $H'' \leq H_0$ (Th) and $\forall H'' \wedge Th \wedge Irr \wedge \bigwedge_{i=1}^{n}(e_i \wedge f_i)$ is consistent, since $\vdash \forall H \equiv \forall H''$. This contradicts the fact that H is a solution. So H is reduced relative to Th.

Define $H' = \{\inf_{Th}$Explainset(C) $\mid C \in H\}$. We see that $H' \subseteq \{\inf_{Th} H_1 \mid H_1 \subseteq H\}$, Complexity($H'$) $\leq$ Complexity(H), Power(H) = Power(H'), and $H \leq H'$ (Th), and $H' \leq H_0$ (Th). It follows from $H \leq H'$ (Th) that $\forall H' \wedge Th \wedge Irr \wedge \bigwedge_{i=1}^{n}(e_i \wedge f_i)$ is consistent. So since H is a solution, Complexity(H') = Complexity(H) and $H \sim H'$ (Th). Suppose H'' satisfies 1), 2) and 3) and $H'' \preceq_{cp} H'$. Then as $H' \preceq_{cp} H$, $H'' \preceq_{cp} H$. So, as H is a solution and $H' \preceq_{cp} H$, $H' \preceq_{cp} H \preceq_{cp} H''$. This establishes condition 4) for H' and concludes the first part of the proof.

We demonstrate the last part of the theorem next. Suppose that H' satisfies conditions 1) to 4). Then it satisfies conditions 1) and 2) for being a solution. Suppose that H'' also satisfies conditions 1) and 2) for being a solution and that $H'' \preceq_{cpg} H'$. We will show that $H' \preceq_{cpg} H''$. Let $H''' = \{\inf_{Th}$ Explainset(C) $\mid C \in H''\}$. We see that

$H'' \le H'''$ (Th), $H''' \prec_{cpg} H''$ and H''' satisfy condition 1). Therefore H''' satisfies conditions 1), 2) and 3). Further $H''' \prec_{cp} H''$. As $H'' \prec_{cpg} H'$, $H'' \prec_{cp} H'$ and so $H''' \prec_{cp} H'$. As H' satisfies condition 4) it follows that $H' \prec_{cp} H'''$. But $H''' \prec_{cpg} H'' \prec_{cpg} H'$. Therefore $H' \le H'''$.

There is therefore a map $\theta : H''' \Rightarrow H'$ such that $\theta(C''') \le C'''$ (Th), for any C''' in H'''. If the map is not onto, there is a C' in H' such that $\theta(C''') \ne C'$ for any C''' in H'''. Then $H' \setminus \{C'\}$ satisfies conditions 1), 2) and 3) which contradicts the fact that H' satisfies condition 4). Therefore θ is onto. Since Complexity(H''') = Complexity(H''), as $H''' \prec_{cp} H'$ and $H' \prec_{cp} H'''$, θ must be a bijection. Therefore as Power (C''') $\le$ Power($\theta(C''')$) for any C''' in H''' and Power(H') = Power(H''') then Power(C''') = Power($\theta(C''')$) for any C''' in H'''. But as $\theta(C''') \le C''$ (Th), Explainset($\theta(C''')$) $\supseteq$ Explainset(C''') for any C''' in H'''. Therefore Explainset($\theta(C''')$) = Explainset(C''') and so $C''' \le \inf_{Th}$Explainset($\theta(C''')$) $\le \theta(C''')$ (Th), as H' satisfies condition 1), for any C''' in H'''. Now, as θ is a bijection, $H''' \sim H'$ (Th) and taking this with $H''' \prec_{cpg} H''$ and $H' \prec_{cp} H'''$ we see that $H' \prec_{cpg} H''$ which concludes the proof that H' is a solution.

Suppose next that H' satisfies conditions 1) to 4) and that H is reduced, relative to Th, and that $H \sim H'$ (Th). Then H' is a solution and so is reduced by the above. Therefore by the remark after the definition of when a set of clauses is relatively reduced, there is a bijection $\theta : H' \Rightarrow H$ such that $\theta(C') \sim C'$ for every C' in H'.

Therefore $H \prec_{cpg} H'$ and since H satisfies conditions 1) and 2) for being a solution as $H \sim H'$ (Th) it follows that H is a solution, concluding the proof.

Let us look at some examples. Here is another appearance of the crows.

f	e
f_1 = Black(crow1) f_2 = Black(crow2)	e_1 = Crow(crow1) e_2 = Crow(crow2)

Table 1

Also Th is empty. We have $H_0 = \{C_1, C_2\}$ where

$C_1 = \{\overline{\text{Crow}}(\text{crow1}), \text{Black}(\text{crow1})\}$,

$C_2 = \{\overline{\text{Crow}}(\text{crow2}), \text{Black}(\text{crow2})\}$.

Now, $C_3 = \inf\{C_1, C_2\} = \{\overline{\text{Crow}}(x), \text{Black}(x)\}$.

Evidently $\forall C_3 \wedge e_1 \wedge e_2 \wedge f_1 \wedge f_2$ is consistent and so $\{C_3\}$ is the only solution. We have induced 'All crows are black'.

Next, we give a less trivial example from Hunt, Marin and Stone (1966). We must learn that all bears or large animals are dangerous. Our observational data consists of a description of various animals, both dangerous and non-dangerous, in terms of the attributes Size, Animality and Colour as given in table 2. Again, Th is empty.

Now, $H_0 = \{C_i \mid 1 \leq i \leq 7\}$. The members of $\{\inf H' \mid H' \subseteq H_0, H' \neq \emptyset\}$ which are consistent with $\bigwedge_{i=1}^{7}(e_i \wedge f_i)$ are, apart from C_1 to C_7,

$$C_8 = \inf\{C_1, C_2\}$$
$$= \{\overline{\text{Size}}(x,s),\ \overline{\text{Colour}}(x,\text{black}),\ \overline{\text{Animal}}(x,\text{bear}),\ \overline{\text{Dangerous}}(x)\}$$

and

$$C_9 = \inf\{C_3, C_6, C_7\}$$
$$= \{\overline{\text{Size}}(x,\text{large}),\ \overline{\text{Colour}}(x,c),\ \overline{\text{Animal}}(x,a),\ \overline{\text{Dangerous}}(x)\}.$$

The solution is $H = \{C_8, C_9, C_4, C_5\}$ and includes the following version of the generalisation to be learnt:

'Anything that has a size and is a black bear is dangerous' and 'Anything that is a large coloured animal is dangerous'.

Notice that if we assume that all animals have a colour and all bears have a size, then this generalisation is equivalent to the original one.

f	e
f_1 = Dangerous(animal1)	e_1 = Size(animal1,small)$\wedge$Colour(animal1,black)$\wedge$ Animal(animal1,bear)
f_2 = Dangerous(animal2)	e_2 = Size(animal2,medium)$\wedge$Colour(animal2,black)$\wedge$ Animal(animal2,bear)
f_3 = Dangerous(animal3)	e_3 = Size(animal3,large)$\wedge$Colour(animal3,brown)$\wedge$ Animal(animal3,dog)
f_4 = $\overline{\text{Dangerous}}$(animal4)	e_4 = Size(animal4,small)$\wedge$Colour(animal4,black)$\wedge$ Animal(animal4,cat)
f_5 = $\overline{\text{Dangerous}}$(animal5)	e_5 = Size(animal5,medium)$\wedge$Colour(animal5,black)$\wedge$ Animal(animal5,horse)
f_6 = Dangerous(animal6)	e_6 = Size(animal6,large)$\wedge$Colour(animal6,black)$\wedge$ Animal(animal6,horse)
f_7 = Dangerous(animal7)	e_7 = Size(animal7,large)$\wedge$Colour(animal7,brown)$\wedge$ Animal(animal7,horse).

Table 2

Note that in fact the solutions have been located in $\{\inf_{Th} H_1 \mid H_1 \subseteq H_0\}$ rather than in $\mathcal{I}_{Th}(H_0)$. One can reword theorem 1 to find the location in $\mathcal{I}_{Th}(H_0)$. $\mathcal{I}_{Th}(H_0)$ was introduced in order to find a well-known mathematical structure in which the solutions could be located; it is expected to make any associated computational problems a little easier. In future, we shall generally confuse $\mathcal{I}_{Th}(H_0)$ with $\{\inf_{Th} H_1 \mid H_1 \subseteq H_0\}$.

<u>Corollary 1</u> For the case where $\prec$ is $\prec_{cpg}$, every problem has a solution.

<u>Proof</u> We need only show that there is an H' satisfying conditions 1), 2) and 3) of theorem 1, for as $\{\inf_{Th} H_1 \mid H_1 \subseteq H_0\}$ is finite, there will then be a suitable minimal one, with respect to $\prec_{cp}$ and this will be a solution by theorem 1.

Now $H_0 \subseteq \{\inf_{Th} H_1 \mid H_1 \subseteq H_0\}$ and certainly $H_0 \leq H_0$ (Th). It is part of the problem conditions that Th $\wedge$ Irr $\bigwedge_{i=1}^{n}(e_i \wedge f_i)$ is consistent. Since $\vdash \bigwedge_{i=1}^{n}(e_i \wedge f_i) \rightarrow H_0$, it follows that H_0 satisfies condition 3), which concludes the proof.

Now that we know there is always a solution we can try to construct an algorithm for finding one. Let us consider the simplest case where Th and Irr are empty. Now $\inf_{Th}$ is calculable and so, therefore, is $\{\inf_{Th} H_1 \mid H_1 \subseteq H_0\} = \{\inf H_1 \| H_1 \subseteq H_0\}$. For each subset, H', of this set we can check whether $H' \leq H_0$ (Th), and if we could only check the consistency condition, we could then isolate the solutions, using theorem 1, since $\prec_{cp}$ is certainly decidable. Now, in general, the consistency of an arbitrary set of clauses is an undecidable property. We might hope, though, to avoid the difficulty since H' has arisen by means of the special process of generalisation. Unfortunately it can be shown that undecidability persists.

<u>Theorem 2</u> Suppose that $\prec$ is $\prec_{cpg}$. There is no algorithm which will, given any f and Ev produce a solution to the resulting generalisation problem. (Here both Th and Irr are empty.)

<u>Proof</u> We will show that if such an algorithm exists then it is possible to tell whether or not the universal closure of an arbitrary set of clauses is consistent. As this is not possible we shall then have demonstrated the non-existence of the algorithm.

To this end let H be an arbitrary set of clauses. We may assume, without loss of generality that no a_{ij} used in the representation theorem, theorem 3.3.1.4, appears in H, that H contains no tautologies and that all clauses in H are standardised apart. For if H does not have these properties we can effectively find another set of clauses which does and whose universal closure is equiconsistent with that of H.

With H we will associate an f and an Ev in such a way that H is consistent iff any solution to the resulting generalisation problem has a certain decidable property. This will complete the proof.

Let $H = \{D_1, \ldots, D_n\}$ and suppose that the predicate symbols appearing in H are $P_j (j=1,p)$. If P_j has degree m, associate with it a new predicate symbol Q_j of degree m+n+2 for j=1,p. (Notice that $n=||H||$.)

It is convenient to temporarily introduce some new syntactic notation. If $t_1, \ldots, t_{m+n+2}$ are terms and $L = (\pm)\, P_j(t_1, \ldots, t_m)$, for some j, we set

$$L[[t_{m+1}, \ldots, t_{m+n+2}]] = (\pm)\, Q_j(t_1, \ldots, t_{m+n+2}).$$

If L does not begin with some P_j, we set

$$L[[t_{m+1}, \ldots, t_{m+n+2}]] = L.$$

In the above $(\pm)\, P_j$ is either P_j or $\bar{P}_j$ and $(\pm)$ is used according to the usual conventions.

Similarly we set $C[[t_{m+1}, \ldots, t_{m+n+2}]] = \{L[[t_{m+1}, \ldots, t_{m+n+2}]] \mid L \in C\}$ for any clause C and $H'[[t_{m+1}, \ldots, t_{m+n+2}]] = \{C[[t_{m+1}, \ldots, t_{m+n+2}]] \mid C \in H'\}$, for any set, H', of clauses.

Notice that if H' does not contain any of the Q_j then $\forall H'$ is equiconsistent with $\forall\, H'[[t_{m+1}, \ldots, t_{m+n+2}]]$.

Let $x_1, x_2, y_1, y_2, u_1, u_2$ be variables not appearing in H and let P be a new n+2-ary predicate symbol.

We define auxiliary terms t_{ik} $(i=1,n;\ k=1,n)$ by $t_{ii} = f(u_2)$ and $t_{ik} = u_1$ if $k \neq i$ and set

$$H^+ = \{D_i[[x_1,f(y_1),t_{i1},\ldots,t_{in}]] \cup \{P(x_1,f(y_1),t_{i1},\ldots,t_{in})\} \mid D_i \in H\}.$$

We also set

$$H^- = \{D_i[[f(x_2),y_2,t_{i1},\ldots,t_{in}]] \cup \{\bar{P}(f(x_2),y_2,t_{i1},\ldots,t_{in})\} \mid D_i \in H\}.$$

Now $\forall(H^+ \cup H^-)$ is equiconsistent with $\forall H$. For suppose $\forall(H^+ \cup H^-)$ is consistent. Notice that for any i, $D_i[[f(x_2),f(y_1),t_{i1},\ldots,t_{in}]]$ is in $\mathcal{R}(H^+ \cup H^-)$. Therefore $\mathcal{R}(H^+ \cup H^-) \leq H[[f(x),\ldots,f(x)]]$. So $\forall H[[f(x),\ldots,f(x)]]$ is consistent and it follows, by a remark above, that $\forall H$ is.

Suppose $\forall H$ is consistent. Then $\forall H[[x_1,f(y_1),u_1,\ldots,u_n]]$ is also consistent, by the remark made above. As $H[[x_1,f(y_1),u_1,\ldots,u_n]]$ generalises H^+ and does not contain any occurrence of the predicate symbol **P**, any extension of a model of $H[[x_1,f(y_1),u_1,\ldots,u_n]]$ to an interpretation of H^+ will be a model of H^+. If we choose that extension which assigns to **P** the empty predicate, of the appropriate degree, then the extension will also be a model of H^-. Thus if $\forall H$ is consistent, so is $\forall(H^+ \cup H^-)$.

We have proved, therefore, that $\forall H$ and $\forall(H^+ \cup H^-)$ are equiconsistent.

Set $Gen_1 = \{inf(D,D') \ \tau_{D,D'} \mid D \neq D'$ and either D and D' are both in H^+ or else they are both in $H^-\}$.

The translations, $\tau_{D,D'}$ must be taken so that all the variables in Gen_1 are new and the clauses in Gen_1 are standardised apart. Notice that every clause in Gen_1 has a single occurrence of the predicate letter P.

Set $Gen_2 = \{D^* \mid D \in Gen_1$ and D^* is D except that the sign of P is changed$\}$.

Recall the γ^i_j used in theorem 3.3.3.1.4 (the representation theorem).

Let $\gamma^i = \gamma^i_{n'}$ where n' is the first integer such that $(H^+ \cup H^- \cup Gen_1 \cup Gen_2) \gamma^i_{n'}$ contains no variable symbols ($i \geq 1$). Notice that any literal has at most one occurrence in this set of clauses.

Define, $H_0 = (H^+ \cup H^-) \gamma^1 \cup (H^+ \cup H^-) \gamma^2 \cup Gen_1 \gamma^3 \cup Gen_2 \gamma^4$.

We may now define an f and an Ev with the properties promised at the beginning of the proof.

Let f be the set of literals with predicate letter P appearing in H_0.

If $L \in f$, there is, by a remark above on the properties of the γ^i, a unique clause, D, in H_0 in which L occurs.

Set $Ev(L) = \bar{M}_1 \wedge \dots \wedge \bar{M}_{k(L)}$ (where $\{M_k \mid k=1, k(L)\} = D \setminus \{L\}$).

Notice that $H_0 = \{\overline{Ev(L)} \cup \{L\} \mid L \in f\}$ and so H_0 has its usual meaning. We must now show that we have a genuine problem, that is that $\bigwedge_{j=1}^{l}(e_j \wedge f_j)$ is consistent. (l depends on $n = ||H||$, and in fact $l = 4n^2$), and H_0 contains no tautologies.

Suppose that $\bar{L}_1$ and $\bar{L}_2$ are contradictory literals in $\bigwedge_{j=1}^{l}(e_j \wedge f_j)$. From the properties of the γ^i, and the fact that all clauses in $H^+ \cup H^- \cup Gen_1$ are standardised apart, L_1 and L_2 must occur in a single clause D in H_0, and so they cannot begin with the predicate letter P.

Since H contains no tautologies, neither does $H^+ \cup H^-$ and so D is in $Gen_1 \gamma^3 \cup Gen_2 \gamma^4$. Since neither literal has predicate symbol P, we may assume, by the construction of Gen_2, that D is in $Gen_1 \gamma^3$. But if D' is in Gen_1 then D' generalises some non-tautologous clause in $H^+ \cup H^-$ and so must itself be non-tautologous. As γ^3 substitutes distinct constants for distinct variables D must also be non-tautologous, which contradicts the assumption that L_1 and L_2 are contradictory.

Therefore, $\bigwedge_{j=1}^{l}(e_j \wedge f_j)$ is consistent. Since P does not occur in H, and $\bigwedge_{j=1}^{n} e_j$ is consistent, H_0 contains no tautologies.

Let $H_{test} = H^+ \cup H^- \cup Gen_1 \gamma^3 \cup Gen_2 \gamma^4$. We are going to demonstrate that if H_{soln} is a solution to the problem defined by Ev and f then H is consistent iff $H_{soln} \sim H_{test}$. As equivalence is decidable we will then be able to tell, effectively whether H is consistent, if an algorithm for producing a solution is available. This will conclude the proof, as we remarked at the beginning. To do this we need two lemmas.

<u>Lemma 2.1</u> $\forall H$ is equiconsistent with $\forall H_{test} \wedge \bigwedge_{j=1}^{l}(e_j \wedge f_j)$.

<u>Proof</u> Suppose $\forall H$ is consistent. As $\vdash \bigwedge_{j=1}^{l}(e_j \wedge f_j) \rightarrow H_0$ and $Gen_2\, \gamma^3 \cup Gen_2\, \gamma^4 \subseteq H_0$, $\forall H_{test} \wedge \bigwedge_{j=1}^{l}(e_j \wedge f_j)$ is equiconsistent with $\forall(H^+ \cup H^-) \wedge \bigwedge_{j=1}^{l}(e_j \wedge f_j)$. Suppose then that $\forall(H^+ \cup H^-) \wedge \bigwedge_{j=1}^{l}(e_j \wedge f_j)$ is inconsistent. Then $\vdash \forall(H^+ \cup H^-) \rightarrow \neg \bigwedge_{j=1}^{l}(e_j \wedge f_j)$.

Now we have shown above that $\bigwedge_{j=1}^{l}(e_j \wedge f_j)$ is consistent. Therefore $C_0 = \{\bar{f}_j \mid j=1,l\} \cup \bigcup_{j=1}^{l}\bar{e}_j$ is not a tautology. It follows from the subsumption theorem (Lee, 1967 and Kowalski, 1970) that there is a clause C in $\mathcal{R}^{n'}(H^+ \cup H^-)$, for some n', which subsumes C_0. If C has been obtained by resolution from more than one member of $H^+ \cup H^-$, it follows from the construction of $H^+ \cup H^-$ that every literal in C will contain at least three occurrences of the unary function symbol f. Now it is impossible that C be $\emptyset$, for then $\forall(H^+ \cup H^-)$ would be inconsistent and this contradicts the fact that $\forall(H^+ \cup H^-)$ is equiconsistent with $\forall H$ taken together with the assumption that $\forall H$ is consistent. Further no literal in C_0 can contain more than two occurrences of the unary function symbol f. Therefore if C has been obtained by resolution from more than one member of $H^+ \cup H^-$, it is impossible for C to generalise C_0. This contradiction establishes the existence of a D in $H^+ \cup H^-$ such that C is in $\mathcal{R}^{n'}(\{D\})$.

Let us assume that D is in H^+. Then C will contain a positive literal L with predicate letter P containing two occurrences of the function letter f, one of which is in the second argument place of L. But a positive literal occurring in C_0 whose predicate letter is P must be the negation of a literal occurring in a clause in $H^- \gamma^1 \cup H^- \gamma^2 \cup Gen_1 \gamma^3 \cup Gen_2 \gamma^4$. Any such literal occurring in $H^- \gamma^1 \cup H^- \gamma^2 \cup Gen_1 \gamma^3$ has no occurrence of the function symbol f in its second argument place. By the construction of Gen_2, any such literal occurring in a clause in $Gen_2 \gamma^4$ will have exactly one occurrence of the function symbol f. Therefore L generalises no literal in C_0 and so $C \not\leq C_0$. This is a contradiction. If D is in H^- the contradiction is established similarly. So we have established that $\forall(H^+ \cup H^-) \wedge \bigwedge_{j=1}^{l}(e_j \wedge f_j)$ and so $\forall H_{test} \wedge \bigwedge_{j=1}^{l}(e_j \wedge f_j)$ is consistent.

Suppose $\forall H_{test} \wedge \bigwedge_{j=1}^{l}(e_j \wedge f_j)$ is consistent. We see, successively, that so are $\forall H_{test}$, $\forall(H^+ \cup H^-)$ and $\forall H$. This concludes the proof of the lemma.

Lemma 2.2 Suppose H_{soln} is a solution to the problem determined by Ev and f, and that $H_{soln} \subseteq \mathcal{G}(H_0)$. Then, $Gen_1 \gamma^3 \cup Gen_2 \gamma^4 \subseteq H_{soln} \subseteq H_0 \cup H_{test}$. (We may assume that $H_{test} \cup H_0 \subseteq \mathcal{G}(H_0)$.)

Proof For this lemma we need the easily proved fact that if $C \leq D\gamma^i_j$ for some i, j and C contains none of the constants a_{ij}, for any $i \geq 1$ and $j \geq 1$, then $C \leq D$. This fact is implicit in the proof of the representation theorem, theorem 3.3.3.1.4.

Suppose **E** is in $Gen_1 \, \gamma^3$ but not in H_{soln}. Now as H_{soln} is a solution $H_{soln} \leq H_0 \supsetneq \{E\}$. So there is a clause, C, in H_{soln} which generalises **E** and, since $H_{soln} \subseteq \mathcal{G}(H_0)$, C must generalise some other member of H_0 and so C does not contain any of the constants a_{ij} for any $i \geq 1$ and $j \geq 1$.

For some D, D' in H^+, $E = \inf\{D,D'\} \, \tau_{D,D'} \, \gamma^3$. Therefore $C \leq \inf\{D,D'\} \, \tau_{D,D'}$, using the fact given at the beginning of the proof. We can find an f_j in f, such that $E^* = \bar{e}_j \cup \{f_j\}$ and so as $\inf\{D,D'\} \, \tau_{D,D'}$ is inconsistent with $e_j \wedge f_j$, so is C and so therefore is H_{soln}. This contradiction establishes the fact that $Gen_1 \, \gamma^3 \subseteq H_{soln}$.

Similarly we can show that $Gen_2 \, \gamma^4 \subseteq H_{soln}$.

Any clause in $\mathcal{G}(H_0)$ which is not in $H_{test} \cup H_0$ must generalise a member of $Gen_1 \cup Gen_2$, by applications of one or both of the representation theorem and the remark made at the beginning of the proof. This clause is inconsistent with the evidence, as was shown above, and so cannot be a member of H_{soln}. This concludes the proof.

We may now finish the proof of theorem 2. We have to show that if H_{soln} is a solution to the problem defined by Ev and f then H is consistent iff $H_{soln} \sim H_{test}$. By theorem 1, we may assume that $H_{soln} \subseteq \mathcal{G}(H_0)$. We may also assume that $H_{test} \cup H_0 \subseteq \mathcal{G}(H_0)$.

Suppose that H is consistent, that H_{soln} is a solution and that $H_{soln} \not\sim H_{test}$. Then by lemma 2.2, $H_{soln} \subseteq H_{test} \cup H_0$. Therefore $H_{test} \leq H_{test} \cup H_0 \leq H_{soln}$. So $H_{soln} \not\leq H_{test}$ and there is a C_{test} in H_{test} such that no clause in H_{soln} generalises C_{test}. Since $Gen_1\ \gamma^3 \cup Gen_2\ \gamma^4 \subseteq H_{soln}$ by lemma 2.2, C_{test} is in $H^+ \cup H^-$. Now there are clauses C_1, C_2 in H_{soln} such that $C_1 \leq C_{test}\ \gamma^1$ and $C_2 \leq C_{test}\ \gamma^2$ since $H_{soln} \leq H_0$.

C_1 cannot generalise any clause in H_0 other than $C_{test}\ \gamma^1$, for then it generalises C_{test}, by the remark made at the beginning of the proof of lemma 2.2. Therefore $C_1 = C_{test}\ \gamma^1$ and similarly $C_2 = C_{test}\ \gamma^2 \neq C_1$. Let $H'_{soln} = (H_{soln} \cup \{C_{test}\}) \setminus \{C_1, C_2\}$.

Then as $H_{test} \leq H'_{soln}$, $\forall H'_{soln} \wedge \bigwedge_{j=1}^{l}(e_j \wedge f_j)$ is consistent by lemma 2.1. Further $H'_{soln} \leq H_{soln} \leq H_0$, $H'_{soln} \preceq_c H_{soln}$ but $H_{soln} \not\preceq_c H'_{soln}$. This contradicts the fact that H_{soln} is a solution.

We have shown that if $\forall$H is consistent, $H_{soln} \sim H_{test}$.

Suppose $\forall$H is inconsistent and $H_{soln} \sim H_{test}$. Then $\forall H_{soln} \wedge \bigwedge_{j=1}^{l}(e_j \wedge f_j)$ is equiconsistent with $\forall H_{test} \wedge \bigwedge_{j=1}^{l}(e_j \wedge f_j)$ which is inconsistent by lemma 2.1 since $\forall$H is. This certainly contradicts the fact that H_{soln} is a solution, which concludes the proof.

The same result can be obtained for a variety of other $\preceq$'s if we change the construction of f and Ev slightly.

The construction proceeds as above until Gen_1 and Gen_2 have been defined. Then one sets:

$$H_0^m = \bigcup_{k=1}^{m} (H^+ \cup H^-)\ \gamma^k \cup \mathrm{Gen}_1\ \gamma^{m+1} \cup \mathrm{Gen}_2\ \gamma^{m+2} \quad (m \geq 2).$$

The construction given above corresponds to the case where m=2. One may then define f^m and Ev^m analogously to the above construction and verify that $\bigwedge_{j=1}^{l_m}(E_j^m \wedge f_j^m)$ is consistent. (Here, $l_m = 4n^2 + 2n(m-2)$.)

One can then define H_{test}^m and prove:

1 $\forall H$ is equiconsistent with $\forall H_{test}^m \wedge \bigwedge_{j=1}^{l_m}(e_j^m \wedge f_j^m)$

2 If $\forall H$ is inconsistent, $\forall H_{soln} \wedge \bigwedge_{j=1}^{l_m}(E_j^m \wedge f_j^m)$ is consistent and $H_{soln} \leq H_0^m$ then $||H_{soln}|| \geq m$.

The proof of 1 is analogous to that of lemma 2.1. To see that $||H_{soln}|| \geq m$ under the assumptions of 2 suppose otherwise and choose a clause C in $H^+ \cup H^-$. There are clauses $C_1, \ldots, C_m$ in H_{soln} such that $C_k \leq C\ \gamma^k$ (k=1,m) as $H_{soln} \leq H_0^m$. Since $||H_{soln}|| < m$, two of these are the same and so there is a C_k in H_{soln} such that $C_k \leq C$, by the representation theorem. Hence $H_{soln} \leq H^+ \cup H^-$, and so $H_{soln} \leq H^+ \cup H^- \cup H_0^m \leq H_{test}^m$. But $\forall H_{test}^m \wedge \bigwedge_{j=1}^{l_m}(e_j^m \wedge f_j^m)$ is inconsistent as $\forall H$ is, using 1. Therefore $\forall H_{soln}^m \wedge \bigwedge_{j=1}^{l_m}(e_j^m \wedge f_j^m)$ is also inconsistent which contradicts one of the assumptions of 2, which therefore, shows that $||H_{soln}|| \geq m$.

Recall the definitions of $\prec_1$, $\prec_1'$, $\prec_s$, and $\prec_s'$ given in chapter 2.

Corollary 2 Let $\mathcal{G}$ be one of $\mathcal{G}_c$, $\mathcal{G}_{cp}$, $\mathcal{G}_1$, $\mathcal{G}_{1'}$, $\mathcal{G}_s$ or $\mathcal{G}_{s'}$. There is no algorithm which will, given any f and **Ev** produce a solution to the resulting generalisation problem, although such a solution exists.

Proof Suppose H is a set of clauses as described at the beginning of the proof of theorem 2. We need merely show an effective way, given an algorithm which always produces a solution, of deciding whether $\forall H$ is consistent.

Suppose $\mathcal{G}$ is $\mathcal{G}_c$. There is always a solution, given m, to the problem defined by Ev^m and f^m, since the complexities of any solution must lie between zero and $||H_0^m||$ as H_0^m satisfies conditions 1 and 2 for being a solution. Choose $m > ||H^m_{test}||$. Find a solution, H_{soln}, to the problem defined by $\mathbf{Ev}^m$ and f^m. If $\forall H$ is inconsistent then $||H_{soln}|| \geq m$ by $\underline{2}$ above. If $\forall H$ is consistent, so is $\forall H^m_{test} \wedge \bigwedge_{j=1}^{m} (e^m_j \wedge f^m_j)$ and as $H^m_{test} \leq H^m_0$, $||H_{soln}|| \leq ||H^m_{test}|| < m$.

Therefore $\forall H$ is consistent iff $||H_{soln}|| < m$, which concludes the proof for $\mathcal{G} = \mathcal{G}_c$.

The proofs for $\mathcal{G}_1$, $\mathcal{G}_{1'}$, $\mathcal{G}_s$ and $\mathcal{G}_{s'}$ are all similar to the above. In each case if $\forall H$ is consistent, H^m_{test} provides an upper bound for the measure being used. If $\forall H$ is inconsistent one obtains a lower bound, which increases to infinity, with m, on the measure of a solution. As one sees that there must be a solution in each case, one can tell in each case whether $\forall H$ is consistent.

When $\mathcal{G}$ is $\mathcal{G}_{cp}$ we see that there is always a solution to the problem defined by $\mathbf{Ev}^m$ and f^m. For H_0^m satisfies conditions 1 and 2 for being a solution. Hence the complexity of any solution must be less than or equal to $||H_0^m||$ and so the power must be less than or equal to $||H_0^m||^2$. These bounds show that a solution exists. If there is an algorithm for producing a solution then the same algorithm will produce a solution when $\prec$ is $\prec_c$, for $\prec_{cp}$ minimal implies that $\prec_c$ is minimal. Together with the result for $\prec_c$, this implies the result for $\prec_{cp}$. This concludes the proof.

This technique will establish the result for any reasonable way of combining integer-valued complexity measures. When the result has been established for some $\prec_1$ then it follows at once for any lexicographic refinement of $\prec_1$ which always allows solutions. (Compare the proof of the last part of the corollary).

This unsolvability result seems rather paradoxical, since we seem to have little, if any, trouble with consistency, when forming generalisations either in ordinary or scientific life. Furthermore, there can no longer be any hope of a general method. Everything may be reconciled, however, by postulating that our set of generalisation problems is too inclusive; we have not succeeded in restricting ourselves to "ordinary" generalisation problems. We will not try to formulate restrictions which give exactly the ordinary problems and no others. To show that some proposed solution captures the generalisation problems faced by any human being could no doubt involve us in uninviting problems of psychology. Similarly, to capture ordinary scientific classificatory generalisation problems may involve comparisons with, say, 19th century zoological practise, another uninviting task.

What we will do is indicate the kind of restrictions that may be formulated, with a view to giving some that are potentially useful for work in A.I.

One could make restrictions on the vocabulary used. Restrictions on the possible predicate symbols and function symbols, could rule out the type of unsolvability proof used above. One extreme possibility is to require that the only function symbols are constant ones. Then every problem where $\rightsquigarrow$ is $\rightsquigarrow_{cpg}$ and Th and Irr are arbitrary sets of clauses (with no function symbols, other than constant ones) is solvable, by the discussion after theorem 1. We would like to single out one particularly simple subcase. Suppose Th is empty and every member of the (finite) Herbrand base of $Th \wedge Irr \wedge \bigwedge_{i=1}^{n}(e_i \wedge f_i)$ or its negation is implied by $Th \wedge Irr \wedge \bigwedge_{i=1}^{n}(e_i \wedge f_i)$. In other words, $Th \wedge Irr \wedge \bigwedge_{i=1}^{n}(e_i \wedge f_i)$ has exactly one Herbrand model. Let $C = \{\bar{L} \mid Th \wedge Irr \wedge \bigwedge_{i=1}^{n}(e_i \wedge f_i)$ implies L and L is in the Herbrand base of $Th \wedge Irr \wedge \bigwedge_{i=1}^{n}(e_i \wedge f_i)\}$. Then, if $H \subseteq \{\inf H' \mid H' \subseteq H_0, H \neq \emptyset\}$, $\forall H \wedge Th \wedge Irr \wedge \bigwedge_{i=1}^{n}(e_i \wedge f_i)$ is consistent iff $H \not\leq \{C\}$. In other words, we test H against the unique Herbrand model.

Another kind of restriction is to fix on a specific Th. We might require that Th always contains a "basic" scientific theory, which includes some universal assumptions (for example meaning postulates or principles of causality). One extreme is to demand that $Th \wedge Irr$ is a decidable theory, or, at any rate, that one can decide the sentences produced by the process of taking relative infima. In this case it follows from the discussion of theorem 1 that an arbitrary problem is decidable if $\preceq$ is equal to $\preceq_{cpg}$ and the vocabulary of $\bigwedge_{i=1}^{n}(e_i \wedge f_i)$ is a subset of that of Th $\wedge$ Irr.

Finally one might place restrictions on the kind of data generalised. Most psychological and taxonomic generalisation problems (Hunt, Marin and Stone, 1966) form classes from attributes, which are maps from objects to finite sets, or to real numbers.

Suppose, then, that some solvable case has been found, when $\preceq$ is $\preceq_{cpg}$. We give some properties of solutions, independent of which subcase is being considered, but which can be of help when looking for solutions.

First we show that solutions are irredundant. H is said to be <u>irredundant</u> iff:

1) If C is in H then Explainset(C)

$$\not\subseteq \bigcup_{D \in H} \text{Explainset}(D).$$

2) If C is in H and D is in H_0 then either $C \leq D$ (Th) or Th $\wedge$ Irr $\wedge \forall H \wedge \forall \text{inf}_{Th}\{C,D\} \wedge \bigwedge_{i=1}^{n}(e_i \wedge f_i)$ is inconsistent.

Intuitively, H is irredundant if it cannot be improved (in the $\preceq_{cpg}$ ordering) by removing clauses or generalising clauses, without violating one of the first two conditions for it to be a solution.

Theorem 3 Any solution is irredundant.

Proof Suppose condition 1 for H to be irredundant fails. Then for some C in H, $H' = H \setminus \{C\} < H_0$ (Th) iff $H \leq H_0$ (Th). Then Complexity(H) > Complexity(H').

Suppose H fails condition 2. Let $H' = (H \setminus \{C\}) \cup \{\inf\{C,D\}\}$ where D is that member of H_0 determined by the failure. Then Complexity(H') = Complexity(H) but Power(H') > Power(H).

In both cases H' satisfies the first two conditions for being a solution iff H does. So H cannot be a solution.

It would, evidently, be helpful if one could divide the solution of the formal problem into the solution of several smaller problems, or at any rate to restrict the search to a smaller part of $\mathcal{G}_{Th}(H_0)$ the set of generalisations of H_0. Our next few lemmas are a step in this direction.

Under some hypotheses, the search space, $\mathcal{G}_{Th}(H_0)$ can be divided into several positive and negative parts.

Let $f^{+P} = \{f_i \in f \mid f_i$ is positive and has predicate symbol $P\}$

$f^{-P} = \{f_i \in f \mid f_i$ is negative and has predicate symbol $P\}$

$H_0^{+P} = \{C_i \in H_0 \mid f_i \in f^{+P}\}$

$H_0^{-P} = \{C_i \in H_0 \mid f_i \in f^{-P}\}$.

Let $P_j (j=1,m)$ be the predicate symbols appearing in f.

We need two lemmas.

Lemma 1 Let P be a predicate symbol not occurring in Th. Suppose $E \leq C$ (Th) and it is not the case that $\vdash_{Th} \forall C$ and P only occurs positively (negatively) in C. Then P can only occur positively (negatively) in E.

Proof Suppose P only occurs positively in C.

Suppose P occurs negatively in E. For some D, $E \leq D$ and $\vdash_{Th} D \equiv C$. P must occur negatively in D.

Let $\mathfrak{A}$ be a model of Th and $\forall D$. Define $\mathfrak{A}'$ to be that structure which is identical to $\mathfrak{A}$ except that it assigns $\emptyset$ to P. Then $\mathfrak{A}'$ is also a model of Th since P does not occur in Th, and of D since P occurs negatively in D. Therefore $\mathfrak{A}'$ is a model of $\forall C$ as $\vdash_{Th} D \equiv C$ implies that $\vdash_{Th} \forall D \equiv \forall C$. Let $C = C_1 \cup C_2$ where C_2 is the set of literals in C whose predicate symbol is P. As $\mathfrak{A}'$ assigns $\emptyset$ to P and is a model of $\forall C$, $\mathfrak{A}'$ is a model of $\forall C_1$. Therefore so is $\mathfrak{A}$ and so $\mathfrak{A}$ is a model of $\forall C$. As $\mathfrak{A}$ is an arbitrary model of Th, this contradicts the fact that $\forall C$ is not a consequence of Th.

When P only occurs negatively in C, the proof is similar.

Lemma 2 Let P be a predicate symbol not occurring in Th. Suppose $C_1 \cup C_2$ is a ground clause where C_1 contains no positive (negative) occurrences of P and every literal in C_2 contains a positive (negative) occurrence of P. Suppose, further that $C_1 \cup C_2$ is not deduceable from Th. If E contains no positive (negative) occurrence of P and $E \leq C_1 \cup C_2$ (Th) then $E \leq C_1$ (Th).

Proof We prove only the positive part of the theorem. The negative part follows similarly. For some E', and σ, $E' = E\sigma$ and $\vdash_{Th} E' \to (C_1 \cup C_2)$. E' contains no positive occurrence of P. If we show that $\vdash_{Th} E' \to C_1$, then it will follow at once that $E \leq C_1$ (Th). Let Th' be the set of Skolemisations of members of Th. It is well-known that Th' is a conservative extension of Th. Suppose that $E' \to C_1$ is not deducible from Th. Then neither is it deducible from Th'. Therefore there is a Herbrand model $\mathcal{A}$, of Th' and a substitution σ' such that $(E' \to C_1)\sigma'$ is ground and false in $\mathcal{A}$. Therefore $E'\sigma'$ is true in $\mathcal{A}$ and $C_1\sigma' = C_1$ is false. Since $\vdash_{Th} E' \to (C_1 \cup C_2)$, $\vdash_{Th'} E' \to (C_1 \cup C_2)$. Therefore $\vdash_{Th'} E'\sigma' \to (C_1\sigma' \cup C_2\sigma')$ and so $C_2\sigma' (= C_2)$ must be true in $\mathcal{A}$. Let $\mathcal{A}'$ be obtained from $\mathcal{A}$ by stipulating:

1) If $L \in C_2$, L is false in $\mathcal{A}'$.

2) If $L \notin C_2$, L has the same value in $\mathcal{A}'$ as in $\mathcal{A}$.

Since C_2 is a set of positive literals with predicate symbol P and the only possible literals in $E'\sigma'$ with predicate symbol P are negative, $E'\sigma'$ must also be true in $\mathcal{A}'$. If a literal, L, in C_1 changes truth value then $\bar{L} \in C_2$ and so $\vdash C_1 \cup C_2$ which contradicts the assumption that $C_1 \cup C_2$ is not deducible from Th. Hence no literal in C_1 changes truth value and so C_1 is false in $\mathcal{A}'$. By definition, C_2 is false in $\mathcal{A}'$. Therefore $(E' \rightarrow C_1 \cup C_2)$ is false in $\mathcal{A}'$ which contradicts the fact that $\vdash_{Th} E' \rightarrow (C_1 \cup C_2)$. Therefore $E' \rightarrow C_1$ is deducible from Th, which completes the proof.

We are now in a position to prove two theorems on division of the search space.

<u>Theorem 4</u> Suppose that no P_j appears in Th, and that $P_1 \ldots P_{m'}$ appear only positively in the e_i if at all $(i=1,n)$ and $P_{m'+1} \ldots P_m$ appear only negatively in the e_i if at all $(i=1,n)$.

Then any solution, $H \subseteq \mathcal{J}_{Th}(H_0)$ is equal to $H^+ \cup H^-$ where

$$H^+ \subseteq \bigcup_{j=1}^{m'} \mathcal{J}_{Th}(H_0^{+P}j) \cup \bigcup_{j=m'+1}^{m} \mathcal{J}_{Th}(H_0^{-P}j) = \mathcal{J}^+_{Th}(H_0), \text{ say, and}$$

$$H^- \subseteq \bigcup_{j=1}^{m'} \mathcal{J}_{Th}(H_0^{-P}j) \cup \bigcup_{j=m'+1}^{m} \mathcal{J}_{Th}(H_0^{+P}j) = \mathcal{J}^-_{Th}(H_0) \text{ say.}$$

<u>Proof</u> We consider first the case where all the P_j appear positively, that is, $m' = m$ and then $\mathcal{J}^+_{Th}(H_0) = \bigcup_j (H_0^{+P}j)$ and similarly for $\mathcal{J}^-_{Th}(H_0)$. It follows from theorem 1 that it is only necessary to show that if $C \in \mathcal{J}_{Th}(H_0) \setminus (\mathcal{J}_{Th}(H_0^+) \cup \mathcal{J}_{Th}(H_0^-))$, then $\forall C \wedge Th \wedge Irr \wedge \bigwedge_i (e_i \wedge f_i)$ is inconsistent.

Such a C is of the form $\inf_{Th}\{D, E\}$ where there are predicate symbols P_j and $P_{j'}$ such that D contains a positive occurrence of P_j and perhaps negative ones of P_j and $P_{j'}$ and E contains a negative occurrence of $P_{j'}$ and perhaps negative ones of P_j and $P_{j'}$. It is possible that $j=j'$. From lemma 1 and $C \leq D$ (Th), C contains only negative occurrences of P_j. Further, D is of the form $\bar{e}_i \cup \{f_i\}$ for some i, where f_i contains a positive occurrence of P_j. Hence by lemma 2, $C \leq \bar{e}_i$ (Th) for some i and so C is indeed inconsistent with $\bigwedge_i (e_i \wedge f_i) \wedge$ Th.

The general case may be proved either by a similar detailed examination of another three cases or else by renaming the predicate symbols $P_j (m' < j \leq m)$ as $\neg Q_j (m' < j \leq m)$ so that the general case reduces to the above using the easily proven fact that, with the evident definitions, the solutions to the renaming of a formal problem, of the class considered, are the renamings of the solutions to the formal problem. This concludes the proof.

Under the assumptions of theorem 4, we may consider the predicate symbols P_j as divided into three classes viz. $P_1, \ldots, P_{m_1}$, the symbols whose occurrences in the e_i are all positive and which do have at least one such occurrence and $P_{m_1+1} \ldots P_{m_2}$, the symbols whose occurrences in the e_i are all negative and which do have at least one such occurrence and the $P_{m_2+1} \ldots P_m$, the symbols which have no occurrence in any e_i, then we see from the argument used in the proof of theorem 4 that if D is in H_0, has an occurrence of $P_{m'} (m_2 < m' \leq m)$ and $m'' \neq m'$, and $E \in H_0^{+P_{m''}} \cup H_0^{-P_{m''}}$ then $\inf_{Th}\{C,D\}$ is inconsistent with Th $\wedge \bigwedge_i (e_i \wedge f_i)$. Combining this observation with theorem 4 we obtain:

<u>Corollary 3</u> If the P_j are categorised as above, and no P_j appears in Th, then any solution H is equal to $H^+ \cup H^- \cup \bigcup_{j=m_1+1}^{m_2} H^{+P_j} \cup \bigcup_{j=m_2+1}^{m} H^{-P_j}$, where:

$$H^{+} \subseteq \bigcup_{j=1}^{m_1} \mathcal{J}_{Th}(H_0^{+P_j}) \cup \bigcup_{j=m_1+1}^{m_2} \mathcal{J}_{Th}(H_0^{-P_j}),$$

$$H^{-} \subseteq \bigcup_{j=1}^{m_1} \mathcal{J}_{Th}(H_0^{-P_j}) \cup \bigcup_{j=m_1+1}^{m_2} \mathcal{J}_{Th}(H_0^{+P_j}),$$

$$H^{+P_j} \subseteq \mathcal{J}_{Th}(H_0^{+P_j}) \quad (m_2 < j \leq m),$$

$$H^{-P_j} \subseteq \mathcal{J}_{Th}(H_0^{-P_j}) \quad (m_2 < j \leq m).$$