High-level languages for fluid approximation of agent-based models

Jane Hillston
LFCS, University of Edinburgh

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Outline

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The Discrete World View

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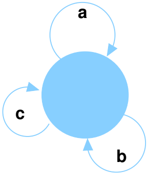

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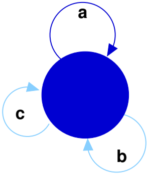

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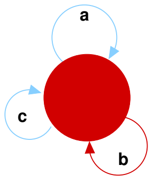

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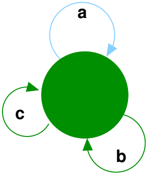

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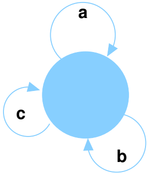


Various formalisms have been designed for capturing such behaviour.

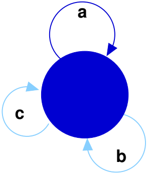
Process Algebra

- Models consist of agents which engage in actions.

$$\alpha.P$$

action type or name

agent/ component



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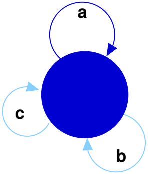

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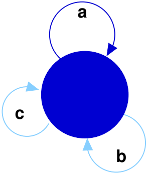

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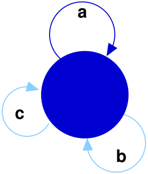

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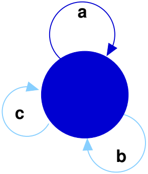

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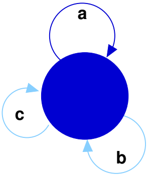

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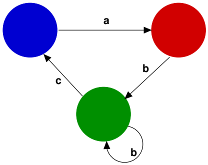

Quantitative Modelling

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These techniques are no longer widely applicable for expressing the dynamic behaviour observed in distributed systems with concurrent behaviour.

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Primary examples include:

- Stochastic Petri Nets and
- Stochastic/Markovian Process Algebras.

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Integrated analysis

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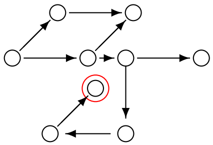
Reachability analysis

How long will it take for the system to arrive in a particular state?



Integrated analysis

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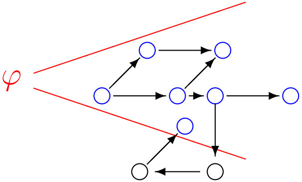
Model checking

Does a given property φ hold within the system with a given probability?

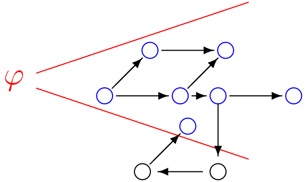


Integrated analysis

Qualitative verification can now be complemented by quantitative verification.

Model checking

For a given starting state
how long is it until
a given property φ holds?



Performance Evaluation Process Algebra

PEPA components perform activities either independently or in co-operation with other components.

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$$P[5] \equiv (P \parallel P \parallel P \parallel P \parallel P)$$

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$$\frac{E \xrightarrow{(\alpha,r)} E'}{E + F \xrightarrow{(\alpha,r)} E'}$$

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Structured Operational Semantics: Cooperation ($\alpha \notin L$)

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Cooperation $$\frac{E \xrightarrow{(\alpha,r_1)} E' \qquad F \xrightarrow{(\alpha,r_2)} F'}{E \underset{L}{\bowtie} F \xrightarrow{(\alpha,R)} E' \underset{L}{\bowtie} F'} \; (\alpha \in L)$$

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where $R = \dfrac{r_1}{r_\alpha(E)} \dfrac{r_2}{r_\alpha(F)} min(r_\alpha(E), r_\alpha(F))$

Apparent Rate

$$
\begin{aligned}
r_\alpha((\beta, r).P) &= \begin{cases} r & \beta = \alpha \\ 0 & \beta \neq \alpha \end{cases} \\
r_\alpha(P + Q) &= r_\alpha(P) + r_\alpha(Q) \\
r_\alpha(A) &= r_\alpha(P) \qquad \text{where } A \stackrel{\mathit{def}}{=} P \\
r_\alpha(P \underset{L}{\bowtie} Q) &= \begin{cases} r_\alpha(P) + r_\alpha(Q) & \alpha \notin L \\ \mathit{min}(r_\alpha(P), r_\alpha(Q)) & \alpha \in L \end{cases} \\
r_\alpha(P/L) &= \begin{cases} r_\alpha(P) & \alpha \notin L \\ 0 & \alpha \in L \end{cases}
\end{aligned}
$$

Structured Operational Semantics: Hiding

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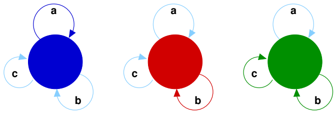
Structured Operational Semantics: Constants

Constant

$$\frac{E \xrightarrow{(\alpha,r)} E'}{A \xrightarrow{(\alpha,r)} E'} \; (A \stackrel{def}{=} E)$$

Solving discrete state models



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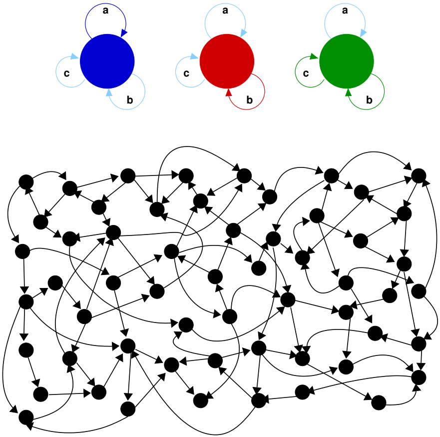

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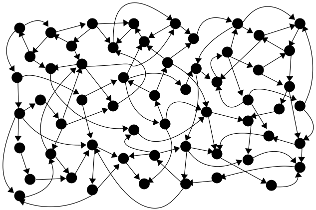

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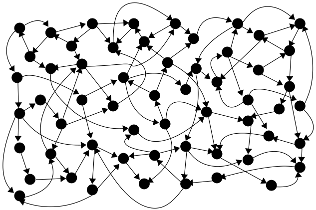

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$$Q = \begin{pmatrix} q_{1,1} & q_{1,2} & \cdots & q_{1,N} \\ q_{2,1} & q_{2,2} & \cdots & q_{2,N} \\ \vdots & \vdots & & \vdots \\ q_{N,1} & q_{N,2} & \cdots & q_{N,N} \end{pmatrix}$$

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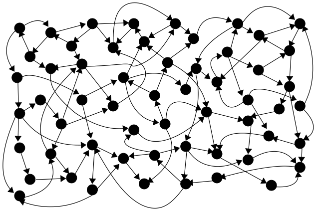

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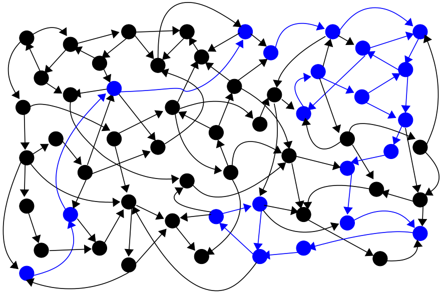
$$\pi(t) = (\pi_1(t), \pi_2(t), \ldots, \pi_N(t))$$

$$\pi(\infty)Q = 0$$

Solving discrete state models

Alternatively they may be studied using stochastic simulation. Each run generates a single trajectory through the state space. Many runs are needed in order to obtain average behaviours.



PEPA applied in practice (discrete)

- QoS protocols for mobile devices

  H.Wang, D.I.Laurenson and J.Hillston. A General Performance Evaluation Framework for Network Selection Strategies in 3G-WLAN Interworking Networks. IEEE TMC 2013.

- Disease spread within populations

  S.Benkirane, R.Norman, E.Scott and C.Shankland. Measles Epidemics and PEPA: An Exploration of Historic Disease Dynamics Using Process Algebra. Formal Methods 2012.

- Clinical pathways in hospitals

  X.Yang, R.Han, Y.Guo, J.T.Bradley, B.Cox, R.Dickinson and R.Kitney. Modelling and performance analysis of clinical pathways using the stochastic process algebra PEPA. BMC Bioinformatics 2012.

- Mobile applications

  N.Arijo, R.Heikel, M.Tribastone and S.Gilmore. Modular performance modelling for mobile applications. ICPE 2011.

- Security application: Key distribution centres

  Y.Zhao and N.Thomas. Efficient solutions of a PEPA model of a key distribution centre. Performance Evaluation 2010.

State space explosion

As the size of the state space becomes large it becomes infeasible to carry out numerical solution and extremely time-consuming to conduct stochastic simulation.

The Fluid Approximation Alternative

When there are repeated instances of agents (populations) in the system there is an alternative: fluid approximation.

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For a large class of models, just as the size of the state space becomes unmanageable, the models become amenable to an efficient, scale-free approximation.

Identity and Individuality

Population systems are constructed from many instances of a set of components.

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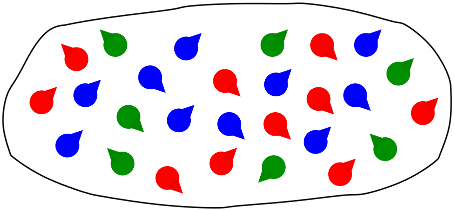

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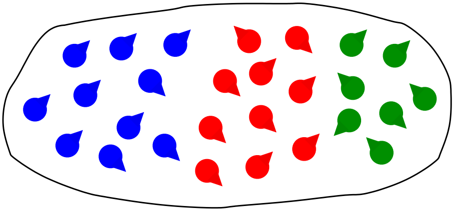

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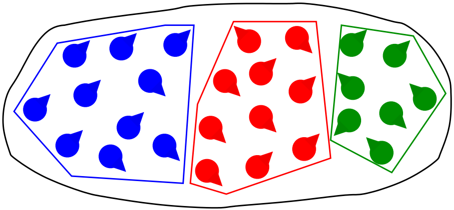

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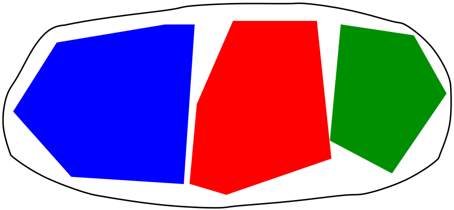

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Even better reductions can be achieved when we no longer regard the components as individuals.

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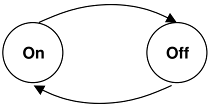
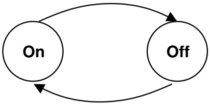
Population models — intuition



$Y(t)$

Population models — intuition



$$Y(t)$$

N copies: $Y_i^{(N)}$

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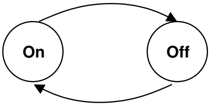

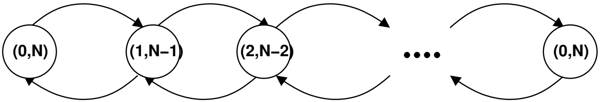

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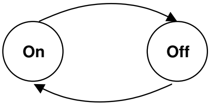

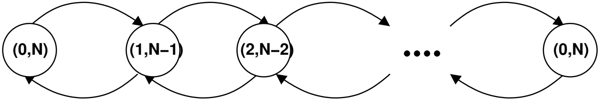

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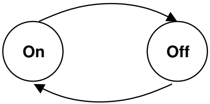

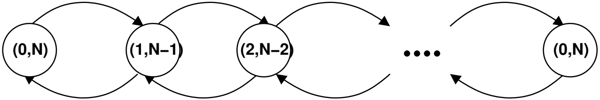

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$$X_j^{(N)} = \sum_{i=1}^{N} \mathbf{1}\{Y_i^{(N)} = j\}$$

- $Y(t)$, $Y_i^{(N)}(t)$ and $\mathbf{X}^{(N)}(t)$ are all CTMCs;
- As N increases we get a sequence of CTMCs, $\mathbf{X}^{(N)}(t)$

Population state space

- The population process $\mathbf{X}^{(N)} = (X_1^{(N)}, \ldots, X_n^{(N)})$ has the dimension of the state space of $Y(t)$.

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- If we make the closed world assumption: $\sum_{j=1}^{n} X_j^{(N)} = N$
- N.B. PEPA models always satisfy the closed world assumption.

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- Each transition is specified by a rate function $r_\tau^{(N)}$, and by an update vector $\mathbf{v}_\tau$, specifying the impact of the event on the population vector.
- The infinitesimal generator matrix $Q^{(N)}$ of $\mathbf{X}^{(N)}(t)$ is defined as:

$$q_{\mathbf{x},\mathbf{x}'} = \sum\{r_\tau(\mathbf{x}) \mid \tau \in \mathcal{T},\ \mathbf{x}' = \mathbf{x} + \mathbf{v}_\tau\}.$$

Population models — summary of notation

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We have N individuals $Y_i^{(N)} \in S$, $S = \{1, 2, \ldots, n\}$ in the system (can have multiple classes).

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Dynamics (system level)

$\mathbf{X}^{(N)}$ is a CTMC with transitions $\tau \in \mathcal{T}$:

$$\tau\text{: } \mathbf{X}^{(N)} \text{ to } \mathbf{X}^{(N)} + \mathbf{v}_\tau \text{ at rate } r_\tau^{(N)}(\mathbf{X})$$

Scaling Conditions

Scaling assumptions

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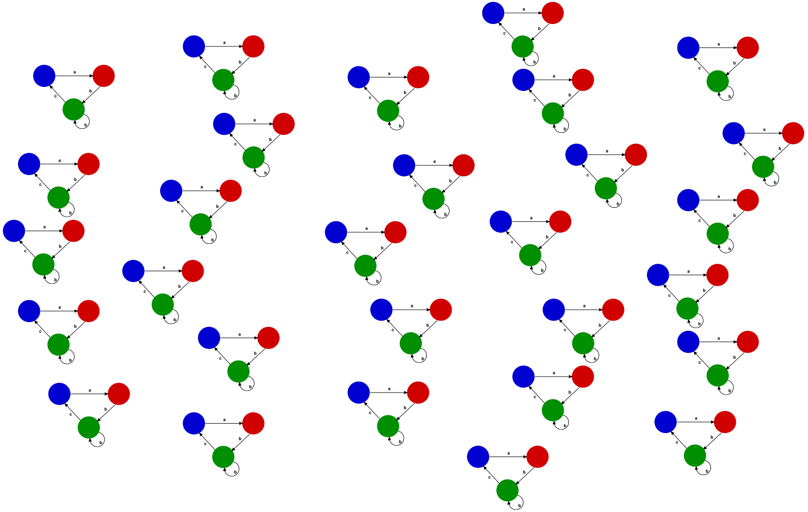
  - the normalised update is $\hat{\mathbf{v}} = \mathbf{v}/N$
  - there is a normalised rate function $\hat{r}_\tau(\hat{\mathbf{X}})$
- $\forall \tau$ assume there exists a bounded and Lipschitz continuous function $f_\tau(\hat{\mathbf{X}})$, the limit rate function on normalised variables, independent of N, such that $\frac{1}{N}\hat{r}_\tau^{(N)}(\mathbf{x}) \to f_\tau(\mathbf{x})$ uniformly.

Normalised process — intuition



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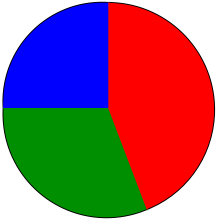
The whole population is represented as a single process.



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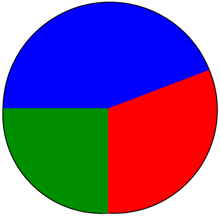

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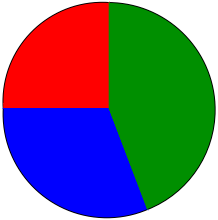

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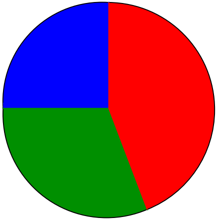

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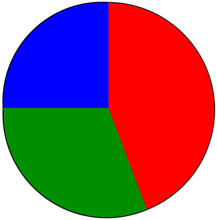

Even when the number of individuals varies ($N \longrightarrow \infty$) the processes remain comparable.

Drift

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The drift $F^{(N)}(\hat{\mathbf{X}})$ — the mean instantaneous increment of model variables in state $\hat{\mathbf{X}}$ — is defined as

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Limit Drift

Let f_τ be the limit rate functions.

The limit drift of the model $\hat{\mathcal{X}}^{(N)}$ is

$$F(\hat{\mathbf{X}}) = \sum_{\tau \in \hat{\mathcal{T}}} \mathbf{v}_\tau f_\tau(\hat{\mathbf{X}}),$$

and $F^{(N)}(\mathbf{x}) \to F(\mathbf{x})$ uniformly as $N \longrightarrow \infty$.

Fluid ODE and Fluid approximation theorem

Fluid ODE

The fluid ODE is

$$\frac{d\mathbf{x}}{dt} = F(\mathbf{x}), \quad \textit{with } \mathbf{x}(0) = \mathbf{x_0} \in S.$$

Since F is Lipschitz (all f_τ are), this ODE has a unique solution $\mathbf{x}(t)$ starting from $\mathbf{x_0}$.

Deterministic Approximation Theorem (Kurtz)

Assume that $\exists\, \mathbf{x_0} \in S$ such that $\hat{\mathbf{X}}^{(N)}(0) \to \mathbf{x_0}$ in probability. Then, for any finite time horizon $T < \infty$, it holds that as $N \longrightarrow \infty$:

$$\mathbb{P}\left\{ \sup_{0 \leq t \leq T} ||\hat{\mathbf{X}}^{(N)}(t) - \mathbf{x}(t)|| > \varepsilon \right\} \to 0.$$

T.G.Kurtz. *Solutions of ordinary differential equations as limits of pure jump Markov processes.* Journal of Applied Probability, 1970.

Fluid Approximation ODEs

The fluid approximation ODEs can be interpreted in two different ways:

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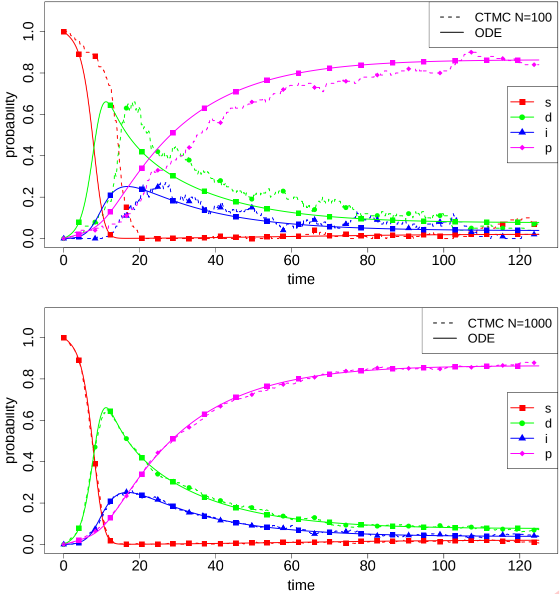
We focus on the second interpretation — a functional version of the Law of Large Numbers.

Instead of having a sequence of random variables, converging to a deterministic value, here we have a sequence of CTMCs for increasing population size, which converge to a deterministic trajectory, the solution of the fluid ODE.

Illustrative trajectories



Comparison of the limit fluid ODE and a single stochastic trajectory of a network epidemic example, for total populations $N = 100$ and $N = 1000$.

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The major impediment is state space explosion and fluid approximation offers a solution to that problem.

Fluid semantics for Stochastic Process Algebras

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- Moreover the derivation of the ODEs can be automated in the implementation of the language.

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The agents whose initial state is in each subset correspond to that component.

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2. Collect the transitions of the reduced context as symbolic updates on the state representation (Jump Multiset)
3. Calculate the rate of the transitions in terms of an arbitrary state of the CTMC.

Once this is done we can extract the vector field $F_{\mathcal{M}}(x)$ from the jump multiset, under the assumption that the population size tends to infinity.

M.Tribastone, S.Gilmore and J.Hillston. Scalable Differential Analysis of Process Algebra Models. IEEE TSE 2012.

Consistency results

- The vector field $\mathcal{F}(x)$ is Lipschitz continuous i.e. all the rate functions governing transitions in the process algebra satisfy local continuity conditions.

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- Thus the hypotheses of the Deterministic Approximation Theorem are satisfied.
- The generated ODEs are the fluid limit of the family of CTMCs and so approximate the discrete behaviour as the size of the system grows.
- Moreover Lipschitz continuity of the vector field guarantees existence and uniqueness of the solution to the initial value problem.

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- Fluid approximation of passage times have been defined.

R.A.Hayden, A.Stefanek and J.T.Bradley. Fluid computation of passage-time distributions in large Markov models. TCS 2012.

PEPA applied in practice (continuous)

- Denial of service attacks

  S.Gilmore, J.Hillston and N.Zon. Abstraction Interpretation of PEPA Models. Semantics, Logics and Calculi 2016.

- Emergency egress

  A.Kumar Singh and S. Kaur. Analysis of Emergency Evacuation of Buildings using PEPA. IDCDIT 2015.

  M.Massink, D.Latella, A.Bracciali and M.Harrion. A scalable fluid flow process algebraic approach to emergency egress. SEFM 2010.

- Security protocols Y.Zhao and N.Thomas. A Simplified Solution of a PEPA model of the Kerberos Protocol. CyberC 2011.

- Energy considerations in computing systems

  A.Stefanek, R.A.Hayden and J.T.Bradley. Fluid computation of the performance/energy trade-off in large scale Markov models. Performance Evaluation Review 2011.

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- Process algebras, such as PEPA, are well-suited to modelling the behaviour of such systems in terms of the individuals and their interactions.
- Continuous approximation allows a rigorous mathematical analysis of the average behaviour of such systems.
- This alternative view of systems has opened up many and exciting new research directions.

Thanks

Thanks

Some of this work was carried out under the auspices of the QUANTICOL project funded by the FET-Proactive FoCAS initiative:

quanticol

www.quanticol.eu