A SIMPLE FPRAS FOR BI-DIRECTED REACHABILITY
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Abstract. Gorodezky and Pak ([Random Struct. Algorithms, 2014]) introduced a “cluster-popping” algorithm for sampling root-connected subgraphs in a directed graph, and conjectured that it runs in expected polynomial time on bi-directed graphs. We confirm their conjecture. It follows that there is a fully polynomial-time randomized approximation scheme (FPRAS) for reachability in bi-directed graphs. Reachability is the probability that, assuming each arc fails independently, all vertices can reach a special root vertex in the remaining graph. A bi-directed graph is one in which each directed arc has a parallel twin oriented in the opposite sense.

1. Introduction

Network reliability problems are extensively studied \#P-hard problems [Col87] (see also [BP83, PB83, KL85]). In fact, these problems are among the oldest \#P-hard problems, and the two-terminals version is listed in Valiant’s original thirteen [Val79]. The general setup of these problems is that in a given (directed or undirected) graph, every edge (or arc) $e$ has an independent probability $p_e$ to fail, and we are interested in various reliability measures of the remaining graph. For example, the two-terminals connectedness [Val79] asks for the probability that $s$ is connected to $t$ for two arbitrary vertices $s$ and $t$ in the remaining graph, and the undirected all-terminals reliability asks the probability of all vertices being connected after edges fail. The latter can also be viewed as a specialization of the Tutte polynomial $T_G(x, y)$ with $x = 1$ and $y > 1$, yet another classic topic whose computational complexity gets extensively studied [JWV90, GJ08, GJ14].

Despite their importance, the approximation complexity of network reliability problems remains elusive. There is no known efficient approximation algorithm (for any variant) and nor is there any evidence of hardness. A notable exception is Karger’s fully polynomial-time randomized approximation scheme (FPRAS) for undirected all-terminals unreliability [Kar99] (see also [HS14, Kar16, Kar17] for more recent developments). Although approximating unreliability is potentially more useful in practice, it does not entail an approximation of its complement.

In this paper, we consider a reliability measure called reachability, introduced by Ball and Provan [BP83]. Given a directed\(^1\) graph $G = (V, A)$ with a special root vertex $r$, $G$ is called root-connected if there is a directed path in $G$ from any non-root vertex to $r$. Let $0 < p_e < 1$ be the failure probability of arc $e$, and define the weight of a subgraph $S$ to be $\text{wt}(S) := \prod_{e \in S} (1 - p_e) \prod_{e \notin S} p_e$. The reachability is defined as

$$Z(G, r; p) := \sum_{S \subseteq A 	ext{ is root-connected}} \text{wt}(S).$$

Here, $p = (p_e : e \in A)$ denotes the vector of failure probabilities. When $(G, r)$ is clear from the context, we also write $Z(p)$ for short. Intuitively, $Z(G, r; p)$ is the probability that if each arc $e$ fails with probability $p_e$ independently, the remaining graph is still root-connected.

The computational problem we are interested in is the following.

Name: Reachability
Instance: A directed graph $G = (V, A)$ with root $r$, and the parameters $p$.
Output: $Z(G, r; p)$.

\(^1\)If the graph is undirected, then reachability is the same as all-terminals reliability.
An efficient and exact algorithm exists when the graph is acyclic [BP83] or has a small number of cycles [Hag91]. However, in general the problem is \#P-hard [PB83]. Moreover, approximating REACHABILITY is at least as hard as approximating undirected all-terminals reliability. This is shown in Theorem 10 via an approximation-preserving reduction (in the sense of [DGGJ04]).

Gorodezky and Pak [GP14] proposed an interesting “cluster-popping” algorithm to approximate REACHABILITY. The algorithm samples root-connected subgraphs with probability proportional to their weights, and then the reduction from counting to sampling is via a sequence of contractions. A cluster is a subset of vertices not including the root and without any out-going arc. The sampling algorithm randomizes all arcs independently, and then repeatedly resamples minimal clusters until none is left, at which point the remaining subgraph is guaranteed to be root-connected. This approach is similar to Wilson’s “cycle-popping” algorithm [Wil96] for rooted spanning trees, and to the “sink-popping” algorithm [CPP02] for sink-free orientations. Gorodezky and Pak [GP14] have noted that cluster-popping can take exponential time in general, but they conjecture that in bi-directed\(^2\) graphs, where the existence of an arc implies the existence of the reversed arc with the same failure probability, the algorithm runs within polynomial-time.

Our main result is that we confirm this conjecture. Note that Gorodezky and Pak [GP14] has shown that solving REACHABILITY exactly remains \#P-hard in bi-directed graphs. Let \(p_{\text{max}}\) be the maximum failure probability of arcs. Let \(m\) be the number of arcs and \(n\) the number of vertices.

**Theorem 1.** There is an FPRAS for REACHABILITY in bi-directed graphs. The running time is \(O\left(\epsilon^{-2p_{\text{max}}(1-p_{\text{max}})^{-1}m^2n^3}\right)\) for an \((1 \pm \epsilon)\)-approximation.

We analyze the “cluster-popping” algorithm [GP14] under the partial rejection sampling framework [GJL17], which is a general approach to sampling from a product distribution conditioned on avoiding a number of “bad” events. Partial rejection sampling is inspired by the Moser-Tardos algorithm for Lovász Local Lemma [MT10]. In particular, all three “popping” algorithms [Wil96, CPP02, GP14] are special cases of partial rejection sampling for extremal instances [GJL17] (in the sense of [KS11, She85]), where these bad events are either disjoint or independent. In case of “cluster-popping”, the bad events are exactly minimal clusters.

One advantage of the partial rejection sampling treatment is that we have an explicit formula for the expected number of resampling events for any extremal instance [GJL17], which equals to the ratio between the probability of having exactly one bad event and the probability of avoiding all bad events. In order to bound this ratio, we use a combinatorial encoding idea and design an mapping from subgraphs with a unique minimal cluster to root-connected subgraphs. To make this mapping injective, we record an extra vertex and an arc so that we can recover the pre-image. This extra cost is upper-bounded by an polynomial in the size of the graph.

Another advantage of this general framework is that we can easily generalize the algorithm to include other distributions on the edges. The distribution of a bi-directed edge is to have both directions fail independently with the same probability. An easy generalization of the algorithm is to include another type of edges, where it is oriented to either direction (exclusively) with \(1/2\) probability. In fact, we can approximate a bi-directed edge using the latter type of edges with only polynomial overhead.

Such a generalization gives some hope of an FPRAS for undirected all-terminals reliability (namely the half-line \(x = 1, \ y > 1\) on the Tutte plane), since it is the same as reachability if every edge is undirected, where the two directions either both fail or are both preserved. Unfortunately, our approach does not seem to generalize to this distribution on edges, and undirected all-terminals reliability remains open.

\(^2\)There are other definitions of “bi-directed graphs” in the literature. Our definition is sometimes also called a symmetric directed graph.
2. Cluster-popping

Let $G$ be a directed graph with root $r$. Let $\pi_G(\cdot)$ (or $\pi(\cdot)$ for short) be the distribution resulting from choosing each arc $e$ independently with probability $1 - p_e$, and conditioning on that the resulting graph being root-connected. In other words, the support of $\pi(\cdot)$ is the collection of all root-connected subgraphs, and the probability of each subgraph $S$ is proportional to its weight $\text{wt}(S)$. Then $Z(p)$ is the normalizing factor of the distribution $\pi(\cdot)$. Gorodezky and Pak [GP14] has shown that approximating $Z(p)$ can be reduced to sampling from $\pi(\cdot)$ when the graph is bi-directed.

The partial rejection sampling framework [GJL17] is a general approach to sample from a product distribution conditioned on a number of undesired events not occurring. It starts with randomizing all variables independently, and then gradually eliminating “bad” events. At every step, we need to find an appropriate set of variables to resample. In particular, we call an instance extremal, if any two bad events are either disjoint or independent [GJL17, She85]. For extremal instances, the resampling set can be simply chosen to be the set of all variables involved in occurring bad events [GJL17].

The “cluster-popping” algorithm of Gorodezky and Pak [GP14], to sample root-connected subgraphs from $\pi(\cdot)$, can be viewed as a special case of partial rejection sampling [GJL17] for extremal instances. With every arc $e$ of $G$ we associate a random variable that records whether that arc has failed. Bad events are characterized by the following notion of clusters.

**Definition 2.** In a directed graph $(V, A)$ with root $r$, a subset $C \subseteq V$ of vertices is called a cluster if $r \notin C$ and there is no arc $u \to v \in A$ such that $u \in C$ and $v \notin C$.

We say $C$ is a minimal cluster if $C$ is a cluster and for any proper subset $C' \subset C$, $C'$ is not a cluster.

If $(V, A)$ contains no cluster, then it is root-connected. For each vertex $v$, let $A_{\text{out}}(v)$ be the set of outgoing arcs from $v$. We also abuse the notation to write $A_{\text{out}}(S) = \bigcup_{v \in S} A_{\text{out}}(v)$ for a subset $S \subseteq V$ of vertices. To “pop” a cluster $C$, we re-randomize all arcs in $A_{\text{out}}(C)$. However, re-randomizing clusters does not yield the desired distribution. We will instead re-randomize minimal clusters.

**Claim 3.** Any minimal cluster is strongly connected.

**Proof.** Let $C$ be a minimal cluster, and $v \in C$ be an arbitrary vertex in $C$. We claim that $v$ can reach all vertices of $C$. If not, let $C'$ be the set of reachable vertices of $v$ and $C' \subset C$. Since $C'$ does not have any outgoing arcs, $C'$ is a cluster. This contradicts to the minimality of $C$. □

**Claim 4.** If $C_1$ and $C_2$ are two distinct minimal clusters, then $C_1 \cap C_2 = \emptyset$.

**Proof.** By Claim 3, $C_1$ and $C_2$ are both strongly connected components. If $C_1 \cap C_2 \neq \emptyset$, then they must be identical. □

For every subset $C \subseteq V$ of vertices, we define a bad event $B_C$, which occurs if $C$ is a minimal cluster. Observe that $B_C$ relies on only the status of arcs in $A_{\text{out}}(C)$. Thus, if $C_1 \cap C_2 = \emptyset$, then $B_{C_1}$ and $B_{C_2}$ are independent, even if some of their vertices are adjacent. By Claim 4, we know that two bad events $B_{C_1}$ and $B_{C_2}$ are either independent or disjoint. Thus the aforementioned extremal condition is met. Moreover, it was shown [GJL17] that if the instance is extremal, then at every step, we only need to resample variables involved in occurring bad events. This leads to the cluster-popping algorithm of Gorodezky and Pak [GP14], which is formally described in Algorithm 1.

An advantage of thinking in the partial rejection sampling framework is that we have a closed form formula for the expected running time of these algorithms on extremal instances. Let $\Omega_k$ be the collection of subgraphs with $k$ minimal clusters, and

$$Z_k := \sum_{S \in \Omega_k} \text{wt}(S).$$

Then $Z_0 = Z(p)$, since any subgraph in $\Omega_0$ has no cluster and is thus root-connected.
Algorithm 1 Cluster Popping

Let $S$ be a subset of arcs by choosing each arc $e$ with probability $1 - p_e$ independently.

while There is a cluster in $(V, S)$.

Let $C_1, \ldots, C_k$ be all minimal clusters in $(V, S)$, and $C = \bigcup_{i=1}^k C_i$.

Re-randomize all arcs in $A_{\text{out}}(C)$ to get a new $S$.

end while

return $S$

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Theorem 5 ([GJL17]). Let $T$ be the number of resampled events of the partial rejection sampling algorithm for extremal instances. Then

$$\mathbb{E}T = \frac{Z_1}{Z_0}.$$ 

In particular, for Algorithm 1, $T$ is the number of popped clusters.

The less-than-or-equal-to direction of Theorem 5 was shown by Kolipaka and Szegedy [KS11].

3. Running time of Algorithm 1 in bi-directed graphs

Gorodezky and Pak [GP14] have given examples of directed graphs in which Algorithm 1 requires exponential time. In the following we focus on bi-directed graphs. A graph is called bi-directed if the existence of $u \rightarrow v$ implies the existence of $v \rightarrow u$, and the failure probability are the same for these two arcs. We use bi-directed Reachability to denote Reachability in bi-directed graphs. For an arc $e = u \rightarrow v$, let $\overline{e} := v \rightarrow u$ denote its reverse arc. Then in a bi-directed graph, $p_e = p_{\overline{e}}$.

Lemma 6. Let $G = (V, A)$ be a root-connected bi-directed graph with root $r$. We have that $Z_1 \leq \max_{e \in A} \left\{ \frac{p_e}{1 - p_{\overline{e}}} \right\} mnZ_0$, where $n = |V|$, and $m = |A|$.

Proof. We construct an injective mapping $\varphi : \Omega_1 \rightarrow \Omega_0 \times V \times A$. For each subgraph $S \in \Omega_1$, $\varphi$ is defined by “fixing” $S$ so that no minimal cluster is present. We choose in advance an arbitrary ordering of vertices and arcs. Let $C$ be the unique minimal cluster in $S$ and $v$ be the first vertex in $C$. Let $R$ denote the set of all vertices which can reach the root $r$. Since $S \in \Omega_1$, $R \neq V$ and let $U = V \setminus R$. Since $G$ is root-connected, there is an arc in $A$ from $U$ to $R$. Let $u \rightarrow u'$ be the first such arc, where $u \in U$ and $u' \in R$. We let $\varphi(S) = (S_{\text{fix}}, v, u \rightarrow u')$,

where $S_{\text{fix}} \in \Omega_0$ is defined next.

Consider the subgraph $H = (U, S[U])$, where

$$S[U] := \{ x \rightarrow y \mid x \in U, \ y \in U, \ x \rightarrow y \in S \}.$$ 

We contract all strongly connected components in $H$, resulting in a new graph $\hat{H}$. (We use the decoration $\sim$ to denote arcs, vertices, etc. in the contracted graph.) To be more precise, we replace each strongly connected component by a single vertex. For a vertex $w \in U$, let $[w]$ denote the strongly connected component containing $w$. For example, $[v]$ is the same as the minimal cluster $C$ by Claim 3. We may also view $[w]$ as a vertex in $\hat{H}$ and we do not distinguish the two views. The arcs in $\hat{H}$ are naturally induced by $S[U]$. Namely, for $[x] \neq [y]$, an arc $[x] \rightarrow [y]$ is present in $\hat{H}$ if there exists $x' \in [x]$, $y' \in [y]$ such that $x' \rightarrow y' \in S$. Since there are no strongly connected components of size at least 2 in $\hat{H}$, $\hat{H}$ is acyclic.

We claim that $\hat{H}$ is root-connected with root $[v]$. This is because $[v]$ must be the unique sink in $\hat{H}$ and $\hat{H}$ is acyclic. If there is another sink $[w]$ where $v \notin [w]$, then $[w]$ is a minimal cluster in $H$. This contradicts $S \in \Omega_1$.

Since $\hat{H}$ is root-connected, there is at least one path from $[u]$ to $[v]$. Let $\hat{W}$ denote the set of vertices of $\hat{H}$ that can be reached from $[u]$ in $\hat{H}$ (including $[u]$), and $\hat{W} := \{ x \mid [x] \in \hat{W} \}$. Then
$W$ is a cluster and $[u]$ is the unique source in $\tilde{H}[W]$. As $\tilde{H}$ is root-connected, $[v] \in \hat{W}$. Define $S_{\text{flip}} := \{ x \rightarrow y \mid x \neq [y], x, y \in W, \text{and } x \rightarrow y \in S \}$. Notice that $S[W]$ is different from $S_{\text{flip}}$, namely all arcs that are inside strongly connected components are ignored in $S_{\text{flip}}$. Now we are ready to define $S_{\text{fix}}$. We reverse all arcs in $S_{\text{flip}}$ and add the arc $u \rightarrow u'$ to fix the minimal cluster. Formally, let $S_{\text{fix}} := S \cup \{ u \rightarrow u' \} \cup \{ y \rightarrow x \mid x \rightarrow y \in S_{\text{flip}} \} \setminus S_{\text{flip}}$. Let $\tilde{H}_{\text{fix}}$ be the graph obtained from $\tilde{H}$ by reversing all arcs induced by $S_{\text{flip}}$. Observe that $[u]$ becomes the unique sink in $\tilde{H}_{\text{fix}}[\hat{W}]$ (and $[v]$ becomes the unique source).

We verify that $S_{\text{fix}} \in \Omega_0$. For any $x \in R$, $x$ can still reach $r$ in $(V, S_{\text{fix}})$ since the path from $x$ to $r$ in $(V, S)$ is not changed. Since $u \rightarrow u' \in S_{\text{fix}}$, $u$ can reach $u' \in R$ and hence $r$. For any $y \in W$, $y$ can reach $u$ as $[u]$ is the unique sink in $\tilde{H}_{\text{fix}}[\hat{W}]$. For any $z \in U \setminus W$, $z$ can reach $v \in W$ since the path from $z$ to $v$ in $(V, S)$ is not changed.

Next we verify that $\varphi$ is injective. To do so, we show that we can recover $S$ given $S_{\text{fix}}$, $u \rightarrow u'$, and $v$. First remove $u \rightarrow u'$ from $S_{\text{fix}}$. The set of vertices which can reach $r$ in $(V, S_{\text{fix}} \setminus \{ u \rightarrow u' \})$ is exactly $R$ in $(V, S)$. Namely we can recover $U$ and $R$. As a consequence, we can recover all arcs in $S$ that are incident at $R$, as these arcs are not changed.

What is left to do is to recover arcs in $S[U]$. To do so, we need to find out which arcs have been flipped. We claim that $\tilde{H}_{\text{fix}}$ is acyclic. Suppose there is a cycle in $\tilde{H}_{\text{fix}}$. Since $\tilde{H}$ is acyclic, the cycle must involve flipped arcs and thus vertices in $\hat{W}$. Let $[x] \in \hat{W}$ be the lowest one under the topological ordering of $\tilde{H}[\hat{W}]$. Since $\hat{W}$ is a cluster, the outgoing arc $[x] \rightarrow [y]$ along the cycle in $\tilde{H}_{\text{fix}}$ must have been flipped, implying that $[y] \in \hat{W}$ and $[y] \rightarrow [x]$ is in $\tilde{H}[\hat{W}]$. This contradicts to the minimality of $[x]$.

Since $\tilde{H}_{\text{fix}}$ is acyclic, strongly connected components of $H_{\text{fix}} := (U, S_{\text{fix}}[U])$ are identical to those of $H = (U, S[U])$. Hence contracting all strongly connected components of $H_{\text{fix}}$ results in exactly $\tilde{H}_{\text{fix}}$. All we need to recover now is the set $\hat{W}$. Let $\hat{W}'$ be the set of vertices reachable from $[v]$ in $\tilde{H}_{\text{fix}}$. It is easy to see that $\hat{W} \subseteq \hat{W}'$. We claim that actually $\hat{W} = \hat{W}'$. For any $[x] \in \hat{W}'$, there is a path from $[v]$ to $[x]$ in $\tilde{H}_{\text{fix}}$. Suppose $[x] \not\in \hat{W}$. Since $[v] \in \hat{W}$, we may assume that $[y]$ is the first vertex along the path such that $[y] \rightarrow [z]$ where $[z] \not\in \hat{W}$. Thus $[y] \rightarrow [z]$ has not been flipped and is present in $\tilde{H}$. However, this contradicts the fact that $\hat{W}$ is a cluster in $\tilde{H}$.

To summarize, given $S_{\text{fix}}$, $u \rightarrow u'$, and $v$, we may uniquely recover $S$. Hence the mapping $\varphi$ is injective. Moreover, flipping arcs does not change the weight as $p_{e'} = p_e$ and only adding the arc $u \rightarrow u'$ would. We have that $\text{wt}(S_{\text{fix}}) = \frac{1-p_{u \rightarrow u'}}{p_u} \text{wt}(S)$. The lemma follows.

We remark that an alternative way of fixing $S$ in the proof above is to reverse all arcs in $S[W]$ without defining $S_{\text{flip}}$. The key point is that doing so leaves the strongly connected components alone. However this makes the argument less intuitive.

Let $p_{\text{max}} = \max_{e \in E} p_e$. Combining Theorem 5 and Lemma 6, we have the following theorem.

**Theorem 7.** Let $T$ be the expected number of popped clusters. For a root-connected bi-directed graph $G = (V, A)$, $\mathbb{E} T \leq \frac{p_{\text{max}}}{1-p_{\text{max}}} mn$, where $n = |V|$, and $m = |A|$.

### 4. Approximate counting

We include the approximate counting algorithm of Gorodezky and Pak [GP14] for completeness. Let $G = (V, A)$ be an instance of Bi-directed Reachability with root $r$ and parameters $p$. We construct a sequence of graphs $G_0, \ldots, G_{n-1}$ where $n = |V|$ and $G_0 = G$. Given $G_{i-1}$, choose two arbitrary adjacent vertices $u_i$ and $v_i$, remove all arcs between $u_i$ and $v_i$ (in either direction), and identify $u_i$ and $v_i$ to get $G_i = (V_i, A_i)$. Namely we contract all arcs between $u_i$ and $v_i$, but parallel arcs in the resulting graph are preserved. If one of $u_i$ and $v_i$ is $r$, the new vertex is labelled $r$. Thus $G_{n-1} = (\{r\}, \emptyset)$. Since $A_i$ is always a subset of $A$, we denote by $p_i$ the parameters $p$ restricted to $A_i$. 


For $i = 1, \ldots, n - 1$, define a random variable $Z_i$ as follows:

$$Z_i := \begin{cases} 1 & \text{if } (V_{i-1}, S_{i-1}) \text{ is root-connected in } G_{i-1}; \\ 0 & \text{otherwise,} \end{cases}$$

where $S_{i-1} \subset A_{i-1}$ is a random root-connected subgraph drawn from the distribution $\pi_{G_{i-1}}(\cdot)$, together with all arcs $e$ between $u_i$ and $v_i$ added independently with probability $1 - p_e$. It is easy to see that

$$\mathbb{E} Z_i = \frac{Z(G_{i-1}, r; p_{i-1})}{Z(G_{i}, r; p_i)},$$

and

$$Z(G, r; p) = \prod_{i=1}^{n-1} \mathbb{E} Z_i.$$

Let $p_{\text{max}} = \max_{e \in E} p_e$ and $s = \lceil 5(1 - p_{\text{max}})^{-2}(n-1)\varepsilon^{-2} \rceil$. We estimate $\mathbb{E} Z_i$ by the empirical mean of $s$ independent samples of $Z_i$, denoted by $\tilde{Z}_i$, and let $\tilde{Z} = \prod_{i=1}^{n-1} \tilde{Z}_i$ and $Z = Z(G, r; p)$. Gorodezky and Park [GP14] showed the following.

**Proposition 8** ([GP14]). $Pr \left( \left| Z - \tilde{Z} \right| > \varepsilon Z \right) \leq 1/4$.

In order to sample $Z_i$, we use Algorithm 1 to draw independent samples of root-connected subgraphs. Since each resampling event involves at most $m$ arcs, Theorem 7 implies that each sample takes at most $\frac{p_{\text{max}}}{1 - p_{\text{max}}} m^2 n$ time. We need $O\left(\frac{n}{\varepsilon^2 (1 - p_{\text{max}})^2}\right)$ samples for each $Z_i$. Putting everything together, we obtain a proof of Theorem 1.

A natural question is what if $1 - p_{\text{max}}$ is close to 0. Intuitively, this means that some arc is very likely to fail. We note that, if $1 - p_e = O(n^{-3})$ for every arc $e$, then with high probability, sampling from the distribution $\pi(\cdot)$ yields a rooted spanning tree (with probability proportional to its weight). Thus, in this case, we can approximate $\pi(\cdot)$ by an efficient rooted spanning tree sampler, for example, the cycle-popping algorithm [Wil96].

5. Generalizing Algorithm 1

It is critical to have $p_e = p_{\text{ref}}$ in the proof of Lemma 6. Also, we need the two directions to be independent to ensure independence of neighbouring minimal clusters. Nonetheless, our method also generalizes to another distribution on the edges.

We consider a more general model. Instead of a directed graph, we assume the underlying graph $G$ is undirected. An edge $(u, v)$ has 4 possible states: $u \nrightarrow v$, $u \rightarrow v$, $u \leftarrow v$, and $u \leftrightarrow v$. A configuration $\sigma$ is an assignment from edges to these 4 states. We say $\sigma$ is root-connected if the corresponding directed graph is. Associate with each edge a distribution of the four possible states, and denote this mapping by $\delta$. Namely, $\delta(e)$ is the distribution on edge $e$. Note that we can easily recover single direction arcs or bi-directed edges in this model by associating the corresponding distributions to edges. Let $w_{\delta}(\sigma)$ be the probability of a configuration $\sigma$ under the product distribution upon all edges under $\delta$, and by abuse of notation,

$$Z(G, r; \delta) := \sum_{\sigma \text{ is root-connected}} w_{\delta}(\sigma).$$

Namely, $Z(G, r; \delta)$ is the probability that if the state of each edge $e$ is drawn from the distribution $\delta(e)$, the remaining graph is root-connected.

In particular, we consider *uni-directional* edges, where the distribution is an uniform orientation. Namely the edge points to one of its endpoints uniformly at random. If all edges are uni-directional, then we are interested in uniform orientations of the graph conditioned on being root-connected. Table 1 summarizes a few distributions on edges.

We consider the following problem.
Assignments of distributions on $E$ guarantees a polynomial expected running time via Theorem.

Proof. Lemma 9.

1. probilities of the same element in the support is within edges. We say two distributions are edge by uni-directional edges. The next lemma is useful for approximating distributions on Approximate bi-directed edges.

5.1. All of the rest of the analysis still applies because we can still flip edges no matter bi-directed instead of adding the arc $u \rightarrow v'$. We verify the extremal condition next. Let $C_1$ and $C_2$ be two distinct minimal clusters that both occurred. Note that Claim 4 holds for directed graphs in general, and therefore $C_1 \cap C_2 = \emptyset$. If there were a uni-directional edge $(u, v)$ such that $u \in C_1$ and $v \in C_2$, then at least one of $C_1$ and $C_2$ cannot be a cluster. Hence all edges between $C_1$ and $C_2$, if any, are bi-directed, which implies that $B_{C_1}$ and $B_{C_2}$ are independent. This confirms the extremal condition.

For the running time analysis, the proof of Lemma 6 still works. The only difference is that instead of adding the arc $u \rightarrow u'$, we may potentially flip this edge if it were uni-directional. All of the rest of the analysis still applies because we can still flip edges no matter bi-directed or uni-directional. Thus for these instances, we have that $Z_1 \leq \max_{e \in A} \left\{ \frac{p_e}{1-p_e}, 1 \right\} mnZ_0$, which guarantees a polynomial expected running time via Theorem 5.

5.1. Approximate bi-directed edges. In fact, we can approximately simulate a bi-directed edge by uni-directional edges. The next lemma is useful for approximating distributions on edges. We say two distributions are $\varepsilon$-close if their supports are the same and the ratio between probabilities of the same element in the support is within $1 \pm \varepsilon$.

Lemma 9. Let $G = (V, E)$ be a graph with root $r$ where $n = |V|$ and $m = |E|$, and $\delta, \delta'$ be two assignments of distributions on $E$. If for every $e \in E$, $\delta(e)$ and $\delta'(e)$ are $n^{-3}$-close, then

$$1 - \frac{1}{n} \leq \frac{Z(G, r; \delta')}{Z(G, r; \delta)} \leq 1 + \frac{1}{n}.$$ 

Proof. Since for every $e \in E$, $\delta(e)$ and $\delta'(e)$ are $n^{-3}$-close, for any subgraph $S \subseteq E$, we have

$$\left(1 - \frac{1}{n^3}\right)^m \leq \frac{wt_{\delta'}(S)}{wt_{\delta}(S)} \leq \left(1 + \frac{1}{n^3}\right)^m.$$ 

Since $m \leq n^2/2$, the lemma follows by summing over all root-connected subgraphs. □

It is standard to amplify the approximation ratio in Lemma 9 by taking identical copies. See, for example, [DGGJ04] for details. Thus by Lemma 9, we only need to get a distribution that is $n^{-3}$-close to the bi-directed one.

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Table 1. The distributions of different types of edges.

Name: Mixed Reachability

Instance: A graph $G = (V, A)$ with root $r$ and a mapping $\delta$ where $\delta(e)$ is either bi-directed or uni-directional.

Output: $Z(G, r; \delta)$.
Next let us define some notations about simulation and gadgets. An edge gadget $G$ is a graph with uni-directional edges and two special vertices. We use $\Pr_G(\cdot)$ to denote the probability space where each edge in $G$ is oriented uniformly and independently. When we want to simulate an edge $(u, v)$, we replace it with $G$ and identify $(u, v)$ with the two special vertices of $G$. Denote by $p_\emptyset$, $p_u$, $p_v$, and $p_{uv}$ the probabilities of the four states: $u \not\rightarrow v$, $u \rightarrow v$, $u \leftarrow v$, and $u \leftrightarrow v$. In a gadget with uni-directional edges, $p_u = p_v$, and we abbreviate these probabilities by a vector $[p_\emptyset; p_u; p_{uv}]$. Using this notation, a bi-directed edge with parameter $p$ is $[p^2; p(1 - p), (1 - p)^2]$ and a uni-directional edge is $[0, 1/2, 0]$. For a gadget $G$, the effective probability of, for example, $u \not\rightarrow v$ is

$$
\Pr_G(u \text{ can reach } v \text{ but } v \text{ cannot reach } u \mid \text{ all vertices in } G \text{ can reach either } u \text{ or } v).
$$

(1) Probabilities for other states are similar. Note that the conditioning is necessary.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{gadget.png}
\caption{The gadget to simulate a bi-directed edge.}
\end{figure}

The gadget to approximate a bi-directed edge is a $k$-stretching composed with an $\ell$-thickening with sufficiently large $k$ and $\ell$ linear in $k$, depicted in Figure 1. To be specific, for a path $P_k$ of $k$ edges (namely $k$-stretching) with endpoints $u$ and $v$, due to the conditioning, there are $k - 1$ orientations such that $u \not\leftrightarrow v$, 1 orientation each so that $u \rightarrow v$ or $u \leftarrow v$, and no orientation making $u \leftrightarrow v$. Thus, the effective distribution of $P_k$ is $\left[\frac{k - 1}{k + 1}, \frac{1}{k + 1}, 0\right]$. Then we put $\ell$ copies of $P_k$ together (namely $\ell$-thickening) with common endpoints $u$ and $v$. The effective distribution is

$$
\left[\left(\frac{k - 1}{k + 1}\right)^\ell, \left(\frac{k}{k + 1}\right)^\ell - \left(\frac{k - 1}{k + 1}\right)^\ell, 1 - 2\left(\frac{k}{k + 1}\right)^\ell + \left(\frac{k - 1}{k + 1}\right)^\ell\right].
$$

(2) For an arbitrary constant $p \in (0, 1)$, let $\ell = \lceil k \ln p^{-1} \rceil$. It is easy to see that the distribution in (2) is $n^{-3}$-close to $[p^2; p(1 - p), (1 - p)^2]$ by taking a polynomially large $k$. By Lemma 9, we can effectively simulate a bi-directed edge with parameter $p$ at the cost of polynomial overhead.

5.2. A reduction from reliability to reachability. Similarly to the results above, we can simulate undirected edges by arcs. This would provide a reduction from RELIABILITY to REACHABILITY. We formally define the problem of undirected all-terminals reliability.

**Name:** RELIABILITY

**Instance:** A (undirected) graph $G = (V, E)$, and the parameters $p = (p_e)_{e \in E}$.

**Output:** The probability that if each edge $e$ fails with probability $p_e$, the remaining graph is connected.

We show that REACHABILITY is at least as hard as RELIABILITY to approximate.

**Theorem 10.** If there is an FPRAS for REACHABILITY, then there is an FPRAS for RELIABILITY.
Proof. Since reachability in undirected graphs is the same as reliability, we only need to simulate undirected edges. To this end, we use a gadget from [Jer81], depicted in Figure 2.

To simulate arcs with 0 probability to fail, we can simply do a $k$-thickening of arcs with failure probability $p$. (If one insists on simple graphs, then we can 2-stretch these parallel edges.) The effective failure probability is $p^k$ which decays exponentially in $k$. Thus we only need to take a polynomially large $k$ to approximate 0 failure probability.

6. Concluding remarks

In this paper we confirm the conjecture of Gorodezky and Pak [GP14] and give an FPRAS for Bi-directed Reachability. The core ingredient of the FPRAS is the cluster-popping algorithm to sample root-connected subgraphs, namely Algorithm 1. We manage to analyze it using the partial rejection sampling framework.

A natural question is how far can we generalize Algorithm 1. We may construct gadgets to simulate distributions other than the two we considered. In all the constructions we have tried, the two directions are always anti-correlated in the resulting effective distribution. To prove this negative correlation seems difficult due to the conditioning in (1). Unfortunately, the interesting case of undirected edges is exactly the opposite. In an undirected edge, the two directions are positively correlated (in a maximum way). Thus, RELIABILITY remains resistant to our approach.

Another direction is how far can we go beyond bi-directed graphs? What about Eulerian directed graphs? Is approximating Reachability NP-hard in general?

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References


