Category Theory for Dummies (I)

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Not quite everything you've ever wanted to know...

- You keep hearing about category theory.
- Cool-sounding papers by brilliant researchers (e.g. Wadler's "Theorems for free!")
- But it's scary and incomprehensible.
- And Category Theory is not even taught here.
- Goal of this series: Familarity with basic ideas, not expertise

Outline

- Categories: Why are they interesting?

- Categories: What are they? Examples.

- Some familiar properties expressed categorically

- Some basic categorical constructions

Category theory

- An abstract theory of "structured things" and "structure preserving function-like things".

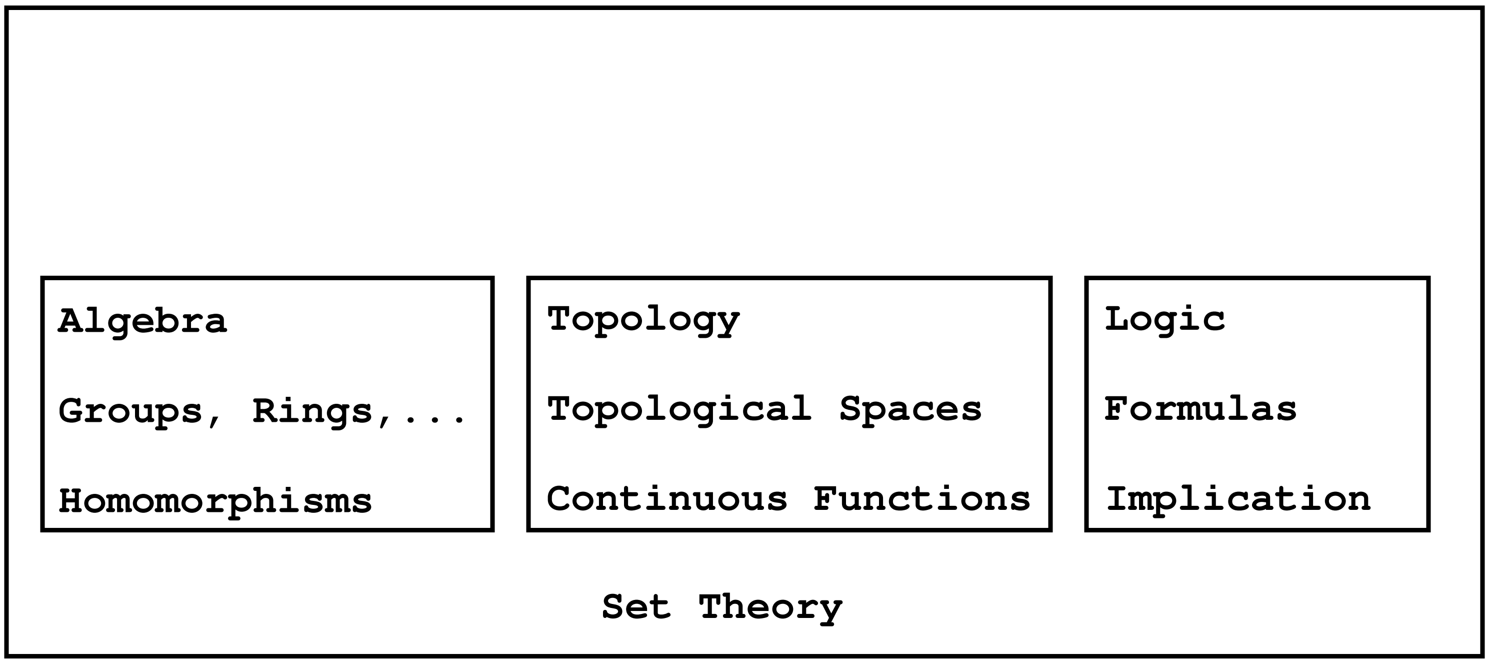
- Independent of the *concrete representation* of the things and functions.

- An alternative foundation for mathematics? (Lawvere)

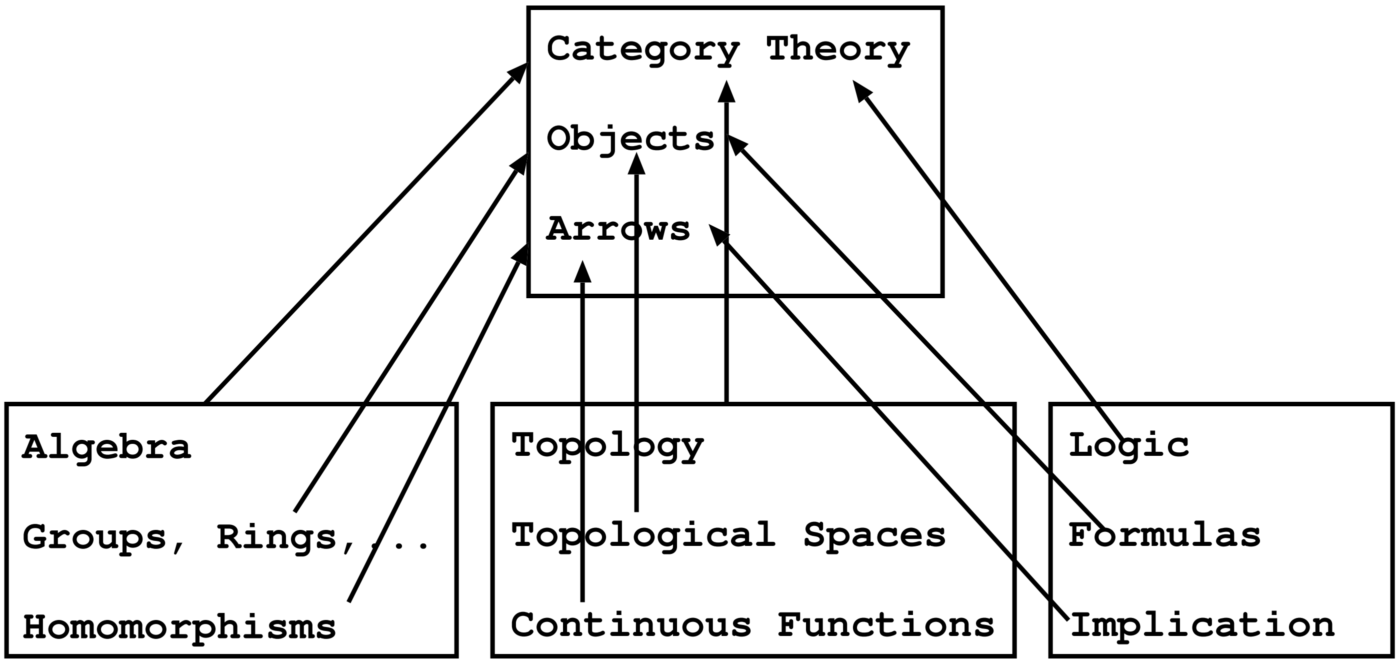
- Closely connected with computation, types and logic.

- Forbiddingly complex notation for even simple ideas.

A mathematician’s eye view of the world



A category theorist's eye view of the world



My view (not authoritative):

- Category theory helps organize thought about a collection of related things
- and identify patterns that recur over and over.
- It may suggest interesting ways of looking at them
- but does not necessarily help understand the things being studied (and may get in the way).

What is a category?

Some structures

- Sets A

- Vector spaces of vectors over $\mathbb{R}$: $(V, + : V \times V \to V, \cdot : \mathbb{R} \times V \to V)$

- ML types $\mathsf{int}, \tau \times \tau', \tau \to \tau', \tau$ list

Some classes of functions

- Set functions $f : A \to B = \{(x, f(x)) \mid x \in A\}$

- Matrices $M : V \to W$ with

$$M(\alpha \cdot_V x +_V \beta \cdot_V y) = \alpha \cdot_W f(x) +_W \beta \cdot_W f(y)$$

- Function terms $\lambda x : A.e : A \to B$

Composition

- Functions are closed under composition (when domain and range match)

- I.E., if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ then $g \circ f : A \longrightarrow C$ is a function too.

- For sets $g \circ f = \{(x, g(f(x))) \mid x \in A\}$.

- For matrices $g \circ f = g \cdot f$ (matrix multiply).

- For ML-terms, $g \circ f = \lambda x : A.g(f(x))$.

Identity

- For every structure A, there is an identity function, let's write it $id_A : A \to A$.

- For sets, $id_A = \{(x, x) \mid x \in A\}$.

- For matrices, $id_V = I$, the identity matrix over V.

- For any ML type τ, $id_\tau = \lambda x : \tau . x : \tau \to \tau$.

Facts

- Composition is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- id_A is a unit for composition: if $f : A \to B$,

$$id_B \circ f = f = f \circ id_A$$

Surprise!

- You now know the definition of a category $\mathcal{C} = (\mathcal{C}, \rightarrow, id, \circ)$

   1. $\mathcal{C}$ is a collection of objects.

   2. If A, B are in $\mathcal{C}$, then $A \rightarrow B$ is a collection of arrows f from A to B.

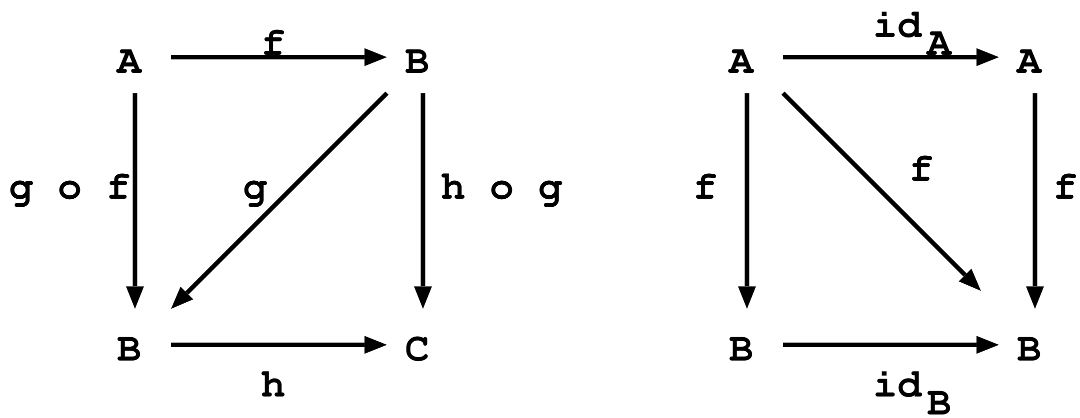
   3. $id_A : A \rightarrow A$ and whenever $f : A \rightarrow B, g : B \rightarrow C$, then $g \circ f : A \rightarrow C$.

   4. $\circ$ is associative, and id_A is a unit with respect to $\circ$.

- Note: Objects and arrows can be anything.

Diagrams

- Equations can be expressed using commutative diagrams:



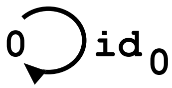
- Idea: every pair of paths with same source and target are equal.

Examples

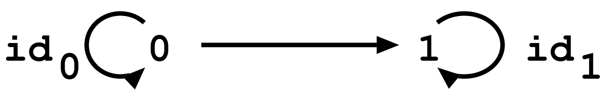
- **Set** is the category of sets and set functions.
- **Vec** is the category of vector spaces and matrices.
- **ML** is the category of ML types and function terms.
- These examples are misleading: They all have more in common than just the category structure.

Numbers as categories

- $\mathbf{0}$ is a category. It's empty.
- $\mathbf{1}$ is a category:



- $\mathbf{2}$ is a category, etc:



Some weird categories

- A *monoid* $(M, \epsilon : M, \cdot : M \times M \to M)$ is a set with an associative operation $\cdot$ with unit ϵ.

- In fact, a monoid is basically a category with one object.
  - It has one object M, and each element $x \in M$ is an arrow $x : M \to M$
  - $id_M = \epsilon$ is a unit, $x \circ y = x \cdot y$ is associative

- And a category with only one object is basically a monoid.

Some weird categories

- Similarly, any graph G can be used to construct a category:
  - Objects are vertices.
  - Arrows are *paths* (sequences of edges).
- Lesson: Objects are not always "really sets", and arrows not always "really functions".
- So what works in $\mathbf{Set}$ doesn't necessarily work in all categories. Not even close.

Categorical properties

Categorical properties

- A categorical property is something that can be defined in the language of category theory

- without reference to the underlying mathematical structure (if any).

- That is, in terms of objects, arrows, composition, identity (and equality)

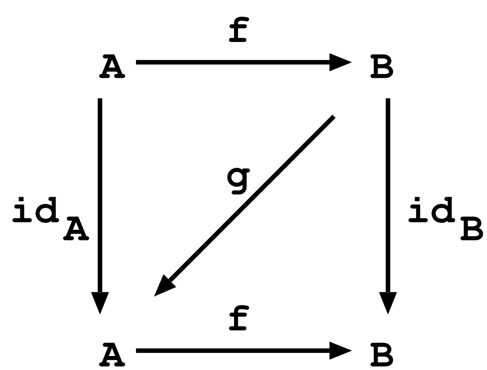
- Why? Categorical properties are meaningful in *any* category

Inverses

- "Having an inverse" is one of the most basic properties of functions.

- In $\mathcal{C}$, $f : A \to B$ has an inverse $g : B \to A$ if

$$f \circ g = id_A \qquad g \circ f = id_B$$



Isomorphism

- Invertible functions are called *isomorphisms*, and A, B are isomorphic ($A \cong B$) if there is an isomorphism in $A \to B$ (or vice versa).

- In $\mathbf{Set}$, $A \cong B$ if $|A| = |B|$.

- In $\mathbf{Vec}$, $V \cong W$ if $dim(V) = dim(W)$.

- What about $\mathbf{ML}$?

  int $\cong$ int $\qquad \tau \times \tau' \cong \tau' \times \tau \qquad \tau \to \tau_1 \times \tau_2 \cong (\tau \to \tau_1) \times (\tau \to \tau_2)$

Isomorphic = “Really the Same”

- Isomorphic objects are interchangeable as far as you can tell in $\mathcal{C}$.

- In category theory, “unique” almost always means “unique up to isomorphism”.

- Category theorists **love** proving that two very different-looking things are isomorphic.

One-to-One Functions, Monomorphisms and an Evil Pun

- In **Set**, a function is 1-1 if $f(x) = f(y)$ implies $x = y$.

- Equivalently, if $f \circ g = f \circ h$ then $g = h$ (why?)

- In $\mathcal{C}$, $f : A \to B$ is *monomorphic* if this is the case.

- Mnemonic for remembering that one-to-one functions are monomorphisms: mono a mono.

- You may groan. But you will not forget.

Onto Functions and Epimorphisms

- In $\mathbf{Set}$, a function $f : A \to B$ is onto if for every $y \in B$ there is an $x \in A$ with $y = f(x)$.

- Equivalently, if $g \circ f = h \circ f$ then $g = h$ (why?)

- In $\mathcal{C}$, $f : A \to B$ is *epimorphic* if this is the case.

- I have no evil pun for this.

Next

- Functors: Structure-preserving maps between categories

- Universal constructions: units, voids, products, sums, exponentials.

- Functions between functors: when are two "implementations of polymorphic lists" equivalent? when are two semantics equivalent?

- Even scarier stuff.