Markov chain algorithms for bounded degree $k$-Sat

Heng Guo (University of Edinburgh)

Joint with Weiming Feng, Yitong Yin (Nanjing University) and Chihao Zhang (Shanghai Jiao Tong University)

LFCS lab lunch, Mar 10th, 2020
Satisfiability

One of the most important problems in computer science

**Input:** A formula in conjunctive normal form, like

\((x_1 \lor \overline{x}_3 \lor x_5) \land (x_2 \lor x_3) \land (\overline{x}_3 \lor \overline{x}_4) \land (x_1 \lor \overline{x}_5 \lor x_6 \lor x_7) \ldots\)

**Output:** Is it satisfiable?

The first NP-complete problem Cook (1971) — Levin (1973)
Sometimes we are not satisfied with finding one solution. We want to generate a uniformly at random solution.

The ability of sampling solutions enables us to

- approximately count the number of solutions;
- estimate the marginal probability of individual variables;
- estimate other quantities of interest ...

And sometimes generating random instances satisfying given constraints can be useful too.

Sampling can be NP-hard even if finding a solution is easy (e.g. under Lovász local lemma conditions).
A natural (but not working) approach

Standard sampling approach: Glauber dynamics / Gibbs sampling
A natural (but not working) approach

Standard sampling approach: Glauber dynamics / Gibbs sampling

Choose a random variable, sample its value conditioned on all others.
A natural (but not working) approach

Standard sampling approach: Glauber dynamics / Gibbs sampling

Choose a random variable, sample its value conditioned on all others.
A natural (but not working) approach

Standard sampling approach: Glauber dynamics / Gibbs sampling

Choose a random variable, sample its value conditioned on all others.
A natural (but not working) approach

Standard sampling approach: Glauber dynamics / Gibbs sampling

Choose a random variable, sample its value conditioned on all others.
A natural (but not working) approach

Standard sampling approach: Glauber dynamics / Gibbs sampling

Choose a random variable, sample its value conditioned on all others.
A natural (but not working) approach

Standard sampling approach: Glauber dynamics / Gibbs sampling

Choose a random variable, sample its value conditioned on all others.
A natural (but not working) approach

Standard sampling approach: Glauber dynamics / Gibbs sampling

Choose a random variable, sample its value conditioned on all others.
Three scenarios for Markov chains

Fast mixing

Slow mixing

Not mixing!
Three scenarios for Markov chains

- Fast mixing
- Slow mixing
- Not mixing!
**Disconnectivity for k-Sat**

Suppose we have $k$ variables, and each clause contains all $k$ variables.

$$\Phi = C_1 \land C_2 \land \cdots \land C_m$$

Each $C_i$ forbids one assignment on $k$ variables.

For example, $C_i = x_1 \lor x_2 \lor \cdots \lor x_k$ forbids the all False assignment.

Thus, if we forbade all assignments of Hamming weight $i$ for some $1 \leq i \leq k - 1$ (using $\binom{k}{i}$ clauses), the solution space is not connected via single variable updates.

For example, to remove Hamming weight $k - 1$ assignments, we only need clauses

$$C_1 = \neg x_1 \lor x_2 \lor \cdots \lor x_k$$
$$C_2 = x_1 \lor \neg x_2 \lor \cdots \lor x_k$$

$$\vdots$$

$$C_k = x_1 \lor x_2 \lor \cdots \lor \neg x_k$$

In this example, the all False assignment is disconnected from the rest.
**Disconnectivity for k-Sat**

Suppose we have $k$ variables, and each clause contains all $k$ variables.

$$\Phi = C_1 \land C_2 \land \cdots \land C_m$$

Each $C_i$ **forbids** one assignment on $k$ variables.

For example, $C_i = x_1 \lor x_2 \lor \cdots \lor x_k$ forbids the all False assignment.

Thus, if we forbade all assignments of Hamming weight $i$ for some $1 \leq i \leq k - 1$ (using $\binom{k}{i}$ clauses), the solution space is not connected via single variable updates.

For example, to remove Hamming weight $k - 1$ assignments, we only need clauses

$$C_1 = \neg x_1 \lor x_2 \lor \cdots \lor x_k$$
$$C_2 = x_1 \lor \neg x_2 \lor \cdots \lor x_k$$
$$\vdots$$
$$C_k = x_1 \lor x_2 \lor \cdots \lor \neg x_k$$

In this example, the all False assignment is disconnected from the rest.
Our solution — projection

Projecting from a high dimension to a lower dimension may improve connectivity.

We will run Glauber dynamics on the projected distribution over a suitably “marked” variables.

The general problem is $\textbf{NP}$-hard, so we will focus on bounded degree cases.
**Bounded degree** $k$-**Sat**
**Theorem (Lovász local lemma)**

Let $E_1, \ldots, E_m$ be a set of “bad” events, such that $\Pr[E_i] \leq p$ for all $i$. Moreover, each $E_i$ is independent from all but at most $\Delta$ events. If $e p \Delta \leq 1$, then

$$\Pr \left[ \bigwedge_{i=1}^{m} E_i \right] > 0.$$ 

In the setting of $k$-SAT, each clause $C_i$ defines a bad event $E_i$, which is the forbidden assignment of $C_i$, and $p = 2^{-k}$.

If every variable appears in at most $d$ clauses, then $\Delta \leq kd$.

$$e p \Delta \leq 1 \iff e 2^{-k} kd \leq 1 \iff k \geq \log d + \log k + C$$
Lovász local lemma

**Theorem (Loász local lemma)**

Let $E_1, \ldots, E_m$ be a set of “bad” events, such that $\Pr[E_i] \leq p$ for all $i$. Moreover, each $E_i$ is independent from all but at most $\Delta$ events. If $ep\Delta \leq 1$, then

$$\Pr \left[ \bigwedge_{i=1}^{m} \overline{E_i} \right] > 0.$$ 

In the setting of $k$-SAT, each clause $C_i$ defines a bad event $E_i$, which is the forbidden assignment of $C_i$, and $p = 2^{-k}$.

If every variable appears in at most $d$ clauses, then $\Delta \leq kd$.

$$ep\Delta \leq 1 \iff e2^{-k}kd \leq 1$$

$$\iff k \geq \log d + \log k + C$$
We consider $k$-CNF formula with variable degree at most $d$.

**Theorem (Moser and Tardos, 2011)**

If $k \geq \log d + \log k + C$, then we can always find a satisfying assignment in polynomial time.

The algorithm is extremely simple: assign variables u.a.r., then keep resample variables in violating clauses.

Unfortunately, sampling is substantially harder.

**Theorem (Bezáková, Galanis, Goldberg, G. and Štefankovič, 2016)**

If $k \leq 2\log d + C$, then sampling satisfying assignments is NP-hard, even if there is no negation in the formula (monotone case).
We consider $k$-CNF formula with variable degree at most $d$.

**Theorem (Moser and Tardos, 2011)**

If $k \geq \log d + \log k + C$, then we can always find a satisfying assignment in polynomial time.

The algorithm is extremely simple: assign variables u.a.r., then keep resample variables in violating clauses.

Unfortunately, sampling is substantially harder.

**Theorem (Bezáková, Galanis, Goldberg, G. and Štefankovič, 2016)**

If $k \leq 2 \log d + C$, then sampling satisfying assignments is NP-hard, even if there is no negation in the formula (monotone case).
Open problem:

Is there an efficient algorithm to sample satisfying assignments of \( k \)-Sat given \( k \gtrsim 2 \log d + C \)?
Hermon, Sly and Zhang (2016) Glauber dynamics mixes in $O(n \log n)$ time if $k \geq 2 \log d + C$ and there is no negation (monotone formula).

G., Jerrum and Liu (2016) “Partial rejection sampling” terminates in $O(n)$ time if $k \geq 2 \log d + C$ and there is no small intersection.

Moitra (2016) An “exotic” deterministic algorithm in $n^{O(k^2 d^2)}$ time if $k \geq 60 (\log d + \log k) + 300$.

Theorem (Our result)

We give a Markov chain based algorithm in $\tilde{O}(n^{1+\delta} k^3 d^2)$ time if $k \geq 20 (\log d + \log k) + \log \delta^{-1}$ where $\delta \leq 1/60$ is an arbitrary constant.
**Results**

**Hermon, Sly and Zhang (2016)** Glauber dynamics mixes in $O(n \log n)$ time if $k \geq 2 \log d + C$ and there is no negation (monotone formula).

**G., Jerrum and Liu (2016)** “Partial rejection sampling” terminates in $O(n)$ time if $k \geq 2 \log d + C$ and there is no small intersection.

**Moitra (2016)** An “exotic” deterministic algorithm in $n^{O(k^2d^2)}$ time if $k \geq 60(\log d + \log k) + 300$.

**Theorem (Our result)**

We give a Markov chain based algorithm in $\tilde{O}(n^{1+\frac{3}{60}}k^3d^2)$ time if $k \geq 20(\log d + \log k) + \log \delta^{-1}$ where $\delta \leq 1/60$ is an arbitrary constant.
Our algorithm
Goal: to sample from the uniform distribution $\mu$ over satisfying assignments

1. Mark a set $M$ of variables;
2. Run Glauber dynamics on the projected distribution $\mu_M$ for $O(n \log n)$ steps. This yields an (approximate) sample $\sigma_M \sim \mu_M$;
3. Use rejection sampling to sample from $\sigma_{V \setminus M}$;
4. Output $\sigma_M \cup \sigma_{V \setminus M}$. 
A set $M$ of variables are marked so that:

1. for any clause $C_i$, $|C_i \cap M| \gtrsim 0.11k$;
2. for any clause $C_i$, $|C_i \setminus M| \gtrsim 0.51k$;

The existence of $M$ is guaranteed by the local lemma, and $M$ can be found by the Moser-Tardos algorithm in linear time.
If $|C_i \cap M|$ is large, then all components are small.

**Lemma**

For almost all $\sigma \in \{0, 1\}^M$, $V \setminus M$ scatters into connected components of size $O(\text{poly}(dk) \log n)$.

If $|C_i \setminus M|$ is large, then all variables are close to the uniform distribution.

**Lemma**

*Conditioned on any assignment of $M$, for any $v \in V \setminus M$,*

$$\left| \Pr_{\sigma \sim \mu_{V \setminus M}}[\sigma(v) = 1] - \frac{1}{2} \right| \leq \exp(-O(k)).$$

So the marking is to balance these two effects.
1. The Glauber dynamics on the marked variables is rapidly mixing;

2. The Glauber dynamics on the marked variables can be implemented efficiently;

3. The rejection sampling step in the end terminates quickly.

Item (1) is shown using the path coupling method.

Items (2) and (3) are shown together. In particular, the Glauber dynamics is implemented using rejection sampling.
Implementing the Glauber dynamics

Glauber dynamics: compute the marginal probability of a variable conditioned on all other marked variables, which defines a smaller instance. ($\#P$-hard in general.)

We approximately implement this by using rejection sampling on

1. all unmarked variables, and
2. the variable to be updated.

Rejection sampling terminates in $O(n^\delta)$ steps with high probability.

(Every clause here has at least 1 marked variable and 2 unmarked variable.)
Overview

The marking

$|C_i \setminus M|$ is large

Local uniformity

Path coupling

$O(n \log n)$ iterations

$\tilde{O}(n^{1+\delta})$ running time

$|C_i \cap M|$ is large

Small components

Rej. sampling

$O(n^\delta)$ per iteration
Why rejection sampling?

Draw $\sigma_M \sim \mu_M$. For each clause, there are at least $\Omega(k)$ variables assigned. Many clauses are satisfied, and the remaining clauses scatter into connected components of size $\approx \log n$.

However, the size is $\Omega(dk \log n)$, so a brute-force enumeration takes time $n^{\Omega(dk)}$ which is too slow to our needs.

We use the local lemma once again here to show that uniform at random assignments satisfies remaining clauses with probability at least $\Omega(n^{-\delta})$. Thus, the rejection sampling succeeds in time $\tilde{O}(n^\delta)$. 
Path coupling condition: given two assignment $\sigma_0$ and $\sigma_1$ which differ on only one variable $v$,

$$\sum_{u \in M, u \neq v} d_{TV}(\mu_u(\cdot | \sigma_0), \mu_u(\cdot | \sigma_1)) < 1.$$ 

Using the coupling inequality, it suffices to show that for a carefully designed coupling $C$ of $\mu_{M\backslash\{v\}}(\cdot | \sigma_0)$ and $\mu_{M\backslash\{v\}}(\cdot | \sigma_1)$ such that

$$\mathbb{E}_{(\tau_0, \tau_1) \sim C} \left| \{u | u \in M \backslash \{v\}, \tau_0(u) \neq \tau_1(u)\} \right| < 1.$$ 

This “disagreement coupling” $C$ is very similar to the one used by G., Liao, Lu and Zhang (2018), which is a refined version of Moitra (2016). However, previous analysis has a whp guarantee, and we need a new analysis to bound the expectation.
The disagreement percolation is similar to a branching process with branching factor $d_k$, and each child survives with probability $\exp(-O(k))$ due to local uniformity.
The disagreement percolation is similar to a branching process with branching factor $d_k$, and each child survives with probability $\exp(-O(k))$ due to local uniformity.
The disagreement percolation is similar to a branching process with branching factor $d_k$, and each child survives with probability $\exp(-O(k))$ due to local uniformity.
The disagreement percolation is similar to a branching process with branching factor $d_k$, and each child survives with probability $\exp(-O(k))$ due to local uniformity.
The disagreement percolation is similar to a branching process with branching factor $d_k$, and each child survives with probability $\exp(-O(k))$ due to local uniformity.
CONCLUDING REMARKS
In another recent work, Galanis, Goldberg, G. and Yang (2019) showed that there is an efficient algorithm to approximately count the number of satisfying assignment of a random $k$-Sat instance with high probability, if the density is at most $2^{k/300}$.

- This improves the previous best algorithm which works for density $\leq \frac{2 \log k}{k}$ (Montanari and Shah 2007).
- The algorithm is based on Moitra (2016), with some extra ingredients to handle $\Omega(\log n)$ degree variables.
- Nonetheless, it is not clear if the Markov chain approach works for random formulas.
Open problems

- Is the conjectured threshold correct?
  - Getting rid of the marking?

- Other CSPs, like hypergraph colouring?

- Other applications of this projection method?
Thank you!
arXiv:1911.01319