An Illative Theory of Relations

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1 Introduction

In a previous paper an intensional theory of relations was formulated [Plo90]. It was intended as a formalisation of some of the ideas of Situation Theory concerning relations, assignments, states-of-affairs and facts; it was hoped it could serve as a springboard for formalising other notions especially those concerning situations and propositions. The method chosen was to present a formal theory in a variation of classical first-order logic allowing terms with bound variables (and also quantification over function variables, but no axioms of choice).

One infelicity of this work was that not every formula corresponded to a state-of-affairs according to a certain notion of internal definability; indeed one could show such correspondences inconsistent with the theory. Jon Barwise suggested changing the logic to allow partial predicates and partial functions. The idea of using a 3-valued approach is an old one: see [Fef84] for general information about results closely related to those given below. Another infelicity, pointed out by Peter Aczel, was that the logic formalised part of the metalanguage of the structures concerned, and these structures already had their own notion of proposition or, better, state-of-affairs. This meant that there was a repetition of logical apparatus; for example the logical conjunction was replicated by a conjunction for soas.

In this paper we present a non-standard logic for our structures. It is a type-free intensional logic, and is also in the tradition of Curry's illative logic [HS86]; see also [AczN, FM87, Smi84, MA88]. The logic has two judgments: that an object is a fact and that an object is a state-of-affairs (cf. truth and proposition). Objects are given using a variant of the traditional situation theory notation which is more standard, logically speaking, with explicit negation and quantification (see also [Bar87]). No metalinguistic apparatus is employed.

In Section 2 we present such illative logic in general. Consistency and completeness theorems hold. Such logics seem appropriate when formalising structures of the kind we consider, which are very close to Aczel's Frege structures [Acz80]. A theory of relations is given in Section 3; it is intended as an illative replacement for the theory in [Plo90]. Since the logical apparatus is built-in the theory looks a good deal simpler than the previous one. In Section 4 we compare illative theories to standard ones and show every illative theory can be conservatively extended to a standard one (although we do not settle the question of the relation of the present theory to the previous one). One can interpret this result in favour of the illative approach: the price paid for the use of a standard theory gains no more power. In Section 5 we (partially) follow Barwise's suggestion and consider a general 3-valued logic with partial predicates. Again we have a conservative extension result over illative logic. What is more we get that every formula is internally definable (actually, in a stronger sense than conjectured by Barwise). We interpret this result as even more favourable to illative logic: not only can one not prove more, one cannot even express more. It would be very interesting to take up the other suggestion of partial functions, especially considering the kind of model-theoretic work in [Bar87].

2 Illative Logic

We need a language of *terms* over a *signature* which consists of a set of *functional symbols*, F, each of given arity $((n_1, \ldots, n_k), n)$ with $n, n_i \geq 0$. The symbols $\perp$ (of arity $((), 0)$), $\wedge, \vee, |, \supset, =$ (of arity $((), 2)$) and $\exists$, $\forall$ (of arity $((1),0)$), are always included. Then the *terms* and *functional terms* on the signature are generated from a given countable set of variables x by stipulating that

1. any variable is a term
2. if F is a functional symbol of arity $((n_1, \ldots, n_k), n)$ and $ft_1, \ldots, ft_k$ are function terms of arities $n_1, \ldots, n_k$ and $t_1, \ldots, t_n$ are terms then $F(ft_1, \ldots, ft_k,\, t_1, \ldots, t_n)$ is a term
3. if t is a term and $x_1, \ldots, x_n$ are distinct variables then $(x_1, \ldots, x_n)t$ is a function term of arity n $(n \geq 0)$

In the last clause $x_1, \ldots, x_n$ bind any free occurrences in t. We do not distinguish α-equivalent terms, by which is meant terms equivalent up to renaming of bound variables. We employ infix and other notational devices as appropriate. For example, we write $\varphi \wedge \psi$ rather than $\wedge(\varphi, \psi)$ and $\exists x\varphi$ rather than $\exists((x)\varphi)$.

Now we explain *judgments*, *sequents* and *rules*. A judgment has either of two forms: φ or $\mathbf{S}\varphi$ where φ is a term (read the first as "φ is true" and the second as "φ is a state-of-affairs"). A sequent has the form $\Gamma \Rightarrow J$ where J is a judgment and Γ is a finite set of judgments. An n-ary rule is a set of $(n+1)$tuples of sequents and is usually given in a schematic form

$$\frac{seq_1, \ldots, seq_n}{seq}$$

where schematic variables and side-conditions may be employed. We also use the form

$$\frac{\begin{matrix} (\Gamma_1) & & (\Gamma_n) \\ J_1 & \ldots & J_n \end{matrix}}{J}$$

meaning seq_i is $\Gamma, \Gamma_i \Rightarrow J_i$ and seq is $\Gamma \Rightarrow J$ (comma means union); empty Γ_i are omitted. The horizontal line is omitted when $n = 0$, in either case.

Structural Rules

Reflexivity $\Gamma, J \Rightarrow J$

Weakening $\dfrac{\Gamma \Rightarrow J}{\Gamma, \triangle \Rightarrow J}$

Transitivity $\dfrac{\Gamma \Rightarrow J' \ (J' \ \texttt{in} \ \Gamma'), \Gamma' \Rightarrow J''}{\Gamma \Rightarrow J''}$

Substitution $\dfrac{\Gamma[x_1, \ldots, x_n] \Rightarrow J[x_1, \ldots, x_n]}{\Gamma[t_1, \ldots, t_n] \Rightarrow J[t_1, \ldots, t_n]}$

Judgments $\dfrac{\varphi}{\mathbf{S}\varphi}$

Absurdity $(\mathbf{S}I \perp)$ $\mathbf{S} \perp$

$(E \perp)$ $\dfrac{\perp}{J}$

Conjunction $(\mathbf{S}I\wedge)$ $\dfrac{\mathbf{S}\varphi \ \ \mathbf{S}\psi}{\mathbf{S}(\varphi \wedge \psi)}$

$(\mathbf{S}E\wedge)$ $\dfrac{\mathbf{S}(\varphi \wedge \psi)}{\mathbf{S}\varphi, \mathbf{S}\psi}$ (meaning two rules)

$(I\wedge)$ $\dfrac{\varphi \ \ \psi}{\varphi \wedge \psi}$

$(E\wedge)$ $\dfrac{\varphi \wedge \psi}{\varphi, \psi}$ (meaning two rules)

Disjunction $(\mathbf{S}I\vee)$ $\dfrac{\mathbf{S}\varphi \ \ \mathbf{S}\psi}{\mathbf{S}(\varphi \vee \psi)}$

$(\mathbf{S}E\vee)$ $\dfrac{\mathbf{S}(\varphi \vee \psi)}{\mathbf{S}\varphi, \mathbf{S}\psi}$ (meaning two rules)

$(I\vee)$ 1. $\dfrac{\mathbf{S}(\varphi \vee \psi) \ \ \varphi}{\varphi \vee \psi}$ 2. $\dfrac{\mathbf{S}(\varphi \vee \psi) \ \ \psi}{\varphi \vee \psi}$

$$(E\vee) \qquad \frac{\varphi \vee \psi \quad \begin{matrix}(\varphi)\\ J\end{matrix} \quad \begin{matrix}(\psi)\\ J\end{matrix}}{J}$$

Restriction

$$(\mathbf{S}I\ |) \qquad \frac{\mathbf{S}\varphi \quad \psi}{\mathbf{S}(\varphi \mid \psi)}$$

$$(\mathbf{S}E\ |) \qquad \frac{\mathbf{S}(\varphi \mid \psi)}{\mathbf{S}\varphi, \mathbf{S}\psi} \text{ (meaning two rules)}$$

$$(I\ |) \qquad \frac{\varphi \quad \psi}{\varphi \mid \psi}$$

$$(E\ |) \qquad \frac{\varphi \mid \psi}{\varphi, \psi} \text{ (meaning two rules)}$$

Sequential Implication

$$(\mathbf{S}I \supset) \qquad \frac{\mathbf{S}\varphi \quad \begin{matrix}(\varphi)\\ \mathbf{S}\psi\end{matrix}}{\mathbf{S}(\varphi \supset \psi)}$$

$$(\mathbf{S}E \supset) \qquad 1.\ \frac{\mathbf{S}(\varphi \supset \psi)}{\mathbf{S}\varphi} \qquad 2.\ \frac{\mathbf{S}(\varphi \supset \psi) \quad \varphi}{\mathbf{S}\psi}$$

$$(I \supset) \qquad \frac{\mathbf{S}\varphi \quad \begin{matrix}(\varphi)\\ \psi\end{matrix}}{\varphi \supset \psi}$$

$$(E \supset) \qquad \frac{\varphi \supset \psi \quad \varphi}{\psi}$$

Law of the Excluded Middle

$$\frac{\mathbf{S}\varphi}{\varphi \vee \neg\varphi} \text{ (where } \neg\varphi =_{def} \varphi \supset \perp)$$

Existential Quantification

$$(\mathbf{S}\exists I) \qquad \frac{\Gamma \Rightarrow \mathbf{S}\varphi[x]}{\Gamma \Rightarrow \mathbf{S}(\exists x \varphi[x])} \ (x \text{ not free in } \Gamma)$$

$$(\mathbf{S}\exists E) \qquad \frac{\mathbf{S}(\exists x\varphi[x])}{\mathbf{S}\varphi[t]}$$

$$(\exists I) \qquad \frac{\mathbf{S}(\exists x\varphi[x]) \;\; \varphi[t]}{\exists x\varphi[x]}$$

$$(\exists E) \qquad \frac{\Gamma \Rightarrow \exists x\varphi[x] \quad \Gamma, \varphi[x] \Rightarrow J}{\Gamma \Rightarrow J} \; (x \text{ not free in } \Gamma \text{ or } J\ )$$

Universal Quantification

$$(\mathbf{S}\forall I) \qquad \frac{\Gamma \Rightarrow \mathbf{S}\varphi[x]}{\Gamma \Rightarrow \mathbf{S}(\forall x\varphi[x])} \; (x \text{ not free in } \Gamma)$$

$$(\mathbf{S}\forall E) \qquad \frac{\mathbf{S}(\forall x\ \varphi[x])}{\mathbf{S}\varphi[t]}$$

$$(\forall I) \qquad \frac{\Gamma \Rightarrow \varphi[x]}{\Gamma \Rightarrow \forall x\varphi[x]} \; (x \text{ not free in } \Gamma)$$

$$(\forall E) \qquad \frac{\forall x\varphi[x]}{\varphi[t]}$$

Equality

Totality $\quad \mathbf{S}(t = u)$

Reflexivity $\quad t = t$

Symmetry $\quad \dfrac{t = u}{u = t}$

Transitivity $\quad \dfrac{t = u \quad u = v}{t = v}$

Congruence $\quad \dfrac{ft_1 = ft'_1 \ldots ft_k = ft'_k \;\; t_1 = t'_1 \ldots t_n = t'_n}{F(ft_1, \ldots, ft_k, t_1, \ldots, t_n) = F(ft'_1, \ldots, ft'_k, t'_1, \ldots, t'_n)}$

(Here the equality judgments between function terms are abbreviations: $(x_1, \ldots, x_n)t = (x_1, \ldots, x_n)t'$ abbreviates $\forall x_1 \ldots \forall x_n \;\; t = t'$).

Judgment Congruence $\quad$ 1. $\dfrac{\mathbf{S}\varphi \;\; \varphi = \psi}{\mathbf{S}\psi}$

2. $\dfrac{\varphi \quad \varphi = \psi}{\psi}$

Finally, a *proof* of a sequent *seq* from a set T of sequents is a list $seq_1, \ldots, seq_n$ with $seq = seq_n$ and where each seq_i either is in T or follows from the previous sequents by an application of a rule: we write $T \vdash seq$ when there is a proof of *seq* from T.

Now we turn to semantics and explain *interpretations* which are minor variants of Aczel's Frege structures and have the form

$$\mathcal{M} = \langle \langle Ob, \mathcal{F}_n \rangle_{n \geq 0}, Soa, Fact,\ H \rangle$$

where $\langle Ob,\mathcal{F}_n \rangle_{n \geq 0}$ is an *explicitly closed family* in Aczel's sense, where $Fact \subset Soa \subset Ob$ and where for any functional symbol F which has arity $((n_1, \ldots, n_k), n)$, $H(F)$: $\mathcal{F}_{n_1} \times \ldots \times \mathcal{F}_{n_k} \times Ob^n \rightarrow Ob$ is an $\mathcal{F}$-functional in (a minor variant of) the sense of Aczel. Furthermore the following *logical schemas* hold (writing $F^{\mathcal{M}}$ for $H(F)$):

Absurdity	$\bot^{\mathcal{M}}$ is in *Soa* but not *Fact*.
Conjunction	1. $a \wedge^{\mathcal{M}} b$ is in *Soa* iff a and b are.
	2. $a \wedge^{\mathcal{M}} b$ is in *Fact* iff a and b are.
Disjunction	1. $a \vee^{\mathcal{M}} b$ is in *Soa* iff a and b are.
	2. $a \vee^{\mathcal{M}} b$ is in *Fact* iff it is in *Soa* and either a or b is in *Fact*.
Restriction	1. $a \mid^{\mathcal{M}} b$ is in *Soa* iff a is and b is in *Fact*.
	2. $a \mid^{\mathcal{M}} b$ is in *Fact* iff a and b are.
Sequential Implication	1. $a \supset^{\mathcal{M}} b$ is in *Soa* iff a is and also if a is in *Fact* then b is in *Soa*.
	2. $a \supset^{\mathcal{M}} b$ is in *Fact*, iff it is in *Soa* and if a is in *Fact* then so is b.
Existential Quantification	1. For any f in $\mathcal{F}_1$, $\exists^{\mathcal{M}}(f)$ is in *Soa* iff every $f(a)$ is.
	2. For any f in $\mathcal{F}_1, \exists^{\mathcal{M}}(f)$ is in *Fact* iff it is in *Soa* and some $f(a)$ is.
Universal Quantification	1. For any f in $\mathcal{F}_1$, $\forall^{\mathcal{M}}(f)$ is in *Soa* iff every $f(a)$ is.
	2. For any f in $\mathcal{F}_1$, $\forall^{\mathcal{M}}(f)$ is in *Fact* iff every $f(a)$ is.
Equality	$a = b$ is in *Soa* and is in *Fact* iff a and b are equal.

An *assignment* is a function, s, from variables to objects. One defines $t^{\mathcal{M}}[s]$ in Ob and $ft^{\mathcal{M}}[s]$ in $\mathcal{F}_n$ (ft a function term of arity n) in a straightforward way (cf. [Bar87, Plo90]). The judgments are interpreted by: $\mathcal{M} \models_s t$ (resp $\mathbf{S}\ t$) iff $t^{\mathcal{M}}[s]$ is in $Fact$(resp Soa). Then sequents are interpreted by: $\mathcal{M} \models (\Gamma \Rightarrow J)$ iff for any s if $\mathcal{M} \models_s J'$ (for every J' in Γ) then $\mathcal{M} \models_s J$. And, finally, logical validity is defined by: $T \models seq$ iff whenever for any $\mathcal{M}$, $\mathcal{M} \models seq'$ (for every seq' in T) then $\mathcal{M} \models seq$.

Then a variation on standard Henkin-type arguments shows that

Consistency and Completeness Theorem: For any set T of sequents and any sequent *seq*

$$T \vdash seq \text{ iff } T \models seq$$

The connectives all have a certain monotone character. Proof-theoretically this can be expressed as a derived rule. Let $\Gamma : \varphi \leq \psi$ abbreviate the three sequents $(\Gamma, \mathbf{S}\varphi \Rightarrow \mathbf{S}\psi)$, $(\Gamma, \varphi \Rightarrow \psi)$ and $(\Gamma, \mathbf{S}\varphi, \psi \Rightarrow \varphi)$. Then the following is a derived rule:

$$\frac{\Gamma : \varphi \leq \psi \qquad \Gamma : \varphi' \leq \psi'}{\Gamma : \varphi\ op\ \varphi' \leq \psi\ op\ \psi'}$$

where *op* is any of $\wedge, \vee, |, \supset$. Also

$$\frac{\Gamma : \varphi[x] \leq \psi[x] \qquad (x \text{ not free in} \Gamma)}{\Gamma : Qx\varphi[x] \leq Qx\psi[x]}$$

is a derived rule where Q is either of $\exists, \forall$. In terms of interpretations, for a, b in Ob, write $a \leq b$ to mean that (1) if a is in Soa then so is b and (2) if a is in Soa then a is in $Fact$ iff b is. Then the connectives are monotone in that if $a \leq b, a' \leq b'$ then $a\ op^{\mathcal{M}} a' \leq b\ \ op^{\mathcal{M}} b'$ and similarly for the quantifiers.

One can adopt other monotone connectives and quantifiers. For example one could have a "parallel" disjunction, $\vee_p$, with the rules

$$(\mathbf{S}I\vee_p) \qquad \Box\frac{\mathbf{S}\varphi \quad \mathbf{S}\psi}{\mathbf{S}(\varphi \vee_p \psi)}$$

$$(\mathbf{S}E\wedge_p) \qquad \frac{\mathbf{S}(\varphi \vee_p \psi) \quad \begin{matrix}(\mathbf{S}\varphi, \mathbf{S}\psi) \\ J\end{matrix} \quad \begin{matrix}(\varphi) \\ J\end{matrix} \quad \begin{matrix}(\psi) \\ J\end{matrix}}{J}$$

$$(I\vee_p) \qquad 1.\ \frac{\varphi}{\varphi \vee_p \psi} \qquad\qquad 2.\ \frac{\psi}{\varphi \vee_p \psi}$$

$$(E\vee_p) \qquad \frac{\varphi \vee_p \psi \quad \begin{matrix}(\varphi)\\ J\end{matrix} \quad \begin{matrix}(\psi)\\ J\end{matrix}}{J}$$

and one could have a "parallel" existential quantification, $\exists_p$, with the rules

$$(\mathbf{S}I\exists_p) \qquad \frac{\Gamma \Rightarrow \mathbf{S}\varphi[x]}{\Gamma \Rightarrow \mathbf{S}(\exists_p x\varphi[x])}\ (x \text{ not free in } T)$$

$$(\mathbf{S}E\exists_p) \qquad \frac{\Gamma \Rightarrow \mathbf{S}(\exists_p x\varphi[x])\Gamma, \varphi[x] \Rightarrow J(x \text{ not free in } \Gamma \text{ or } J)\Gamma, \mathbf{S}\exists x\varphi[x] \Rightarrow J}{\Gamma \Rightarrow J}$$

$$(I\exists_p) \qquad \frac{\varphi[t]}{\exists_p x\varphi[x]}$$

$$(E\exists_p) \qquad \frac{\Gamma \Rightarrow \exists_p x\varphi[x] \quad \Gamma, \varphi[x] \Rightarrow J \quad (x \text{ not free in } \Gamma\text{or } J)}{\Gamma \Rightarrow J}$$

This corresponds to the parallel view of unsaturated states-of-affairs in [Plo90]. The use of $\mathbf{S}\exists x\ \varphi[x]$ in $(\mathbf{S}E\exists_p)$ is a trick to get the effect of "$\forall x\mathbf{S}\varphi[x]$", so to say. Semantically the appropriate logical schemas would be

Parallel Disjunction	1. $a \vee_p^{\mathcal{M}} b$ is in Soa iff either both are or at least one is in $Fact$. 2. $a \vee_p^{\mathcal{M}} b$ is in $Fact$ iff one of a, b is.
Parallel Existential Quantification	1. For any f in $\mathcal{F}_1$, $\exists_p^{\mathcal{M}}(f)$ is in Soa iff either every $f(a)$ is or at least one is in $Fact$. 2. For any f in $\mathcal{F}_1$, $\exists_p^{\mathcal{M}}(f)$ is in $Fact$ if some $f(a)$ is.

The extended system is still consistent, we conjecture it is also complete.

3 A Theory of Relations

First we need a signature with functional symbols as above together with $0, NIL$ (of arity ((),0)) $ARITY, +1, ASS$ (of arity (( ), 1)), $\leq$, $\prec$, ::, REL, $PRED$ (of arity (( ), 2)) and AC (of arity ((1),2)), ρ (of arity ((1),1)) and LC (of arity ((2),2)). Here $0, ARITY, +1, \leq, AC$ are for arities (taken to be natural numbers); we write (*cases* t *zero* u *succ* $x.v$) for $AC((x)v, t, u)$. And ASS, ::, $\prec, LC$ are for assignments (taken to be sequences of objects); we write (*cases* t *nil* u *cons* $x, y.v$) for $LC((x, y)v, t, u)$ and $REL, PRED, \rho$

are for relations; we write $\rho_u x.\varphi$ for $\rho((x)\varphi, u)$. We also adopt some other evident notational conventions for symbols of arity of the form $((), n)$.

The theory is given as a set of sequents and we write J for $\emptyset \Rightarrow J$.

Arities	*Type*	$\mathbf{S}ARITY(x), ARITY(0), ARITY(x) \supset ARITY(x+1)$
	Cases	$cases\ 0\ zero\ u\ succ\ x.v = u$
		$cases\ (t+1)\ zero\ u\ succ\ x.v[x] = v[t]$
	Induction	$\varphi[0], \forall x ARITY(x) \supset (\varphi[x] \supset \varphi[x+1]),\ ARITY(t) \Rightarrow \varphi[t]$

One can formulate a *rule* of induction which seems to be stronger

Order Definition

$$\begin{aligned} t \leq u \quad \equiv \quad & ARITY(t) \wedge ARITY(u) \wedge \\ & ((t = 0) \vee \exists x \exists y\ t = x + 1 \wedge u = y + 1 \wedge x \leq y) \end{aligned}$$

Here $\varphi \equiv \psi$ abbreviates $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$

Assignments	*Assignment Definition*	$ASS(t, u) \equiv ARITY(u)\ \wedge$
		$((t = NIL \wedge u = 0) \vee \exists x \exists y \exists z\ t = x :: y\ \wedge$
		$u = z + 1 \wedge ASS(y, z))$
	Cases	$cases\ NIL\ nil\ u\ cons\ x, y.v = u$
		$cases\ t :: t'\ nil\ u\ cons\ x, y.v[x, y] = v[t, t']$
	Prefix	$t \prec u \equiv ASS(t) \wedge ASS(u)\ \wedge$
		$(t = NIL \vee \exists x \exists y \exists y' t = x :: y \wedge$
		$u = x :: y' \wedge y \prec y')$

Here $ASS(v)$ abbreviates $\exists m ASS(v, m)$. One can derive an induction axiom:

$$\varphi[NIL], \forall l ASS(l) \supset (\varphi[l] \supset \forall x \varphi[x :: l]), ASS(u) \Rightarrow \varphi[u].$$

Relations	$REL(t, u) \wedge REL(t, v) \supset u = v \wedge ARITY(u)$
Predication	$\mathbf{S}PRED(t, u) \Rightarrow\ \exists l REL(t, l) \wedge ASS(u, l)$
Individuation	$PRED(t, u) = PRED(t', u') \supset t = t' \wedge u = u'$
Abstraction	$REL(\rho_u x \varphi, v) \equiv (u = v) \wedge ARITY(v)$
	$ASS(t, u) : PRED(\rho_u x \varphi[x], t) \sim \varphi[t]$

Here $\Gamma : \varphi \sim \psi$ abbreviates four sequents: $(\Gamma, \varphi \Rightarrow \psi)$, $(\Gamma, \psi \Rightarrow \varphi)$, $(\Gamma, \mathbf{S}\varphi \Rightarrow \mathbf{S}\psi)$, $(\Gamma, \mathbf{S}\psi \Rightarrow \mathbf{S}\varphi)$ (and the colon is omitted if Γ is empty).

The traditional predication and relation abstraction notation of situation theory can be rederived; our idea is that the negation and quantification implicit there is here implicit in the logic. First we get positive unsaturated basic states-of-affairs back:

$$\langle\langle t, u\rangle\rangle =_{def} \exists l \exists m \exists y (l \leq m \wedge REL(t, m) \wedge ASS(u, l) \wedge ASS(y, m) \wedge u \prec y) \\ \& PRED(t, y)$$

where the sequential conjunction operator $\varphi \& u$ is $\varphi \wedge (\varphi \supset \psi)$: it has derived introduction rules

$$\frac{\mathbf{S}\varphi \quad \overset{(\varphi)}{\mathbf{S}\psi}}{\mathbf{S}(\varphi \& \psi)} \qquad \text{and} \qquad \frac{\varphi \quad \psi}{\varphi \& \psi}$$

Then the general case is

$$\langle\langle t, u; v\rangle\rangle =_{def} (v = 0 + 1 \& \langle\langle t, u\rangle\rangle) \vee (v = 0 \& \neg \langle\langle t, u\rangle\rangle)$$

Abstraction is given by:

$$[x_1, \ldots, x_n \mid \varphi] =_{def} \rho_{\underline{n}} y \ \mathit{let}\ x_1, \ldots, x_n \ \mathit{be}\ y\ \mathit{in}\ \varphi$$

using numerals, $\underline{n}$ and where ($\mathit{let\ be}\ u\ \mathit{in}\ \varphi$) stands for φ and $\mathit{let}\ x_1, \ldots, x_{n+1}$ $\mathit{be}\ \ u\ \ \ \mathit{in}\ \varphi$) stands for whatever ($\mathit{cases}\ u\ \mathit{nil}\ \perp\ \mathit{cons}\ x_1, z\ \mathit{let}\ x_2, \ldots, x_{n+1}$ $\mathit{be}\ z\ \mathit{in}\ \varphi$) does. It is a theorem that

$$(*) \langle\langle [x_1, \ldots, x_n \mid \varphi], x_1 :: \ldots :: x_n :: NIL\rangle\rangle \sim \ \varphi$$

for any $n \geq 0$.

The theory can be developed much as in [Plo90]. There is a logical fixed-point theorem: for any $t[x]$ there is a φ such that $\varphi \sim t[\varphi]$. To see this, take φ to be $\langle\langle \triangle, \triangle :: NIL\rangle\rangle$ where $\triangle$ is $[x \mid t[\langle\langle x, x :: NIL\rangle\rangle]]$, and use $(*)$. Now one can see the undefinability of facticity which here means that for any $\psi[x]$ the three sequents $\mathbf{S}\psi[x], \psi[x] \Rightarrow x, x \Rightarrow \psi[x]$ are inconsistent with the theory (cf. [Plo90] where we can assert undefinability within the theory). For taking $\varphi \sim \neg\psi[\varphi]$ one gets first that $\mathbf{S}\varphi$ (as $\mathbf{S}\psi[\varphi]$) and second that $\varphi \sim \neg\varphi$ (as $\varphi \sim \neg\psi[\varphi]$ and $\psi[\varphi] \sim \varphi$). But $\varphi \sim \neg\varphi$ yields $\varphi \Rightarrow \perp$, which yields $\neg\varphi$ as $\mathbf{S}\varphi$ and then we get the contradiction, $\perp$, from $\neg\varphi$ and $\varphi \sim \neg\varphi$.

Other undefinability results can be derived. For example there is no $\varphi[x]$

defining soahood in that the sequents $\mathbf{S}\varphi[x]$ and $\varphi[x] \Leftrightarrow \mathbf{S}x$ are inconsistent with the theory. (Here $\psi \Leftrightarrow \psi'$ abbreviates the two sequents $\psi \Rightarrow \psi'$ and $\psi' \Rightarrow \psi$.) For otherwise, since $\mathbf{S}((0 = 0) \mid x) \Leftrightarrow x$ is derivable, we could then define facticity by $\psi[(0 = 0) \mid x]$. One can get intensionality results. For example it is inconsistent to assert that there are at most n facts (for any $n \geq 0$). Formally, it is inconsistent to have n constants $a_1, \ldots, a_n$ and the sequents: $x \Leftrightarrow (x = a_1) \vee \ldots \vee (x = a_n)$ as this immediately gives a definition of facticity. Similarly, it is inconsistent to assert that there are at most n soas.

Adding non-monotone connectives to the logic also ressults in inconsistencies in that the theory becomes inconsistent. For example, suppose the implication introduction rule were changed to the usual one:

$$\frac{\begin{array}{c}(\varphi)\\ \psi\end{array}}{\varphi \supset \psi}$$

Suppose too we keep the elimination rule as it is and drop the **S**-rules (to avoid other inconsistencies!). Then we would still get an inconsistency by taking a logical fixed-point $\varphi \sim (\varphi \supset \perp)$. This is a variant of the usual inconsistencies in illative logic. It should be possible to formulate a general result along these lines (cf. [Plo90]).

Finally we remark that it is straightforward to construct a model of the theory using the technique employed by Aczel in [Acz80] to construct Frege structures, say along the lines of [Plo90].

4 Illative Logic and Standard Logic

By "standard" we mean the logic advocated in [Plo90] which is a first-order logic allowing variable-binding to occur in terms - except that we forbid function variables, in keeping with the illative logic. Let Σ be a signature as above. We show that every illative theory can be conservatively extended to a standard one over $\overline{\Sigma}$ which is Σ, extended by unary predicate symbols $fact$ and soa.

Specifically let L be the following set of "logical axioms" (obtained by writing down the logical schemas formally).

Judgments	$fact(x) \supset soa(x)$
Sequential Implication	$soa(x \supset y) \equiv soa(x) \wedge (fact(x) \supset soa(y))$
	$soa(x \supset y) \supset (fact(x \supset y) \equiv fact(x) \supset fact(y))$
Existential Quantification	$soa(\exists xt[x]) \equiv \forall x soa(t[x])$
	$soa(\exists xt[x]) \supset (fact(\exists xt[x]) \equiv \exists x fact(t[x]))$
Equality	$soa(x = y)$
	$fact(x = y) \equiv (x = y)$

plus corresponding axioms for $\perp, \wedge, \vee, |, \forall$. Note the overloading of the symbols $\perp, \wedge, \ldots$; this could be avoided by using the dot convention with $\dot{\wedge}, \dot{\vee}, \ldots$ in $\overline{\Sigma}$.

Now a translation of judgments and sequents is given by:

$$\begin{aligned} \overline{\varphi} &=_{def} fact(\varphi) \\ \overline{\mathbf{S}\varphi} &=_{def} soa(\varphi) \\ (J_1, \ldots, J_n \Rightarrow J)^{-} &=_{def} \overline{J}_1 \wedge \ldots \wedge \overline{J}_n \supset \overline{J} \end{aligned}$$

Conservative Extension Theorem: For any set T of sequents and any sequent seq

$$T \vdash seq \text{ iff } \overline{T}, L \vdash \overline{seq}$$

(where $\overline{T} =_{def} \{\overline{seq} | seq \in T\}$)

PROOF: The theorem is proved semantically using the consistency and completeness theorems for the two logics. To any interpretation, $\mathcal{M} = \langle \mathcal{F}, Soa, Fact, H \rangle$ of Σ one assigns an interpretation $\overline{\mathcal{M}} = \langle \mathcal{F}, \overline{H} \rangle$ which validates the logical axioms, $L : \overline{H} \mid \Sigma = H \mid \Sigma$ and $\overline{H}(fact) = Fact$, $\overline{H}(soa) = Soa$. Conversely to any interpretation $\mathcal{N} = \langle \mathcal{F}, H \rangle$ which validates L we can assign an interpretation $\overline{\mathcal{N}} = \langle \mathcal{F}, Soa, Fact, \overline{H} \rangle$ in the evident way. Note that $\overline{\overline{\mathcal{M}}} = \mathcal{M}$ and $\overline{\overline{\mathcal{N}}} = \mathcal{N}$. Note too that for any sequent, seq, and any $\mathcal{M}$: $\mathcal{M} \models seq$ iff $\overline{\mathcal{M}} \models \overline{seq}$. Using the correspondences one then proves directly that $T \models seq$ iff $\overline{T}, L \models \overline{seq}$ $\square$

Applying this result to the above theory of relations we get a theory closer to that in [Plo90]. As it stands the latter theory seems too strong: one should weaken the induction scheme for arities to:

$$fact(t[0]) \wedge (\forall l\ arity(l) \supset fact(t[l] \supset t[l+1])) \supset \forall l\ arity(l) \supset fact(t[l])$$

and should also change the axioms for unsaturated *soas* so that the implicit quantification is in accord with the one employed in the illative logic. There is then an evident translation of the present relational theory, T, to the (amended) old one, T', such that if $T \vdash seq$ then $T' \vdash \overline{seq}$ but we could not prove it conservative.

In the standard theory, $\overline{T}, L$ one has strictly greater expressive power than T. For example one cannot express the formula $\neg fact(x)$ by a closed term, NF in that for any closed term NF, $\neg fact(x) \equiv fact(\langle\!\langle NF, x\rangle\!\rangle)$ is inconsistent with the theory (take a logical fixed-point, $t \sim\langle\!\langle NF, t\rangle\!\rangle$ and derive $fact(t) \equiv \neg fact(t)$). This means that we cannot assign to every formula φ of the theory a Gödel-term, "φ" with the same free variables such that

$$\varphi \equiv fact(``\varphi")$$

holds for otherwise we could take $NF = [x \mid ``\neg fact(x)"]$ in the above and get a contradiction.

5 Illative Logic and Three-Valued Logic

Barwise suggested amending the standard logic to a three-valued one, and we now show that this idea works in general. First we sketch a three-valued variation on the standard logic. We retain the syntax of the standard logic except that we forbid function variables. Interpretations are structures $\mathcal{N} = \langle \mathcal{F}, H\rangle$ just as before except that for any predicate symbol P of arity n, $H(P) = \langle D, T\rangle$ with $T \subset D \subset Ob^n$. One defines $t^{\mathcal{N}}[s]$ as before but now as well as the satisfaction relation $\models$, one also has a *decision* relation $\Vdash$ (cf. Kamp and Veltmann's $\models^+$ and $\models^-$ (see [vB85]). The clauses for the truth conditions correspond to the above logical schemas, we just give examples.

Atomic Formulas	$\mathcal{N}(\!\models P(t_1, \ldots, t_n)[s]$ $(resp \models)$ iff $\langle t_1^{\mathcal{N}}[s], \ldots, t_n^{\mathcal{N}}[s] \rangle \in P_2^{\mathcal{N}}$ (resp $P_1^{\mathcal{N}}$)
Equality	$\mathcal{N} (\!\models t = u[s]$, and $\mathcal{N} \models t = u[s]$ iff $t^{\mathcal{N}}[s] = u^{\mathcal{N}}[s]$
Sequential Implication	$\mathcal{N}(\!\models \varphi \supset \psi[s]$ iff $\mathcal{N}(\!\models \varphi[s]$ and if$\mathcal{N} \models \varphi[s]$ then $\mathcal{N}(\!\models \psi[s]$ $\mathcal{N} \models \varphi \supset \psi[s]$ iff $\mathcal{N}(\!\models \varphi \supset \psi[s]$ and if $\mathcal{N} \models \varphi[s]$ then $\mathcal{N} \models \psi[s]$
Existential Quantification	$\mathcal{N}(\!\models \exists x\varphi[s]$ iff $\mathcal{N}(\!\models \varphi[s[a/x]]$ for every a in Ob $\mathcal{N} \models \exists x\varphi[s]$ iff $\mathcal{N}(\!\models \exists x\varphi[s]$ and $\mathcal{N} \models \varphi[s[a/x]]$ for some a in Ob

Then *sequents* have the form $seq = \varphi_1, \ldots, \varphi_n \Rightarrow \varphi$; one defines $\mathcal{N} \models seq$ to mean that for any s if $\mathcal{N} \models \varphi_i[s]$ (all i)then $\mathcal{N} \models \varphi[s]$. Then $T \models seq$ is defined in the evident way—we not not give an axiomatisation.

Now let Σ be an illative signature, and consider three-valued extensions over $\overline{\Sigma}$ which is Σ extended by a unary predicate symbol $fact$. Let now L_3 be the set of sequents:

Sequential Implication	$fact(x \supset y) \sim fact(x) \supset fact(y)$
Existential Quantification	$fact(\exists x t[x]) \sim \exists x fact(t[x])$
Equality	$fact(x = y) \sim x = y$
Restriction	$fact(x \mid y) \Leftrightarrow fact(x) \wedge fact(y)$ $fact(x \mid y) \downarrow \Leftrightarrow fact(x) \downarrow \wedge fact(y)$

plus corresponding sequents for $\bot, \wedge, \vee, \forall$ (here $\varphi \sim \psi$ abbreviates the four sequents $\varphi \Rightarrow \psi$, $\psi \Rightarrow \varphi$, $\varphi \downarrow \Rightarrow \psi \downarrow$ and $\psi \downarrow \Rightarrow \varphi \downarrow$, where in turn, $\varphi \downarrow =_{def} \varphi \vee \neg\varphi$).

Now a translation of judgments and sequents is given by:

$$\begin{array}{rcl} \overline{\varphi} & =_{def} & fact(\varphi) \\ \overline{\mathbf{S}\varphi} & =_{def} & fact(\varphi) \downarrow \\ (J_1, \ldots, J_n \Rightarrow J)^- & =_{def} & \overline{J_1}, \ldots, \overline{J_n} \Rightarrow \overline{J} \end{array}$$

Semantic Conservative Extension Theorem: For any set, T, of sequents and any sequent, seq:

$$T \models seq \text{ iff } \overline{T}, L_3 \models \overline{seq}$$

Proof: (Outline) As before one gives a one-one correspondence between illative interpretations and three-valued models of L_3. To $\mathcal{M} = \langle \mathcal{F}, Soa, Fact, H \rangle$ one assigns $\overline{\mathcal{M}} = \langle \mathcal{F}, \overline{H} \rangle$ with $\overline{H} \mid \Sigma = H \mid \Sigma$ and $H(fact) = \langle Soa, Fact \rangle$, with the evident converse assignment. One notes that $\mathcal{M} \models seq$ iff $\overline{\mathcal{M}} \models \overline{seq}$ and the result follows. □

In the three-valued logic to every formula φ corresponds a Gödel-term, "φ" obtained by replacing every occurrence of $fact(t)$ by t.

Internalisation Theorem: For any formula φ: $L_3 \vdash \varphi \sim fact(\text{“}\varphi\text{”})$

Proof: One shows by induction on φ that for any model $\mathcal{N}$ of L_3 and any s that $\mathcal{N} \models \varphi[s]$ iff $\mathcal{N} \models fact(\text{“}\varphi\text{”})[s]$, and $\mathcal{N} \models\!\!\!\!/\; \varphi[s]$ iff $\mathcal{N} \models\!\!\!\!/\; fact(\text{“}\varphi\text{”})[s]$. □

The internalisation theorem shows that the three-valued logic is equally as expressive as the illative logic (in that they express the same partial relations in a given model). The price paid for internalisation is that the predicate, $fact$, is necessarily partial: with T being the above relational theory there is a term t (take $t \sim \neg t$) such that $T, L_3 \models fact(t) \downarrow \Rightarrow \perp$

There are other possible choices of three-valued logic. One can vary the connectives and the definitions of the quantifiers and also one can vary the definition of the validity of a sequent. It is not at all known what would result from variations of the second kind. Variations of the first kind should be able to be accommodated by varying the illative logic: one would still expect a conservative extension theorem and an internalisation theorem. However in general one would expect that, if the logic admitted a non-monotone connective then internalisation would be inconsistent with a theory, like those above, with logical fixed-points.

For example suppose we added to the above logic a symbol, f, for an n-ary non-monotone connective and consider illative signatures Σ which always contain f as a functional symbol of arity $((), n)$. Then with $\overline{\Sigma}$ as above, if we were to take L_3 as above, and add

$$fact(f(x_1, \ldots, x_n)) \sim f(fact(x_1), \ldots, fact(x_n))$$

then the resulting theory would be inconsistent if we took T to be the theory of relations considered above.

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