A Simple FPRAS for Bi-directed Reachability

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Counting

THE COMPLEXITY OF COMPUTING QUANTITIES

Complexity class **#P** by Valiant (1979):

a counting analogue of **NP**.

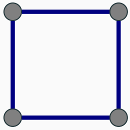
Evaluation of probabilities;
Multivariate integration;
Counting discrete structures …

Network reliability

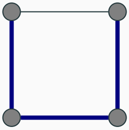
Reliability: in a graph (or network) $G = (V, E)$, suppose each edge fails with probability p. What's the probability that the remaining graph is connected?

In other words, we want to compute

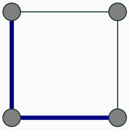
$$Z_{rel}(G, p) := \sum_{R \subseteq E : (V, R) \text{ is connected}} p^{|E \setminus R|}(1-p)^{|R|}.$$



$(1-p)^4$



$p(1-p)^3$



Disconnected!

Computational complexity of reliability

The unweighted case (namely, $p = 0.5$) is among the original 17 #**P**-complete problem in [Valiant '79].

Exact evaluation is #**P**-complete [Jerrum '81] [Provan, Ball '83].

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REACHABILITY

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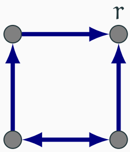

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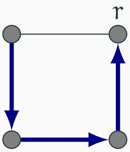

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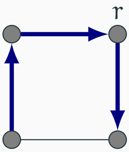

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$$Z_{reach}(G, p) := \sum_{R \subseteq E : (V, R) \text{ is root-connected}} p^{|E \setminus R|}(1-p)^{|R|}.$$

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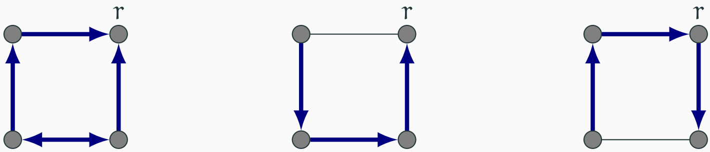

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Cluster popping

If G is bi-directed, approximating $Z_{reach}(G, p)$ can be reduced to sampling root-connected subgraphs [Gorodezky, Pak 14].

Cluster: no edge going out.

Cluster popping [Gorodezky, Pak 14]: randomize edges and repeatedly pop minimal clusters.

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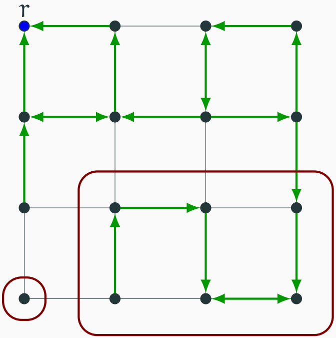

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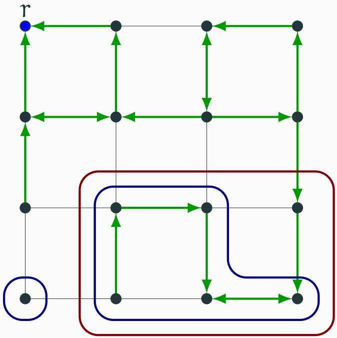

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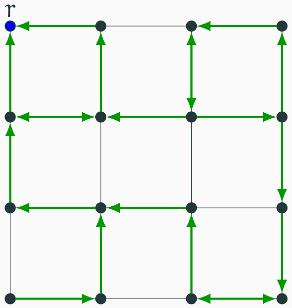

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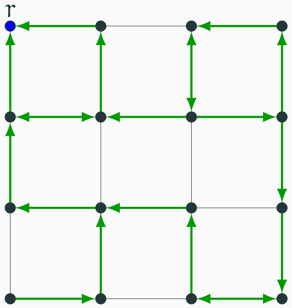

Partial rejection sampling

(why is cluster-popping correct and efficient?)

A random walk SAT-solver

The prototypical NP-complete problem:
given a CNF formula, does it have a satisfying assignment?

$$(x_1 \vee \overline{x_3} \vee x_5) \wedge (x_2 \vee x_3) \wedge (\overline{x_3} \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_5} \vee x_6 \vee x_7) \dots$$

Rejection sampling: assign each variable uniformly at random and independently. If not satisfying, reject and repeat.

Walk-SAT: while there is a violated clause, re-randomize all its variables.

It is optimal in a very general setting! [Moser, Tardos 10]

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Variable framework of the Lovász Local Lemma

Find a "perfect" assignment of the variables avoiding all "bad" events.

Variables $X_1, \ldots, X_n$ "Bad" events $A_1, \ldots, A_m$

Dependency graph: A_i and A_j are adjacent if $\mathtt{var}(A_i) \cap \mathtt{var}(A_j) \neq \emptyset$.

Erdős and Lovász (1975): $4p\Delta \leqslant 1 \Rightarrow$ existence of a perfect assignment.

p: max probability of A_i Δ: max degree of the dependency graph

Lovász (1977) improved the condition to $ep(\Delta + 1) \leqslant 1$.

Shearer (1985) gave the optimal condition of LLL.

LLL only guarantees an exponentially small probability.

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Beck (1991) showed that an algorithmic version is possible, starting a long line of research.

Moser and Tardos (2010) found a very elegant algorithm:

1. Initialize all variables randomly.
2. While there exists an occurring bad event:
   pick one (various rules) and resample all its variables.

Many developments since then:

[Haeupler, Saha, Srinivasan 11], [Kolipaka, Szegedy 11], [Harris, Srinivasan 13], [Achlioptas, Iliopoulos 16], [Harvey, Vondrak 15], [He, Li, Liu, Wang, Xia 17].

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Searching vs. Sampling

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Instead of finding a solution, can we generate a random solution?

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Sampling independent sets?

Target distribution: uniform on independent sets.

Adapting Moser-Tardos:

1. Randomize each vertex.
2. Resample all connected component of size at least 2, until there is none.

This does not draw from the target distribution:

- Once a vertex is unoccupied, it will stay unoccupied till the end. Hence the empty set is overly favored.
- The process converges too fast. However uniformly sampling independent set is NP-hard (even approximately).

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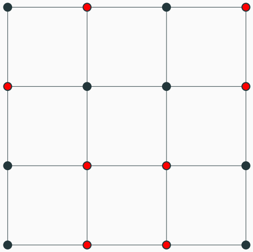

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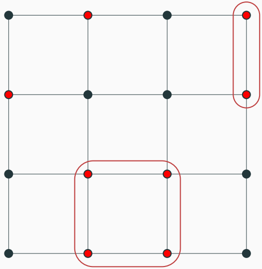

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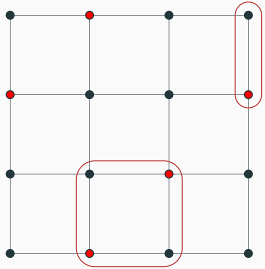

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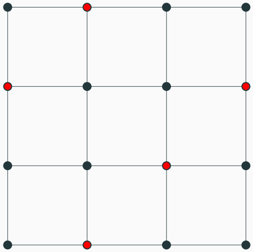

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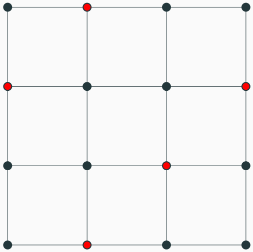

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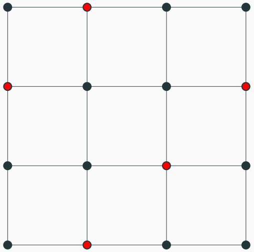

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Wilson's "cycle-popping" algorithm (1996)

Goal: sample a uniform spanning tree with root r.

1. For each $v \neq r$, assign a random arrow from v to one of its neighbours.
2. While there is a (directed) cycle in the current graph, resample all arrows along all cycles.
3. Output.

No cycle + $n-1$ edges $\Rightarrow$ Spanning Tree

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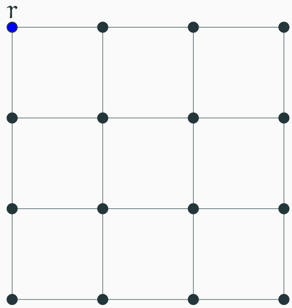

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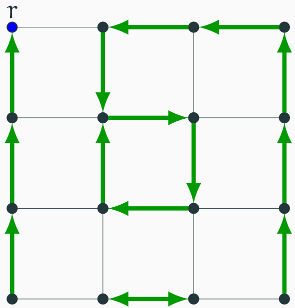

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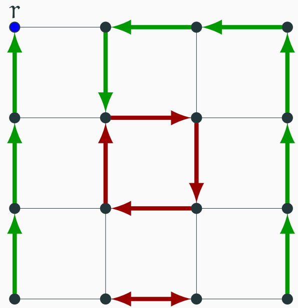

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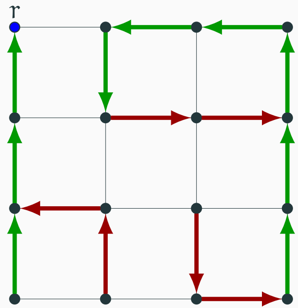

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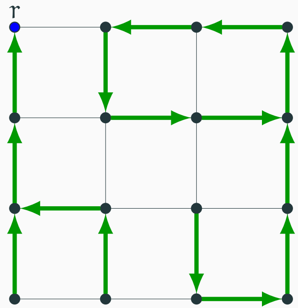

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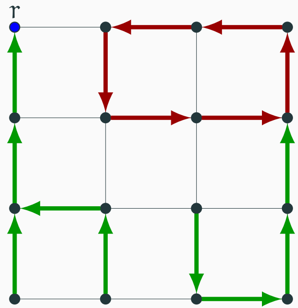

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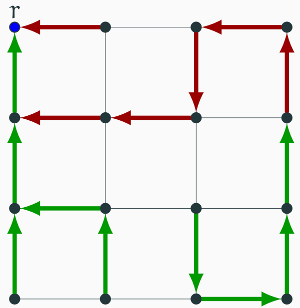

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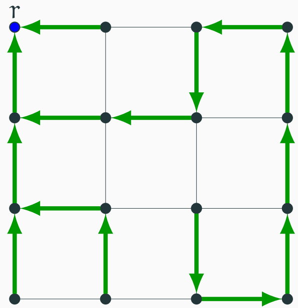

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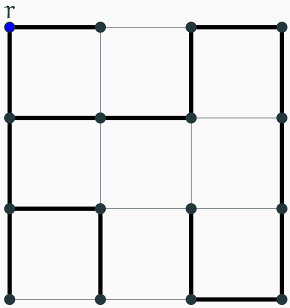

No cycle + $n - 1$ edges $\Rightarrow$ Spanning Tree

Wilson's "cycle-popping" algorithm (1996)

Goal: sample a uniform spanning tree with root r.

1. For each $v \neq r$, assign a random arrow from v to one of its neighbours.
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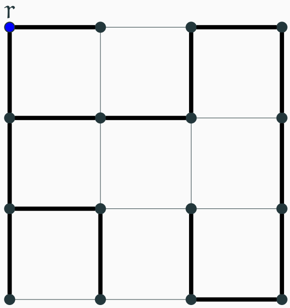

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We call an instance extremal:

if any two "bad" events A_i and A_j are either independent or disjoint.

- Extremal instances minimize the probability of solutions (in some precise sense) [Shearer 85].
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Wilson's setup is extremal:

If two cycles share a vertex (dependent) and they both occur (overlapping), then these two cycles must be identical by following the arrow!

Other extremal instances:

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For extremal instances, replacing a perfect assignment with another one will not change the resampling history!

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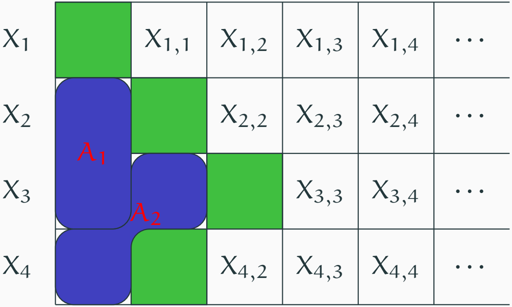

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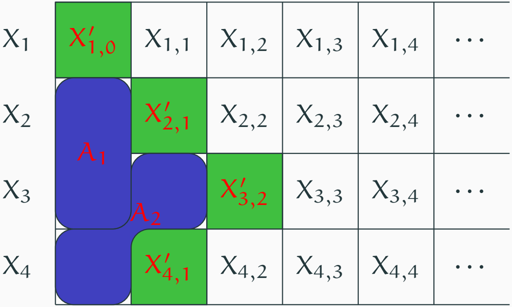

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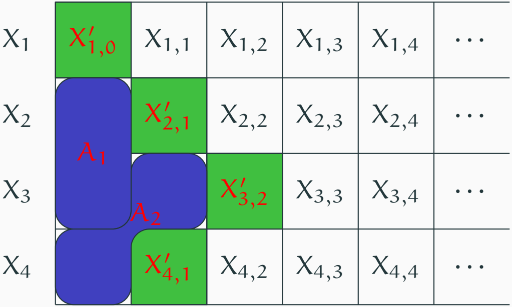

For any output σ and τ, there is a bijection between trajectories leading to σ and τ.

Running time on extremal instances

Theorem (G., Jerrum, Liu 17)

Under Shearer's condition, for extremal instances,

$$\mathbb{E}\,T = \sum_{i=1}^{m} \frac{q_i}{q_\emptyset} = \frac{\#\text{ near-perfect assignments}}{\#\text{ perfect assignments}}.$$

(Shearer's condition: $q_S \geqslant 0$ for all $S \subseteq V$, where q_S is the independence polynomial on $G \setminus \Gamma^+(S)$ with weight $-p_j$ on vertex j.)

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Cluster-popping: repeatedly resample minimal clusters.

Let Ω_k be the set of subgraphs with k minimal clusters.

$$Z_k := \sum_{S \in \Omega_k} p^{|E \setminus S|}(1-p)^{|S|} \qquad\qquad \mathbb{E}\,T = \frac{Z_1}{Z_0}$$

[G., Jerrum 17]: for bi-directed graphs, $Z_1 \leqslant \frac{mn}{1-p} Z_0$.

We show this by designing an injective mapping $\Omega_1 \to \Omega_0 \times V \times E$.

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There is an FPRAS for Reachability *in bi-directed graphs. The running time is* $O\left(\varepsilon^{-2}p(1-p)^{-3}m^2n^3\right)$ *for an* $(1 \pm \varepsilon)$*-approximation.*

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A correct algorithm for sampling independent sets

1. Randomize each vertex.
2. Let $\texttt{Bad}$ be the set of vertices whose connected component has size $\geqslant 2$.
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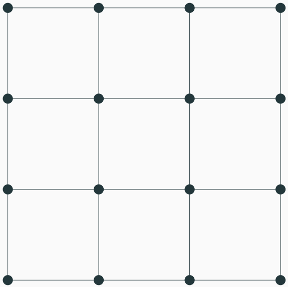
4. Resample $\texttt{Res}$.
   Check independence.



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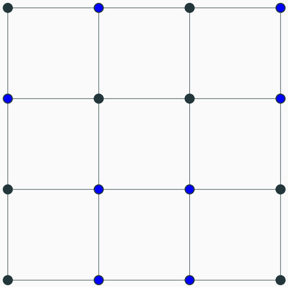

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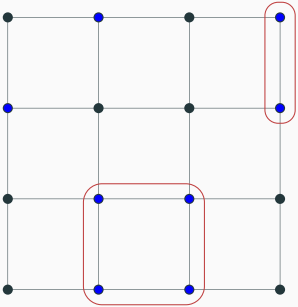

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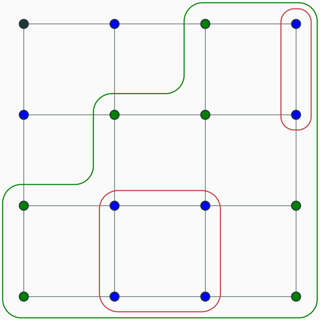

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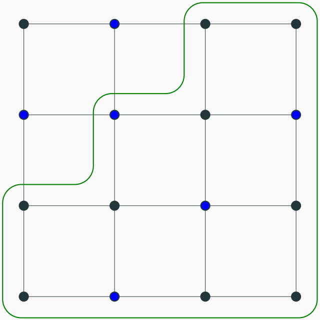

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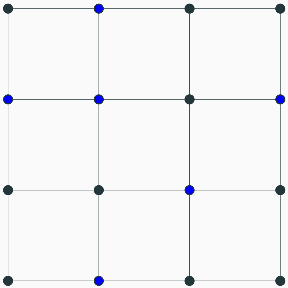

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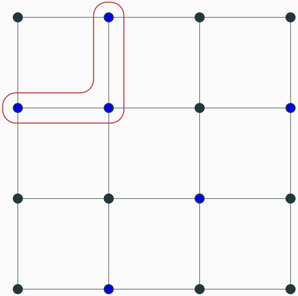

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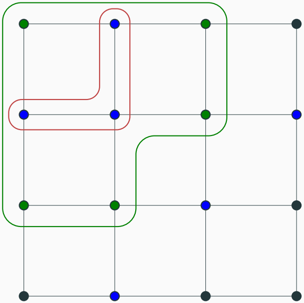

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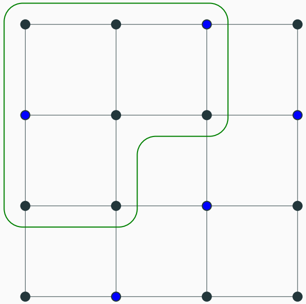

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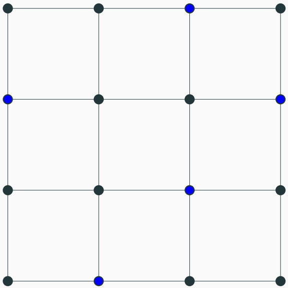

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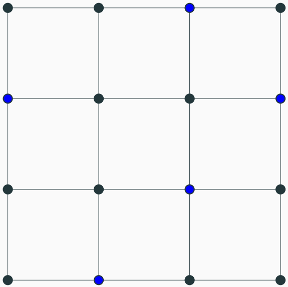

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We also gave a general algorithm, and is optimal in certain restricted cases, up to constants [G., Jerrum, Liu 17].

But it falls short in general. My conjecture is that there is an efficient algorithm whenever $p\Delta^2 \leqslant C$ for some constant C.

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Concluding remarks

Open problems

- How to sample connected subgraphs (or approximate *reliability*)?
- What is the optimal sampling algorithm in the local lemma setting in general?
- Can we do this for perfect matchings - resampling permutations???

A professor is one who can speak on any subject for precisely fifty minutes.

— Norbert Wiener

Thank you!

arXiv:1611.01647

arXiv:1709.08561