Formal languages for stochastic modelling

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1. Introduction: Performance Modelling and Process Algebras
   - Performance Modelling
   - Stochastic Process Algebra

2. Tackling State Space Explosion
   - Lumpability and Bisimulation
   - Fluid Approximation

3. Beyond Performance Modelling
The PEPA project started in Edinburgh in 1991.

- It was motivated by problems encountered when carrying out performance analysis of large computer and communication systems, based on numerical analysis of Markov chains.

- Process algebras offered a compositional description technique supported by apparatus for formal reasoning.

- Performance Evaluation Process Algebra (PEPA) sought to address these problems by the introduction of a suitable process algebra.

- We have sought to investigate and exploit the interplay between the process algebra and the continuous time Markov chain (CTMC).
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**Performance Modelling**

Performance modelling is concerned with the *dynamic behaviour* of systems and quantified assessment of that behaviour.

There are often conflicting interests at play:

- Users typically want to optimise external measurements of the dynamics such as response time (as small as possible), throughput (as high as possible) or blocking probability (preferably zero);
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There is sometimes a perception in software development that performance does not matter much, or that it is easily fixed later by buying a faster machine.

On the contrary — studies have shown that response time is a key feature in user satisfaction and trust in systems.

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There is a long association between queueing networks and continuous time Markov chains (CTMCs) more generally and performance modelling.

This dates back to Erlang’s Loss Formula for the performance of telephone exchanges in the early 20th century.

In the 1960s and 1970s queueing networks were used extensively, but the advent of distributed systems in the 1980s meant that many systems no longer fit the assumptions of queueing networks.

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Performance Modelling using CTMC

Model Construction

\[
Q = \begin{pmatrix}
-\Sigma & \cdots \\
\cdots & -\Sigma & \cdots \\
\cdots & \cdots & -\Sigma & \cdots \\
\cdots & \cdots & \cdots & -\Sigma
\end{pmatrix}
\]

\text{MARKOV PROCESS}

\text{TRANSITION DIAGRAM}

\text{STATE}

\text{SYSTEM}
Performance Modelling using CTMC

Model Construction

- describing the system using a high level modelling formalism
- generating the underlying CTMC
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**Model Manipulation**
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**Model Solution**
- solving the CTMC to find steady state or transient probability distribution
- deriving performance measures
Process Algebra

- Models consist of agents which engage in actions.

\[ \alpha \cdot P \]

- The structured operational (interleaving) semantics of the language is used to generate a labelled transition system.
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  - action type or name
  - agent/component

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Process algebra model
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Process algebra model $\xrightarrow{\text{SOS rules}}$
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Process algebra model \(\xrightarrow{\text{SOS rules}}\) Labelled transition system
Process algebra operators

Process algebras generally have a number of different operators for combining actions and components, typically including:

- **Prefix**: \( \cdot \) – designated first action;
- **Choice**: \( + \) – selection between alternative components;
- **Parallel composition**: \( \parallel \) – components working concurrently;

These operators have rules associated with them such as

\[
P \parallel (Q \parallel R) = (P \parallel Q) \parallel R
\]

and

\[
P + P = P
\]
Bisimulation and congruence

Process algebras are usually equipped with an equivalence relation, termed a **bisimulation**, meaning that one component is equivalent to another if it can copy or simulate any action of the other component and vice versa.

Languages are designed so that these relations are designed so that these equivalence relations are **congruences** with respect to the operators of the language.

For example, if $P \sim Q$ then

- $\alpha.P \sim \alpha.Q$,
- $P + R \sim Q + R$ and
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Stochastic process algebras

Stochastic process algebra

Process algebras where models are decorated with quantitative information used to generate a stochastic process are stochastic process algebras (SPA).
Stochastic Process Algebra

- Models are constructed from components which engage in activities.

\[(\alpha, r).P\]

- The language is used to generate a Continuous Time Markov Chain (CTMC) for performance modelling.
Stochastic Process Algebra

- Models are constructed from **components** which engage in activities.

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  - action type or name
  - activity rate (parameter of an exponential distribution)
  - component/derivative

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- The language is used to generate a **Continuous Time Markov Chain (CTMC)** for performance modelling.
PEPA components perform activities either independently or in co-operation with other components.

\[
\begin{align*}
(\alpha, r).P & \quad \text{Prefix} \\
P_1 + P_2 & \quad \text{Choice} \\
P_1 \lhd P_2 & \quad \text{Co-operation} \\
P/L & \quad \text{Hiding} \\
X & \quad \text{Variable}
\end{align*}
\]

\[P_1 \parallel P_2\] is a derived form for \[P_1 \lhd P_2\].

When working with large numbers of entities, we write \(P[n]\) to denote an array of \(n\) copies of \(P\) executing in parallel.

\[P[5] = (P \parallel P \parallel P \parallel P \parallel P)\]
Performance Evaluation Process Algebra

**PEPA components** perform activities either independently or in co-operation with other components.

\[(\alpha, r).P\] Prefix

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Introduction: Performance Modelling and Process Algebras

Performance Evaluation Process Algebra

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A simple example: processors and resources

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\[\text{Proc}_0 \Downarrow \langle \text{task1} \rangle \text{ Res}_0\]

\[
Q = \begin{pmatrix}
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Why use a process algebra?

- **High level description** of the system eases the task of model construction.
- **Formal language** allows for unambiguous interpretation and automatic translation into the underlying mathematical structure.
- **Properties of that mathematical structure** may be deduced by the construction at the process algebra level.
- **Formal reasoning techniques** such as equivalence relations and model checking can be used to manipulate or interrogate models.
- **Compositionality** can be exploited both for model construction and (in some cases) for model analysis.
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Benefits of process algebra

For example,

- The correspondence between the congruence, Markovian bisimulation, in the process algebra and the lumpability condition in the CTMC, allows exact model reduction to be carried out compositionally.

- Characterisation of product form structure at the process algebra level allows decomposed model solution based on the process algebra structure of the model.

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Deriving performance measures

Under the SOS semantics a SPA model is mapped to a **CTMC** with global states determined by the local states of all the participating components.
Deriving performance measures

When the size of the state space is not too large they are amenable to numerical solution (linear algebra) to determine a steady state or transient probability distribution.

\[ Q = \begin{pmatrix}
q_{1,1} & q_{1,2} & \cdots & q_{1,N} \\
q_{2,1} & q_{2,2} & \cdots & q_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
q_{N,1} & q_{N,2} & \cdots & q_{N,N}
\end{pmatrix} \]

\[ \pi(t) = (\pi_1(t), \pi_2(t), \ldots, \pi_N(t)) \]
Alternatively they may be studied using **stochastic simulation**. Each run generates a single trajectory through the state space. Many runs are needed in order to obtain average behaviours.
State space explosion

As the size of the state space becomes large it becomes infeasible to carry out numerical solution and extremely time-consuming to conduct stochastic simulation.
Model Manipulation

Model simplification: use a model-model equivalence to substitute one model by another which is more attractive from a solution point of view, e.g. smaller state space, special class of model, etc.

Model aggregation: use a state-state equivalence to establish a partition of the state space of a model, and replace each set of states by one macro-state, i.e. take a different stochastic representation of the same model.
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- In the early 1960’s Kemeny and Snell established the conditions under which it was possible to aggregate a Markov chain and still have a Markov chain afterwards.

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Equivalence Relations

In process algebra equivalence relations are defined based on the notion of observability:
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\[ (a,r) \xrightarrow{(b,s)} (c,t) \xrightarrow{(d,u)} (a,r) \]

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\[
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\end{align*}
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In PEPA observation is assumed to include the ability to record timing information over a number of runs.

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The formal definition means this can be applied automatically and compositionally.

State space explosion

As the size of the state space becomes large it becomes infeasible to carry out numerical solution and extremely time-consuming to conduct stochastic simulation.

In these cases we would like to take advantage of the mean field or fluid approximation techniques.

Use continuous state variables to approximate the discrete state space and ordinary differential equations to represent the evolution of those variables over time.

Appropriate for models in which there are large numbers of components of the same type, i.e. models of populations and situations of collective dynamics.
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Fluid approximation theorem

Hypothesis

- $\overline{X}^{(N)}(t)$: a sequence of normalized population CTMC, residing in $E \subset \mathbb{R}^n$
- $\exists x_0 \in S$ such that $\overline{X}^{(N)}(0) \rightarrow x_0$ in probability (initial conditions)
- $x(t)$: solution of $\frac{dx}{dt} = F(x)$, $x(0) = x_0$, residing in $E$.

(Density dependent CTMCs are a special case.)

Theorem

For any finite time horizon $T < \infty$, it holds that:

$$\mathbb{P}\left( \sup_{0 \leq t \leq T} \| \overline{X}^{(N)}(t) - x(t) \| > \varepsilon \right) \rightarrow 0.$$
Simple example revisited

\[
\begin{align*}
    \text{Proc}_0 & \overset{\text{def}}{=} (\text{task}_1, r_1).\text{Proc}_1 \\
    \text{Proc}_1 & \overset{\text{def}}{=} (\text{task}_2, r_2).\text{Proc}_0 \\
    \text{Res}_0 & \overset{\text{def}}{=} (\text{task}_1, r_1).\text{Res}_1 \\
    \text{Res}_1 & \overset{\text{def}}{=} (\text{reset}, r_4).\text{Res}_0
\end{align*}
\]

\[
\text{Proc}_0[N_P] \begin{array}{c}
\ltimes \vphantom{1}
\end{array} \{\text{task}_1\} \text{Res}_0[N_R]
\]
Simple example revisited

\[
\begin{align*}
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\end{align*}
\]

**CTMC interpretation**

<table>
<thead>
<tr>
<th>Processors ($N_P$)</th>
<th>Resources ($N_R$)</th>
<th>States ($2^{N_P+N_R}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
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</tr>
<tr>
<td>10</td>
<td>9</td>
<td>524288</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>1048576</td>
</tr>
</tbody>
</table>
Simple example revisited

\[
\begin{align*}
Proc_0 & \overset{\text{def}}{=} (task1, r_1).Proc_1 \\
Proc_1 & \overset{\text{def}}{=} (task2, r_2).Proc_0 \\
Res_0 & \overset{\text{def}}{=} (task1, r_1).Res_1 \\
Res_1 & \overset{\text{def}}{=} (reset, r_4).Res_0
\end{align*}
\]

- \(task1\) decreases \(Proc_0\) and \(Res_0\)
- \(task1\) increases \(Proc_1\) and \(Res_1\)
- \(task2\) decreases \(Proc_1\)
- \(task2\) increases \(Proc_0\)
- \(reset\) decreases \(Res_1\)
- \(reset\) increases \(Res_0\)

\[Proc_0[N_P] \boxtimes_{\{task1\}} Res_0[N_R]\]
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\[
\frac{dx_1}{dt} = -\min(r_1 x_1, r_3 x_3) + r_2 x_2 \\
x_1 = \text{no. of } \text{Proc}_1
\]

- task1 decreases Proc_0
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- task2 is performed by Proc_1
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ODE interpretation

\[ \frac{dx_1}{dt} = -\min(r_1 x_1, r_3 x_3) + r_2 x_2 \]
\[ x_1 = \text{no. of Proc}_1 \]
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100 processors and 80 resources (simulation run A)
100 processors and 80 resources (simulation run B)
100 processors and 80 resources (simulation run C)
100 processors and 80 resources (average of 10 runs)
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100 processors and 80 resources (average of 1000 runs)
Deriving a Fluid Approximation of a PEPA model

The aim is to represent the CTMC *implicitly* (avoiding state space explosion), and to generate the set of ODEs which are the fluid limit of that CTMC.

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![Diagram](image)
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Fluid Structured Operational Semantics

In order to get to the implicit representation of the CTMC we need to:

1. **Context Reduction**: Remove excess components to find the abstract state representation $\xi$.

2. **Jump Multiset**: Collect the transitions $\alpha$ of the reduced context in terms of update vectors $l$.

3. **Generating Functions**: Calculate the rate of the transitions in terms of an arbitrary state of the CTMC, $f(\xi, l, \alpha)$.

Once this is done we can extract the vector field $F_M(x)$ from the jump multiset.

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Rate properties of PEPA models

Density dependence of parametric transition rates

The transition rates scale in the same way as the population, i.e. if $ P \xrightarrow{r(\xi)} Q $ then, for any $ n \in \mathbb{N} $, $ r(\xi) = n \cdot r(\xi/n) $.

Generating functions give rise to density dependent rates

Let $ M $ be a PEPA model with generating functions $ f(\xi, l, \alpha) $. Then the corresponding sequence of CTMCs will be density dependent.

Lipschitz continuity of parametric apparent rates

Let $ r^*_\alpha (P, \xi) $ be the parametric apparent rate of action type $ \alpha $ in process $ P $. There exists a constant $ L \in \mathbb{R} $ such that for all $ x, y \in \mathbb{R}^d, x \neq y $,

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Kurtz’s Theorem

Kurtz’s Theorem for PEPA

Let $x(t), 0 \leq t \leq T$ satisfy the initial value problem
\[
\frac{dx}{dt} = F(x(t)), \quad x(0) = \delta,
\]
specified from a PEPA model.

Let $\{X_n(t)\}$ be a family of CTMCs with parameter $n \in \mathbb{N}$ generated as explained and let $X_n(0) = n \cdot \delta$. Then,
\[
\forall \varepsilon > 0 \lim_{n \to \infty} \mathbb{P}\left( \sup_{t \leq T} \|X_n(t)/n - x(t)\| > \varepsilon \right) = 0.
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Moreover Lipschitz continuity of the vector field guarantees existence and uniqueness of the solution to the initial value problem.

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In some case there have been new languages developed to support particular features of the application domain. These have included stochastic process algebras for modelling hybrid systems, spatial temporal systems and ecological processes.

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### Molecular processes as concurrent computations

<table>
<thead>
<tr>
<th>Concurrency</th>
<th>Molecular Biology</th>
<th>Metabolism</th>
<th>Signal Transduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concurrent computational processes</td>
<td>Molecules</td>
<td>Enzymes and metabolites</td>
<td>Interacting proteins</td>
</tr>
<tr>
<td>Synchronous communication</td>
<td>Molecular interaction</td>
<td>Binding and catalysis</td>
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</tr>
<tr>
<td>Transition or mobility</td>
<td>Biochemical modification or relocation</td>
<td>Metabolite synthesis</td>
<td>Protein binding, modification or sequestration</td>
</tr>
</tbody>
</table>

**Bio-PEPA modelling**

- The **state of the system** at any time consists of the **local states** of each of its sequential/species components.

- The local states of components are **quantitative** rather than functional, i.e. biological changes to species are represented as distinct components.

- A component varying its state corresponds to it varying its amount.

- This is captured by an integer parameter associated with the species and the effect of a reaction is to vary that parameter by a number corresponding to the **stoichiometry** of this species in the reaction.
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- Each species $i$ is described by a species component $C_i$.

- Each reaction $j$ is associated with an action type $\alpha_j$ and its dynamics is described by a specific function $f_{\alpha_j}$.

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The syntax

**Sequential component (species component)**

\[
S ::= (\alpha, \kappa) \text{ op } S \mid S + S \mid C \quad \text{where } \text{op} = \downarrow \mid \uparrow \mid \oplus \mid \ominus \mid \odot
\]

**Model component**

\[
P ::= P \triangleleft P \mid S(I)
\]

Each action \(\alpha_j\) is associated with a rate \(f_{\alpha_j}\)

The list \(\mathcal{N}\) contains the numbers of levels/maximum concentrations
The syntax

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where \( \text{op} = \downarrow \mid \uparrow \mid \oplus \mid \ominus \mid \odot \)

**Model component**

\[ P ::= P \triangleleft P \mid S(I) \]

Each action \( \alpha_j \) is associated with a rate \( f_{\alpha_j} \)

The list \( \mathcal{N} \) contains the numbers of levels/maximum concentrations
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Model component

\[ P ::= P \triangleleft L P | S(I) \]

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**Model component**

\[ P ::= P \text{ op}_L P \mid S(l) \]

Each action $\alpha_j$ is associated with a rate $f_{\alpha_j}$

The list $\mathcal{N}$ contains the numbers of levels/maximum concentrations
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The list \( N \) contains the numbers of levels/maximum concentrations
The semantics is defined by two transition relations:

- First, a **capability relation** — is a transition possible?
- Second, a **stochastic relation** — gives rate of a transition, derived from the parameters of the model.

The labelled transition system generated by the stochastic relation formally defines the underlying CTMC.

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Example — in Bio-PEPA

\[ k_s = 0.5; \]
\[ k_r = 0.1; \]

\[ \text{kineticLawOf spread} : k_s * I * S; \]
\[ \text{kineticLawOf stop1} : k_r * S * S; \]
\[ \text{kineticLawOf stop2} : k_r * S * R; \]

\[ I = (\text{spread},1) \downarrow ; \]
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\]

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Conclusions

- **Stochastic process algebras** provide high-level description languages which can ease the task of model construction for large CTMC models.

- The formal nature of the language allows for unambiguous interpretation and automatic CTMC generation.

- Properties of the underlying mathematical structure can be detected at the syntax level and proof obligations can be carried out once and for all in the semantics of the language.

- Languages can be tailored to particular application domains making it easier for non-experts to build and analyse Markovian models.
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Thank you