Holographic Algorithms Beyond Matchgates

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(joint work with Jin-Yi Cai and Tyson Williams)

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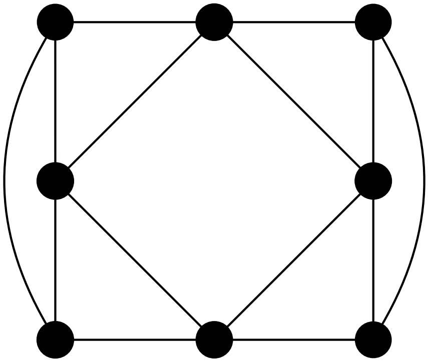
København

July 11th 2014

Counting Perfect Matchings

Perfect Matchings



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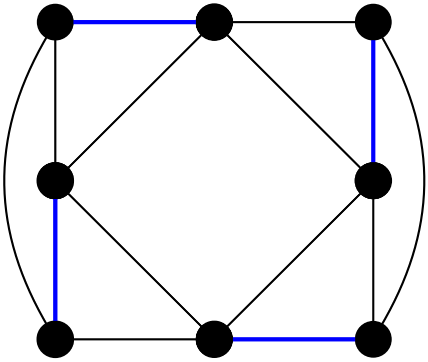

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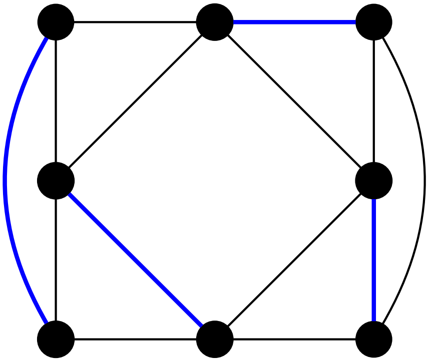

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- The FKT algorithm is based on Pfaffian orientations of planar graphs.

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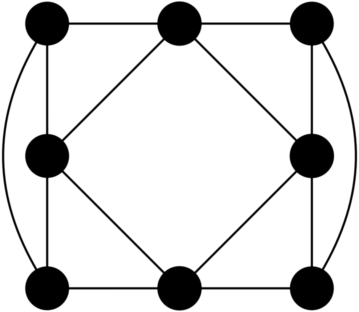

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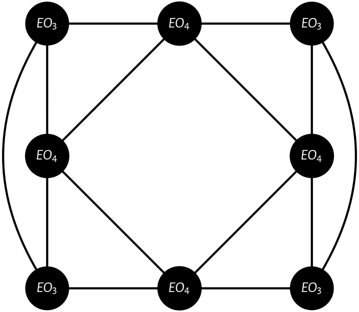
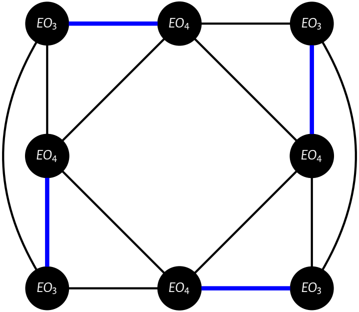


Counting Perfect Matchings Revisited

A systematic way to view $\#\mathrm{PM}$.

- Put functions $\textsc{Exact-One}$ (EO) on nodes and make edges variables.
- $\#\mathrm{PM}$ is just the partition function:

$$\#\mathrm{PM} = \sum_{\sigma : E \mapsto \{0,1\}} \prod_{v \in V} \mathrm{EO}_d(\sigma \mid_{E(v)}).$$



Holant Problems

The (Boolean) Holant problem on instance Ω is to evaluate

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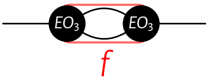

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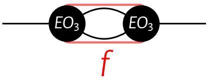

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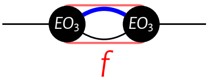

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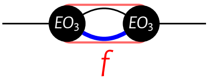

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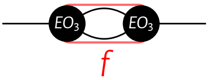

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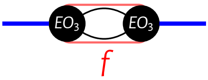

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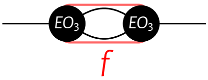

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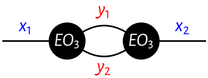

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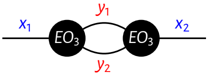

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- Note that $\text{Holant}(f) \equiv \text{Holant}(f \mid =_2)$.

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This work provides some answer to the question.

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- Functions expressible by symmetric functions are not necessarily symmetric.

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- $\neq_2$ imposes an orientation and $[0,0,1,0,0]$ requires it to be Eulerian.
- Holant$([3,0,1,0,3])$ in fact counts the number of Eulerian orientations on 4-regular graphs (up to an easy to compute factor).

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- Affine type, denoted $\mathscr{A}$.
  - Parity functions define an affine system and the number of solutions is easy to compute via computing the rank. The family $\mathscr{A}$ generalizes such functions.

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- In particular, this family contains Clifford gates in quantum computation as a special case.

$\mathscr{A}$- and $\mathscr{P}$-transformable Functions

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- $\mathscr{A}$ (or $\mathscr{P}$)-transformable is a proper super set of $\mathscr{A}$ (or $\mathscr{P}$).
- Fibonnaci gates [Cai, Lu, Xia 08] are in fact $\mathscr{P}$-transformable.

Holant Dichotomy for General Graphs

Theorem (Cai, G., Williams 13)

Let f be a symmetric function. $\text{Holant}(f)$ *is* $\#\mathrm{P}$*-hard unless f is*

1. *degenerate or binary,*
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Corollary

The dichotomy theorem for symmetric Holant problems is decidable in polynomial time.

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- Recall that for $f \in \mathscr{A}$,

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- First check whether the support S of f is an affine subspace:

  Build a basis inductively.

Deciding General $\mathscr{A}$ Functions

- Recall that for $f \in \mathscr{A}$,

$$f(\mathbf{x}) = \chi_{\mathbf{x}A=0} \cdot \sqrt{-1}^{\mathbf{x}B\mathbf{x}^{\mathrm{T}}}.$$

  For a fixed arity n, there are $2^{O(n^2)}$ distinct functions in $\mathscr{A}$.

- First check whether the support S of f is an affine subspace:

  Build a basis inductively.

- If so, decide B by solving entries one at a time.

  Then check if it is consistent with f.

Deciding General $\mathscr{A}$-transformable

- We want to decide whether there exists $T \in \mathbf{GL}_2(\mathbb{C})$ such that $fT^{\otimes n} \in \mathscr{A}$ (or $\mathscr{P}$), with the additional restriction $\left((T^{-1})^{\otimes 2} =_2\right) \in \mathscr{A}$ (or $\mathscr{P}$).

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- Consider the stabilizer group of $\mathscr{A}$:

$$\mathrm{Stab}(\mathscr{A}) := \{T \in \mathbf{GL}_2(\mathbb{C}) | T\mathscr{A} \subseteq \mathscr{A}\}.$$

In fact, $\mathrm{Stab}(\mathscr{A})$ is generated by matrices $\left[\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ up to a constant.

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- Normalize a valid transformation T using matrices in $\mathrm{Stab}(\mathscr{A})$ such that either $T \in \mathbf{SO}_2(\mathbb{C})$ or $\begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{\pi \mathrm{i}}{4}} \end{bmatrix} T \in \mathbf{SO}_2(\mathbb{C})$.

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- Diagonal transformations are easy to check.

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- Let $v_0 = \left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right]$ and $v_1 = \left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]$. Define

$$\theta(v_0, v_1) := \left(\frac{a_0 a_1 + b_0 b_1}{a_1 b_0 - a_0 b_1}\right)^2.$$

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- $\mathscr{A}$-transformable $\Rightarrow \theta(v_0, v_1) = 0, -1$ or $-\frac{1}{2}$.
- When all these are satisfied, valid transformations are restricted to polynomially many.

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Let $f(\mathbf{x}) = \prod_i f_i(\mathbf{x}_i)$ where $\{\mathbf{x}_i\}$ is a partition.

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Then f_i's are unique up to permutations and can be computed in polynomial time.

- Function product factorizations are not unique, that is, f_i's are not unique if some $\mathbf{x}_i$ and $\mathbf{x}_j$ overlap.
- Deciding membership of $\mathscr{P}$ is straightforward.

Deciding $\mathscr{P}$-transformable

- For general functions, using ideas similar to $\mathscr{A}$-transformable, we can restrict to orthogonal and related transformations. Then check them in the $\left[\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right]$ basis.

Deciding $\mathscr{P}$-transformable

- For general functions, using ideas similar to $\mathscr{A}$-transformable, we can restrict to orthogonal and related transformations. Then check them in the $\left[\begin{smallmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{smallmatrix}\right]$ basis.
- For symmetric functions, the procedure is also similar to deciding symmetric $\mathscr{A}$-transformable functions. We can check if f is a sum of two tensor powers and then check $\theta(v_0, v_1)$. When both checks pass, the number of valid transformations are restricted.

Thank you!

Papers are available on my homepage:

`pages.cs.wisc.edu/~hguo/`