Edge Coloring, Siegel's Theorem, and a Holant Dichotomy

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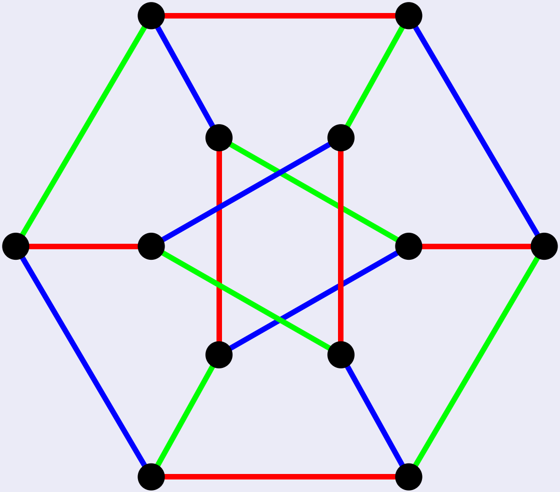
Edge Coloring

Definition



Deciding Edge Colorings

Theorem (Vizing's Theorem)

Edge coloring using at most $\Delta(G) + 1$ colors exists in simple graphs.

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Optimization version for multi-graphs.

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INPUT: A graph G.

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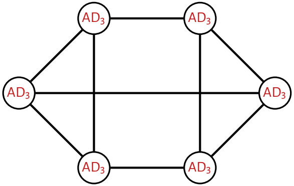
(There is no regular planar graph of degree larger than or equal to 6 by counting the average degree of a triangulation. Our result is actually for multi-graphs when $r > 5$.)

Counting Edge Colorings as a Holant Problem

Put the local constraint function AD_3 on each node.

$$\mathrm{AD}_3(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [\kappa] \text{ are distinct} \\ 0 & \text{otherwise.} \end{cases}$$



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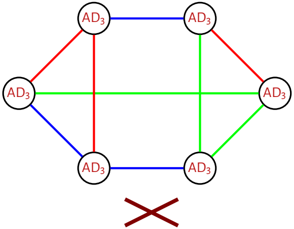

- A configuration is a proper coloring if and only if it satisfies all constraints, that is,

$$w(\sigma) = \prod_{v \in V(G)} \mathrm{AD}_3\left(\sigma|_{E(v)}\right) = 1.$$

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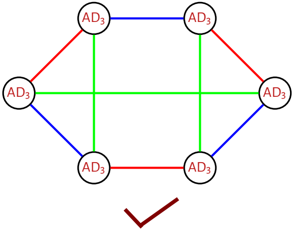

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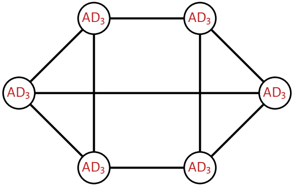

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- To compute $\#\kappa$-EdgeColoring, we sum over all configurations:

$$\mathrm{Holant}(G; \mathrm{AD}_3) = \sum_{\sigma : E(G) \to [\kappa]} w(\sigma).$$

Holant Problems

In this talk, we consider all local constraint functions

$$f(x,y,z) = \begin{cases} a & \text{if } x = y = z & \text{(all equal)} \\ b & \text{otherwise} & \\ c & \text{if } x \neq y \neq z \neq x & \text{(all distinct).} \end{cases}$$

Denote f by $\langle a, b, c \rangle$. Then $\mathrm{AD}_3 = \langle 0, 0, 1 \rangle$.

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$$\mathrm{Holant}_\kappa(G; f) = \sum_{\sigma: E(G) \to [\kappa]} \prod_{v \in V(G)} f\left(\sigma \mid_{E(v)}\right).$$

aka: tensor network contraction, factor graphs, ...

Main Theorem

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For any domain size $\kappa \geqslant 3$ and any $a, b, c \in \mathbb{C}$, the problem of computing $\text{Holant}_\kappa(-; \langle a, b, c \rangle)$ is either #**P**-hard or in polynomial time, even when the input is restricted to planar graphs.

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- #κ-EdgeColoring in 3-regular graphs is the special case $\langle a, b, c \rangle = \langle 0, 0, 1 \rangle$, and it is #**P**-hard.

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Some tractable cases are not so obvious, for example,
$\kappa = 3$ and $\text{Holant}_3(-; \langle -5, -2, 4 \rangle)$;
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$\kappa = 4$ and $\text{Holant}_4(-; \langle -3 - 4i, 1, -1 + 2i\rangle)$.
We have a simple procedure to verify.

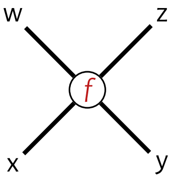
Proof of #3-EdgeColoring on 3-regular graphs

The hardness of $\text{Holant}_3(-; \text{AD}_3)$ is shown by the following reduction chain:

$$\begin{aligned}\#\mathbf{P} &\leqslant_T \text{Holant}_3(-; \langle 2, 1, 0, 1, 0\rangle) \\ &\leqslant_T \text{Holant}_3(-; \langle 0, 1, 1, 0, 0\rangle) \\ &\leqslant_T \text{Holant}_3(-; \text{AD}_3)\end{aligned}$$

$\langle a, b, c, d, e\rangle$ denotes an arity-4 function f

$$f\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \begin{cases} a & \text{if } w = x = y = z \\ b & \text{if } w = x \neq y = z \\ c & \text{if } w = y \neq x = z \\ d & \text{if } w = z \neq x = y \\ e & \text{otherwise.} \end{cases}$$



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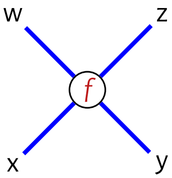

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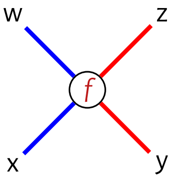

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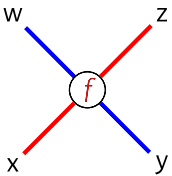

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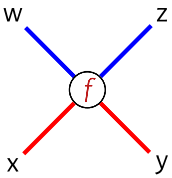

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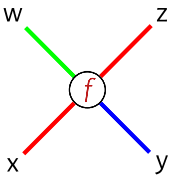

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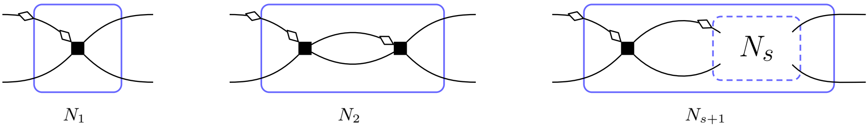
Polynomial Interpolation Step: Recursive Construction

$$\text{Holant}_3(G; \langle 2, 1, 0, 1, 0\rangle) \leqslant_T \text{Holant}_3(G_s; \langle 0, 1, 1, 0, 0\rangle)$$



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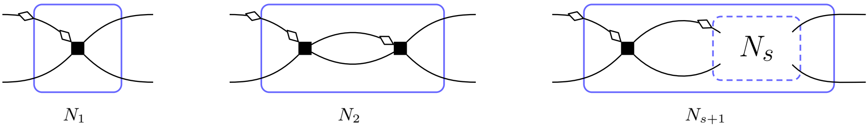

Vertices are assigned $\langle 0, 1, 1, 0, 0 \rangle$. Inputs are ordered anti-clockwise.
Let f_s be the function corresponding to N_s. Then $f_s = M^s f_0$, where

$$M = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad f_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Note that $f_1 = \langle 0, 1, 1, 0, 0 \rangle$.

Polynomial Interpolation Step: Eigenvectors and Eigenvalues

Spectral decomposition $M = P\Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \Lambda = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

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Let $x = 2^{2s}$. Then

$$f_{2s} = P\Lambda^{2s}P^{-1}f_0 = P \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}f_0 = \begin{bmatrix} \frac{x+2}{3} \\ \frac{x-1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

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Note $f(4) = \langle 2, 1, 0, 1, 0 \rangle$.

Polynomial Interpolation Step: The Interpolation

$$\text{Holant}_3(-; \langle 2, 1, 0, 1, 0 \rangle) \leqslant_T \text{Holant}_3(-; \langle 0, 1, 1, 0, 0 \rangle)$$

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\end{aligned}$$

If G has n vertices, then

$$p(G,x) = \mathrm{Holant}_3(G;f(x)) \in \mathbb{Z}[x]$$

has degree n.

Polynomial Interpolation Step: The Interpolation

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If G has n vertices, then

$$p(G,x) = \text{Holant}_3(G; f(x)) \in \mathbb{Z}[x]$$

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Let G_s be the graph obtained by replacing every vertex in G with N_s. Then $\text{Holant}_3(G_{2s}; \langle 0,1,1,0,0\rangle) = p(G, 2^{2s})$.

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Using oracle for $\mathrm{Holant}_3(-; \langle 0, 1, 1, 0, 0\rangle)$, evaluate $p(G, x)$ at $n + 1$ distinct points $x = 2^{2s}$ for $0 \leqslant s \leqslant n$.

By polynomial interpolation, efficiently compute the coefficients of $p(G, x)$.

More on Polynomial Interpolations

- We may not be able to interpolate all vectors, but a subspace of them.
- #3-EdgeColoring has a fixed domain size and a fixed function. For the dichotomy, we need to deal with multivariate polynomial interpolations.
- The key to the proof is to show certain linear systems are non-degenerate.

Interpolation Lemma

Lemma

Suppose there is a recurrence construction implemented by $\mathcal{F}$. Let $s \in \mathbb{C}^n$ be the initial vector and $M \in \mathbb{C}^{n \times n}$ be the recurrence matrix. If s and M satisfy the following conditions:

1. M is diagonalizable with n linearly independent eigenvectors;
2. s is not orthogonal to ℓ of these linearly independent row eigenvectors of M with eigenvalues $\lambda_1, \ldots, \lambda_\ell$;
3. $\lambda_1, \ldots, \lambda_\ell$ satisfy the lattice condition;

then

$$\operatorname{Holant}_\kappa(-; \mathcal{F} \cup \{f\}) \leqslant_T \operatorname{Holant}_\kappa(-; \mathcal{F})$$

for any signature f that is orthogonal to the $n - \ell$ of these linearly independent eigenvectors of M to which s is also orthogonal.

Lattice Condition

Definition

We say that $\lambda_1, \lambda_2, \ldots, \lambda_\ell \in \mathbb{C} - \{0\}$ satisfy the lattice condition if

$$\forall x \in \mathbb{Z}^\ell - \{\mathbf{0}\} \qquad \text{with} \qquad \sum_{i=1}^{\ell} x_i = 0,$$

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$$\prod_{i=1}^{\ell} \lambda_i^{x_i} \neq 1.$$

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- For our application, λ_i's are eigenvalues, that is, roots to the characteristic polynomial of the recurrence matrix.
- Our characteristic polynomial is $(x - \kappa^3)^4 p(x, \kappa) =$

$$(x - \kappa^3)^4 \left(x^5 - \kappa^6(2\kappa - 1)x^3 - \kappa^9(\kappa^2 - 2\kappa + 3)x^2 + (\kappa - 2)(\kappa - 1)\kappa^{12}x + (\kappa - 1)^3\kappa^{15}\right).$$

Some Galois Theory

Lemma

Let $q(x) \in \mathbb{Q}[x]$ be a polynomial of degree $n \geqslant 2$. If

1. the Galois group of q over $\mathbb{Q}$ is S_n or A_n and
2. the roots of q do not all have the same complex norm,

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For any integer $\kappa \geqslant 3$, if $p(x, \kappa)$ is irreducible in $\mathbb{Q}[x]$, then the roots of $p(x, \kappa)$ satisfy the lattice condition.

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Lemma

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Proof.

By discriminant, $p(x, \kappa)$ has 3 distinct real roots and 2 complex roots. Three distinct real roots do not have the same norm. An irreducible polynomial of prime degree n with exactly two nonreal roots has S_n as its Galois group over $\mathbb{Q}$. □

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We show that these are the only integer solutions.

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Theorem (Siegel's Theorem)

Any smooth algebraic curve of genus $g > 0$ defined by a polynomial $f(x, y) \in \mathbb{Z}[x, y]$ has only finitely many integer solutions.

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- Bad news is that Siegel's theorem is not effective.
- There are several effective versions, but the best bound we can find (applied to our polynomial) is 10^{20000}.
- Integer solutions could be enormous.

Diophantine Equations with Enormous Solutions

Pell's Equation (genus 0)

$$x^2 - 61y^2 = 1$$

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Next solution:

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Puiseux Series

Let $y = \kappa + 1$. This simplifies our polynomial:

$$p(x, y) = x^5 - (2y+1)x^3 - (y^2+2)x^2 + (y-1)yx + y^3.$$

Puiseux series converges to the actual roots for polynomials in two variables.

Puiseux series for $p(x, y)$ are

$$\begin{aligned}
y_1(x) &= x^2 + 2x^{-1} + 2x^{-2} - 6x^{-4} - 18x^{-5} + O(x^{-6}),\\
y_2(x) &= \quad x^{3/2} - \frac{1}{2}x + \frac{1}{8}x^{1/2} - \frac{65}{128}x^{-1/2} - x^{-1} - \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2}),\\
y_3(x) &= -x^{3/2} - \frac{1}{2}x - \frac{1}{8}x^{1/2} + \frac{65}{128}x^{-1/2} - x^{-1} + \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2}).
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Runge's method

We want to pick polynomials $g_i(x, y)$ so that when we substitute $y_i(x)$ in $g_i(x, y)$, it is $o(1)$.
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We pick $g_2(x,y) = y^2 + xy - x^3 + x$.

$$\begin{aligned} g_2\left(x, y_2(x)\right) &= -\Theta\left(\sqrt{x}\right), \\ g_2\left(x, y_3(x)\right) &= \Theta\left(\sqrt{x}\right). \end{aligned}$$

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We pick $g_2(x,y) = \frac{y^2+xy-x^3+x}{x} = \frac{y^2}{x} + y - x^2 + 1$.

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But this $g_2(x,y)$ is not really a polynomial.

An Effective Siegel's Theorem for Our Polynomial

Let (a, b) be an integer solution to $p(x, y) = 0$ with $a \neq 0$.

Let p be a prime factor of a.
By considering the order of any such p in b, we can show that $a|b^2$.

So $g_2(a, b) = \frac{b^2}{a} + b - a^2 + 1$ is always an integer.

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So $g_2(a, b) = \frac{b^2}{a} + b - a^2 + 1$ is always an integer.

For (say) $y_2(x)$, we truncate the Puiseux series to get $y_2^+(x)$ and $y_2^-(x)$ such that

$$p\left(x, y_2^-(x)\right) < 0 \text{ and } p\left(x, y_2^+(x)\right) > 0.$$

Then for $x > 16$,

$$-1 < g_2\left(x, y_2^-(x)\right) \leqslant g_2\left(x, y_2(x)\right) \leqslant g_2\left(x, y_2^+(x)\right) < 0.$$

Similarly for $y_1(x)$ and $y_3(x)$.
Hence if $x > 16$, there is no integer solution.
For $x \leqslant -3$, there is only one real root which is not an integer.
Otherwise $-2 \leqslant x \leqslant 16$ it is easy to verify.

Back to Lattice Condition

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Proof.

By previous Lemma, no linear factor over $\mathbb{Z}$.
By Gauss' Lemma, no linear factor over $\mathbb{Q}$.
Then $p(x, y)$ factors as a product of two irreducible polynomials of degrees 2 and 3.
We use some Galois theory to show the lattice condition. $\square$

Summary

- #**P**-hardness of #κ-EdgeColoring in r-regular planar graphs ($\kappa \geqslant r \geqslant 3$).
- A Holant dichotomy with arbitrary domain size.
- Interpolation is a powerful technique in proving counting dichotomies.

  Interesting algebraic problems may rise.

Thank You!

Papers and slides on my webpage:

`www.cs.wisc.edu/~hguo/`